The Double Riemann Zeta Function

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(a joint work with N. Kurokawa)

Double Zeta Function

= A function whose zero is given by a sum of zeros of zeta functions.

Possible Keys to the R.H.

[1] Additive structure of zeros
[2] Fixed width of the critical strip
(The width is always 1.)

Motivation

Z_j: (usual) zeta function with zeros ρ_j
$Z_1 \otimes Z_2$: a multiple zeta with zeros $\rho_1 + \rho_2$

	Deligne's Theorem (Weil Conjecture)	Ramanujan Conj (Selberg e.v. conj)
Z	congruence zeta	$L(s,\varphi)$ φ: a Maass form
ρ	nontrivial zeros nontrivial poles	$s = \pm ir$ (trivial zeros)
R.H.	$\mathrm{Re}(\rho) = \frac{k}{2}$	$\mathrm{Re}(ir) = 0$ $\lambda = \frac{1}{4} + r^2$
critical strip	$\left\|\mathrm{Re}(s) - \frac{k}{2}\right\| < \frac{1}{2}$	$\|\mathrm{Re}(ir)\| < 1/2$ (Jacquet-Shalika)
$Z \otimes Z$	congruence zeta	$L\left(s, \mathrm{Sym}^2(\varphi)\right)$
By [1]	zeros at $s = 2\rho$	$s = \pm 2ir$
R.H.	$\mathrm{Re}(2\rho) = k$	$\mathrm{Re}(2ir) = 0$
By [2]	$\|\mathrm{Re}(2\rho) - k\| < \frac{1}{2}$ $\left\|\mathrm{Re}(\rho) - \frac{k}{2}\right\| < \frac{1}{4}$	$\|\mathrm{Re}(2ir)\| < 1/2$ $\|\mathrm{Re}(ir)\| < 1/4$

We first calculated the simplest case:

Double Hasse zeta function for finite fields

p, q: prime numbers

$$\zeta(s, \mathbf{F}_p) = (1 - p^{-s})^{-1},$$

$$\zeta(s, \mathbf{F}_q) = (1 - q^{-s})^{-1}.$$

Poles are at

$$2\pi i \frac{k}{\log p} \quad \text{and} \quad 2\pi i \frac{n}{\log q} \quad (k, n \in \mathbf{Z}).$$

We constructed a new zeta function $\zeta_{p,q}(s)$ having zeros at

$$s = 2\pi i \left(\frac{k}{\log p} + \frac{n}{\log q} \right) \quad (k, n \geq 0),$$

and poles at

$$s = 2\pi i \left(\frac{k}{\log p} + \frac{n}{\log q} \right) \quad (k, n < 0).$$

Theorem 1 (Composit. Math. (2004)) *Let p, q be distinct primes. We define in* $\mathrm{Re}(s) > 0$

$$\begin{aligned}\zeta_{p,q}(s) \;:=\; \exp\Bigg(&-\frac{\sqrt{-1}}{2}\sum_{n=1}^{\infty}\frac{\cot\left(\pi n\frac{\log p}{\log q}\right)}{n}p^{-ns}\\ &-\frac{\sqrt{-1}}{2}\sum_{n=1}^{\infty}\frac{\cot\left(\pi n\frac{\log q}{\log p}\right)}{n}q^{-ns}\\ &-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}p^{-ns}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}q^{-ns}\Bigg).\end{aligned}$$

(0) It converges absolutely in $\mathrm{Re}(s) > 0$.

(1) The function $\zeta_{p,q}(s)$ has an analytic continuation to all $s \in \mathbf{C}$ as a meromorphic function of order two.

(2) All zeros and poles of $\zeta_{p,q}(s)$ are simple and located at

$$s = 2\pi\sqrt{-1}\left(\frac{m}{\log p}+\frac{n}{\log q}\right),$$

where (m,n) is a pair of nonnegative integers or a pair of negative integers. Indeed it gives a zero or pole according as they are nonnegative or negative.

(3) We have the identification

$$\begin{aligned}\zeta_{p,q}(s) &\cong \zeta(s,\mathbf{F}_p)\otimes\zeta(s,\mathbf{F}_q)\\ &= e^{Q(s)}\zeta(s,\mathbf{F}_p)\otimes\zeta(s,\mathbf{F}_q)\\ &\quad \text{for some polynomial } Q(s).\end{aligned}$$

(4) The function $\zeta_{p,q}(s)$ satisfies a functional equation:

$$\begin{aligned}\zeta_{p,q}(-s) = \zeta_{p,q}(s)^{-1}(pq)^{\frac{s}{2}}(1-p^{-s})(1-q^{-s})\\ \times \exp\left(\frac{\sqrt{-1}\log p\log q}{4\pi}s^2\right.\\ \left.-\frac{\pi\sqrt{-1}}{6}\left(\frac{\log q}{\log p}+\frac{\log p}{\log q}+3\right)\right).\end{aligned}$$

When $p = q$ the result is as follows:

Theorem 2 *Let*

$$\zeta_{p,p}(s) := \exp\left(\frac{\sqrt{-1}}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}p^{-ns} - \left(1 - \frac{\sqrt{-1}\log p}{2\pi}s\right)\sum_{n=1}^{\infty}\frac{1}{n}p^{-ns}\right)$$

in $\mathrm{Re}(s) > 0$. *Then*

(0) It converges absolutely in $\mathrm{Re}(s) > 0$.

(1) The function $\zeta_{p,p}(s)$ *has an analytic continuation to all* $s \in \mathbf{C}$ *as a meromorphic function of order two.*

(2) All zeros and poles of $\zeta_{p,p}(s)$ *are located at*

$$s = \frac{2\pi\sqrt{-1}n}{\log p},$$

which gives a zero or pole of order $|n+1|$, according as n is a nonnegative or negative integer.

(3) *We have the identification*

$$\zeta_{p,p}(s) \cong \zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_p).$$

(4) *The function $\zeta_{p,p}(s)$ satisfies a functional equation:*

$$\zeta_{p,p}(-s) = \zeta_{p,p}(s)^{-1} p^s (1-p^{-s})^2 \exp\left(\frac{\sqrt{-1}(\log p)^2}{4\pi} s^2 - \frac{5\pi\sqrt{-1}}{6}\right).$$

Theorem 3 *The product $\prod_{p,q} \zeta_{p,q}(s)$ diverges for any s. The diagonal product*

$$\prod_p \zeta_{p,p}(s)$$

is absolutely convergent in $\mathrm{Re}(s) > 1$, *and has an analytic continuation with singularities to* $\mathrm{Re}(s) > 0$ *with the natural boundary* $\mathrm{Re}(s) = 0$.

Ideas:

The usual explicit formula

Euler product $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$

Cauchy's theorem $\downarrow\uparrow$ specialize $h(s)$

Explicit formula $$\sum_\rho h(\rho) = \frac{1}{2\pi i}\int_C h(s)\frac{\zeta'}{\zeta}(s)ds. = \sum_p \text{``}\widehat{h}(p)\text{''}$$

The double explicit formula

$$\sum_{\mathrm{Im}(\rho_1),\mathrm{Im}(\rho_2)>0} h(\rho_1+\rho_2)$$
$$= \frac{1}{(2\pi i)^2}\int_{C_+}\int_{C_+} h(s_1+s_2)\frac{\zeta'}{\zeta}(s_1)\frac{\zeta'}{\zeta}(s_2)ds_1 ds_2$$
$$= \sum_{(p,q)} \text{``}\widehat{h}(p,q)\text{''}.$$

When we define the zeta function, take

$$h(t) = \frac{1}{(t+s)^2} - \frac{1}{(t-s)^2}.$$

Settings for the Double Explicit Formula

$h(t)$: an odd function.

$$H_\alpha(t) := h(2\alpha + it) \quad (1/2 < \alpha < 1)$$

$$\widetilde{H}(u) := \int_{-\infty}^{\infty} H(t) e^{itu} dt$$

Assume

$$H_\alpha(t) = O(|t|^{-2-\delta}) \qquad (|t| \to \infty)$$

for some $\delta > 0$.

Theorem 4 *Let $1/2 < \alpha < 1$. We have*

$$\sum_{\mathrm{Re}(\gamma_1),\mathrm{Re}(\gamma_2)>0} H_0(\gamma_1+\gamma_2)$$
$$= \sum_{\substack{p,q\\ p\neq q}} \mathcal{H}^{\alpha}_{p,q} + \sum_{p} \mathcal{H}^{\alpha}_{p,p} + \sum_{p} \mathcal{H}^{\alpha}_{p,\infty} + \mathcal{H}^{\alpha}_{\infty,\infty} + \mathcal{H}^{\alpha}_{0},$$

where the sum in the left hand side is taken over pairs $(\frac{1}{2}+i\gamma_1,\ \frac{1}{2}+i\gamma_2)$ of nontrivial zeros of the Riemann zeta function, the sum in the right hand side is taken over pairs of distinct primes p, q or primes p, and we define for pairs of distinct primes p, q as

$$\begin{aligned}
\mathcal{H}^{\alpha}_{p,q} &= \frac{i}{4\pi^2} \sum_{m,n} \frac{\log p \log q}{\log(p^m q^n)} \frac{1}{p^{m(\alpha+\frac{1}{2})} q^{n(\alpha+\frac{1}{2})}} \\
&\quad \left(\widetilde{H_0}(-m\log p) + \widetilde{H_0}(-n\log q)\right) \\
&+ \frac{i}{4\pi^2} \sum_{m,n} \frac{\log p \log q}{\log(\frac{p^m}{q^n})} \frac{1}{p^{m(\alpha+\frac{1}{2})} q^{n(\alpha+\frac{1}{2})}} \\
&\quad \left(\widetilde{H_\alpha}(-m\log p) - \widetilde{H_\alpha}(-n\log q)\right),
\end{aligned}$$

and for a prime p,

$$\begin{aligned}
\mathcal{H}^{\alpha}_{p,p} &= \frac{i}{4\pi^2}\sum_{m,n}\frac{\log p}{(m+n)p^{(m+n)(\alpha+\frac{1}{2})}} \\
&\quad \left(\widetilde{H_0}(-m\log p)+\widetilde{H_0}(-n\log p)\right) \\
&+ \frac{i}{4\pi^2}\sum_{m\neq n}\frac{\log p}{(m-n)p^{(m+n)(\alpha+\frac{1}{2})}} \\
&\quad \left(\widetilde{H_\alpha}(-m\log p)-\widetilde{H_\alpha}(-n\log p)\right) \\
&+ \frac{1}{4\pi^2}(\log p)^2\sum_{m=1}^{\infty}p^{-2m(\alpha+\frac{1}{2})} \\
&\quad \widetilde{tH_\alpha(t)}(-m\log p),
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}^{\alpha}_{p,\infty} &= -\frac{1}{2\pi^2}\sum_{m=1}^{\infty}\frac{\log p}{p^{m(\alpha+\frac{1}{2})}}\int_{-\infty}^{\infty}p^{-imt}H_\alpha(t) \\
&\quad \int_0^t p^{imt'}\frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac{1}{2}+it'\right)dt'dt \\
&- \frac{1}{2\pi^2}\sum_{m=1}^{\infty}\frac{\log p}{p^{m(\alpha+\frac{1}{2})}}\int_{-\infty}^{\infty}H_0(t) \\
&\quad \int_0^t \mathrm{Re}\left(p^{im(t-t')}\frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac{1}{2}+it'\right)\right)dt'dt,
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}^{\alpha}_{\infty,\infty} &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} H_\alpha(t)\int_0^t \frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac{1}{2}+it_1\right) \\
&\quad \frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac{1}{2}+i(t-t_1)\right)dt_1dt \\
&+ \frac{1}{4\pi^2}\int_{-\infty}^{\infty} H_0(t)\int_0^t \frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac{1}{2}+it_1\right) \\
&\quad \frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac{1}{2}-i(t-t_1)\right)dt_1dt,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{H}^{\alpha}_{0} &= -\frac{\alpha}{\pi}\int_0^{\pi}\sum_{\mathrm{Re}(\gamma_1)>0} h(i\gamma_1+\alpha e^{i\theta})\frac{\xi'}{\xi}(\alpha e^{i\theta})e^{i\theta}d\theta \\
&- \frac{\alpha^2}{4\pi^2}\int_0^{\pi}\int_0^{\pi} h(\alpha e^{i\theta_1}+\alpha e^{i\theta_2}) \\
&\quad \frac{\xi'}{\xi}(\alpha e^{i\theta_1})\frac{\xi'}{\xi}(\alpha e^{i\theta_2})e^{i(\theta_1+\theta_2)}d\theta_1 d\theta_2,
\end{aligned}$$

where $m,n\in\mathbf{Z}$, $m,n\geq 1$.

Definition: The (p,q)-Euler factor of the double Riemann zeta function is defined as:

$$\zeta^{\alpha}_{p,q}(s) = \exp\left(\iint \mathcal{H}^{\alpha}_{p,q}(s)\,ds\,ds\right).$$

Theorem 5 *The* (p,q)*-Euler factors of the double Riemann zeta function are as follows:*

(1) For distinct primes p *and* q*, we have*

$$\zeta^{\alpha}_{p,q}(s) \cong \exp\left(\frac{1}{\pi i}\sum_{m,n}\frac{(\log p)(\log q)}{(m\log p)^2-(n\log q)^2}\right.$$
$$\left(\frac{\cosh(m\alpha\log p)}{q^{n(\alpha+\frac{1}{2})}}p^{-m(s-\frac{1}{2})}\right.$$
$$+\frac{n\log q}{m\log p}\frac{\sinh(m\alpha\log p)}{q^{n(\alpha+\frac{1}{2})}}p^{-m(s-\frac{1}{2})}$$
$$-\frac{m\log p}{n\log q}\frac{\sinh(n\alpha\log q)}{p^{m(\alpha+\frac{1}{2})}}q^{-n(s-\frac{1}{2})}$$
$$\left.\left.-\frac{\cosh(n\alpha\log q)}{p^{m(\alpha+\frac{1}{2})}}q^{-n(s-\frac{1}{2})}\right)\right)$$

in $\mathrm{Re}(s) > \alpha + \frac{1}{2}$*, where the sum is taken over all pairs of all positive integers* m *and* n*. It has an analytic continuation to the entire plane.*

(2) *For a prime* p*, we have in* $\mathrm{Re}(s) > \alpha + \frac{1}{2}$

$$\zeta^{\alpha}_{p,p}(s) \cong \exp\left(\frac{2}{\pi i}\sum_{m\neq n}\frac{p^{-m(s-\frac{1}{2})-n(\alpha+\frac{1}{2})}}{m^2-n^2}\right.$$
$$\left(\cosh(m\alpha\log p)+\frac{n}{m}\sinh(m\alpha\log p)\right)$$
$$+ \frac{1}{2\pi i}\Big((\log p)(s-1-2\alpha)\log(1-p^{-s})$$
$$\left.-\mathrm{Li}_2\left(p^{-s}\right)+\mathrm{Li}_2\left(p^{-s-2\alpha}\right)\Big)\right),$$

It has an analytic continuation to the entire plane.

(3) *The* (p,∞)*-factor* $\zeta^{\alpha}_{p,\infty}(s)$ *and the* (∞,∞)*-factor* $\zeta^{\alpha}_{\infty,\infty}(s)$ *have an analytic continuation to the entire plane, and moreover* $\prod_p \zeta^{\alpha}_{p,\infty}(s)$ *has an analytic continuation to the entire plane with possible singularities at* $s = \frac{1}{2} - 2k \pm \alpha$ $(k \geq 0)$*,* $1-2k$ $(k \geq 0)$*,* $-2k$ $(k \geq 1)$*,* $\rho - 2k$ $(k \geq 0)$*,* $\frac{1}{2}+\rho\pm\alpha$*,* $\frac{3}{2}\pm\alpha$ *with* ρ *any nontrivial zero of* $\zeta(s)$*.*

Notation:

$$\mathsf{Li}_r(u) := \sum_{n=1}^{\infty} \frac{u^n}{n^r} \quad (r = 1, 2, 3, ...)$$

is the polylogarithm.

Theorem 6 *The remaining factor* $\zeta_0^\alpha(s)$ *of the double Riemann zeta function is an analytic function with possible singularities at* $s = 1 \pm \rho$ *with* ρ *any nontrivial zero of* $\zeta(s)$*.*

Notation:

$$\zeta_+(s) := \prod_{\substack{\rho:\ \zeta(\rho)=0 \\ \mathrm{Im}(\rho)>0}} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}$$

is the half Riemann zeta function introduced by Hirano-Kurokawa-Wakayama (2003).

Notation:

$$\Gamma_r(z) := \exp\left(\frac{\partial}{\partial s}\zeta_r(s,z)\Big|_{s=0}\right)$$

is the multiple gamma function by Barnes, where

$$\zeta_r(s,z) := \sum_{n_1,\ldots,n_r=0}^{\infty}(n_1+\cdots+n_r+z)^{-s}.$$

Theorem 7 *Let* $1/2 < \alpha < 1$. *The Euler product for the double Riemann zeta function*

$$\zeta^{\otimes 2}(s) := \left(\prod_{p,q}\zeta_{p,q}^{\alpha}(s)\right)\zeta_0^{\alpha}(s)\left(\frac{\prod_{m=1}^{\infty}\zeta_+(s+2m)}{\zeta_+(s-1)}\right)^2$$

$$\Gamma_2\left(\frac{s}{2}\right)^{-1}\Gamma_1(s)^2 s(s-2),$$

where (p,q) *runs through pairs of all (finite or infinite) places, has the following properties:*

(0) It converges absolutely in $\mathrm{Re}(s) > \alpha + \frac{3}{2}$.

(1) $\zeta^{\otimes 2}(s)$ *has an analytic continuation with singularities to the entire plane.*

(2) Zeros of $\zeta^{\otimes 2}(s)$ *are located at* $s = 2$ *and*

$$s = \rho_1 + \rho_2 \quad (\mathrm{Im}(\rho_1), \mathrm{Im}(\rho_2) > 0),$$

where ρ_j $(j = 1, 2)$ *are zeros of* $\zeta(s)$.

(3) Poles of $\zeta^{\otimes 2}(s)$ *are located at* $s = 1 + \rho$ $(\mathrm{Im}(\rho) \geq 0)$ *and*

$$s = \rho_1 + \rho_2 \quad (\mathrm{Im}(\rho_1), \mathrm{Im}(\rho_2) < 0).$$

(4) We have $\zeta^{\otimes 2}(s) \cong \zeta(s) \otimes \zeta(s)$.

(5) The function $\zeta^{\otimes 2}(s)$ *satisfies a functional equation between* s *and* $2 - s$.

Remark If the Euler product in Theorem 7 is absolutely convergent in $\mathrm{Re}(s) > \frac{3}{2}$, then

$$\zeta(s) \neq 0 \text{ in } \mathrm{Re}(s) > \frac{3}{4}.$$