# Mollified and amplified moments of the Riemann zeta function

## Some new theorems and conjectures

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Two of the most important aspects of research in analytic number theory involve mollifying and amplifying moments of zeta functions.

$$\int_0^T |\zeta(\tfrac{1}{2} + \mathrm{i}t)|^{2k} |M(\tfrac{1}{2} + \mathrm{i}t)|^2 \, \mathrm{d}t$$

where $M(s)$ is a Dirichlet polynomial of length $\theta$, that is

$$M(s) = \sum_{n \leq T^\theta} \frac{a(n)}{n^s}$$

Mollification is to overcome the "loss of logs" problem:

$$\int_0^T |\zeta(\tfrac{1}{2}+\mathrm{i}t)|\,\mathrm{d}t \not\asymp \left(T\int_0^T |\zeta(\tfrac{1}{2}+\mathrm{i}t)|^2\,\mathrm{d}t\right)^{1/2} \sim T\sqrt{\log T}$$

while, hopefully,

$$\int_0^T |M\zeta(\tfrac{1}{2}+\mathrm{i}t)|\,\mathrm{d}t \asymp \left(T\int_0^T |M\zeta(\tfrac{1}{2}+\mathrm{i}t)|^2\,\mathrm{d}t\right)^{1/2} \asymp T$$

Choose the Dirichlet coefficients to be approximately $\mu(n)$

- Levinson, Asymptotic 2nd moment, with Dirichlet polynomial having coefficients like $\mu(n)$, $\theta < 1/2$.
- Balasubramanian, Conrey, Heath-Brown: Asymptotic 2nd moment multiplied by general Dirichlet polynomial, $\theta < 1/2$.
- Conrey: Asymptotic 2nd moment, with Dirichlet polynomial having coefficients like $\mu(n)$, $\theta < 4/7$.
- Iwaniec, Deshouillers: Upper bound for 4th moment times general polynomial, $\theta < 1/5$.
- Watt: As above, but $\theta < 1/4$.
- Gaggero Jara: Asymptotic main term of the 4th moment with general Dirichlet polynomial, $\theta < 4/589$.
- We give a full asymptotic shifted 4th moment for $\theta < 5/27$.

Balasubramanian, Conrey, Heath-Brown (1985)

Let $a(n) \ll n^{\epsilon}$,

$$A(s) = \sum_{h \le T^{\theta}} \frac{a(h)}{h^s}$$

For $\theta < 1/2$,

$$\frac{1}{T}\int_0^T \left|\zeta(\tfrac{1}{2}+\mathrm{i}t)\right|^2 \left|A(\tfrac{1}{2}+\mathrm{i}t)\right|^2 \,\mathrm{d}t$$

$$\sim \sum_{h,k \le T^{\theta}} \frac{a(h)\overline{a(k)}(h,k)}{hk}\left(\log\left(\frac{T(h,k)^2}{2\pi hk}\right) + 2\gamma - 1\right)$$

Let $a(n) \ll n^{\epsilon}$, $b(n) \ll n^{\epsilon}$,

$$A(s) = \sum_{h \le T^{\theta}} \frac{a(h)}{h^s} \quad , \quad B(s) = \sum_{k \le T^{\theta}} \frac{b(k)}{k^s}$$

**Theorem:** For $\theta < 1/2$,

$$\frac{1}{T}\int_0^T \mathcal{Z}(t + \mathrm{i}\alpha)\mathcal{Z}(t + \mathrm{i}\beta)A(\tfrac{1}{2} + \mathrm{i}t)B(\tfrac{1}{2} - \mathrm{i}t)\,\mathrm{d}t \sim$$

$$\sum_{h,k \le T^{\theta}} \frac{a(h)b(k)(h,k)}{hk} \frac{1}{T}\int_0^T \left[ \left(\frac{t}{2\pi}\right)^{\frac{1}{2}(-\alpha+\beta)} \frac{k^{\alpha}h^{-\beta}}{(h,k)^{\alpha-\beta}} \zeta(1-\alpha+\beta) \right.$$

$$\left. + \left(\frac{t}{2\pi}\right)^{\frac{1}{2}(\alpha-\beta)} \frac{k^{\beta}h^{-\alpha}}{(h,k)^{-\alpha+\beta}} \zeta(1+\alpha-\beta) \right] \mathrm{d}t$$

**Theorem:** For $\theta < 5/27$,

$$\int_0^T \mathcal{Z}(t+\mathrm{i}\alpha_1)\mathcal{Z}(t+\mathrm{i}\alpha_2)\mathcal{Z}(t+\mathrm{i}\beta_1)\mathcal{Z}(t+\mathrm{i}\beta_2)A(\tfrac{1}{2}+\mathrm{i}t)B(\tfrac{1}{2}-\mathrm{i}t)\,\mathrm{d}t$$

$$\sim \sum_{h,k\le T^\theta} \frac{a(h)b(k)(h,k)}{hk} \int_0^T \Big\{ G(\{\alpha_1,\alpha_2\};\{\beta_1,\beta_2\})$$

$$+ G(\{\beta_1,\alpha_2\};\{\alpha_1,\beta_2\}) + G(\{\beta_2,\alpha_2\};\{\beta_1,\alpha_1\})$$

$$+ G(\{\alpha_1,\beta_1\};\{\alpha_2,\beta_2\}) + G(\{\alpha_1,\beta_2\};\{\beta_1,\alpha_2\})$$

$$+ G(\{\beta_1,\beta_2\};\{\alpha_1,\alpha_2\}) \Big\}\,\mathrm{d}t$$

where we have written $H = h/(h,k)$ and $K = k/(h,k)$.

$$G(\{\alpha_1,\alpha_2\};\{\beta_1,\beta_2\}) = \left(\frac{t}{2\pi}\right)^{\frac{1}{2}(-\alpha_1-\alpha_2+\beta_1+\beta_2)}$$
$$\times D_{\{\alpha_1,\alpha_2\},\{\beta_1,\beta_2\}}(1)\mathcal{F}_{K,\{\alpha_1,\alpha_2\};H,\{\beta_1,\beta_2\}}(1)$$

where, for $\vec{\alpha} = \{\alpha_1, \alpha_2\}$ and $\vec{\beta} = \{\beta_1, \beta_2\}$,

$$D_{\vec{\alpha},\vec{\beta}}(s) = \frac{\zeta(s-\alpha_1+\beta_1)\zeta(s-\alpha_1+\beta_2)\zeta(s-\alpha_2+\beta_1)\zeta(s-\alpha_2+\beta_2)}{\zeta(2s-\alpha_1-\alpha_2+\beta_1+\beta_2)}$$

and $\mathcal{F}_{K,\vec{\alpha};H,\vec{\beta}}(s)$ is essentially $d_{\vec{\alpha}}(K)d_{-\vec{\beta}}(H)$, where $d_{\vec{\alpha}}(n)$ is the shifted divisor function

$$d_{\vec{\alpha}}(n) := \sum_{m_1 m_2 = n} m_1^{\alpha_1} m_2^{\alpha_2}$$

Let $\vec{\alpha} = \{\alpha_1, \ldots, \alpha_k\}$ and $\vec{\beta} = \{\beta_1, \ldots, \beta_k\}$.

$$d_{\vec{\alpha}}(n) := \sum_{m_1 \cdots m_k = n} m_1^{\alpha_1} \cdots m_k^{\alpha_k}$$

$$\mathcal{F}_{K,\vec{\alpha};H,\vec{\beta}}(s) :=$$
$$\prod_{p|K} \frac{\sum_{j=0}^{\infty} d_{\vec{\alpha}}(p^{j+K_p}) d_{-\vec{\beta}}(p^j)/p^{js}}{\sum_{j=0}^{\infty} d_{\vec{\alpha}}(p^j) d_{-\vec{\beta}}(p^j)/p^{js}} \prod_{p|H} \frac{\sum_{j=0}^{\infty} d_{\vec{\alpha}}(p^j) d_{-\vec{\beta}}(p^{j+H_p})/p^{js}}{\sum_{j=0}^{\infty} d_{\vec{\alpha}}(p^j) d_{-\vec{\beta}}(p^j)/p^{js}}$$

$$D_{\vec{\alpha},\vec{\beta}}(s) = \sum_{n=1}^{\infty} \frac{d_{\vec{\alpha}}(n) d_{-\vec{\beta}}(n)}{n^s}$$

$$G(\{\alpha_1, \ldots, \alpha_k\}; \{\beta_1, \ldots, \beta_k\}) = \left(\frac{t}{2\pi}\right)^{\frac{1}{2}\sum_{j=1}^{k}(-\alpha_j + \beta_j)}$$
$$\times D_{\{\alpha_1,\ldots,\alpha_k\};\{\beta_1,\ldots,\beta_k\}}(1) \mathcal{F}_{K,\{\alpha_1,\ldots,\alpha_k\};H,\{\beta_1,\ldots,\beta_k\}}(1)$$

**Conjecture:** For some $0 < \theta < 1$,

$$\frac{1}{T}\int_0^T A(\tfrac{1}{2}+\mathrm{i}t)B(\tfrac{1}{2}-\mathrm{i}t)\prod_{j=1}^{k}\mathcal{Z}(t+\mathrm{i}\alpha_j)\mathcal{Z}(t+\mathrm{i}\beta_j)\,\mathrm{d}t$$

$$\sim \sum_{h,k\le T^\theta}\frac{a(h)b(k)(h,k)}{hk}\frac{1}{T}\int_0^T \Bigg\{ G(\{\alpha_1,\ldots,\alpha_k\};\{\beta_1,\ldots,\beta_k\}) + \cdots + G(\{\beta_1,\ldots,\beta_k\};\{\alpha_1,\ldots,\alpha_k\}) \Bigg\}\,\mathrm{d}t$$

where we have written $H = h/(h,k)$ and $K = k/(h,k)$, and there are $\binom{2k}{k}$ terms in the sum.

By a lemma of Conrey, Farmer, Keating, Rubinstein and Snaith, the conjecture can be rewritten as

$$\int_0^T A(\tfrac{1}{2}+\mathrm{i}t)B(\tfrac{1}{2}-\mathrm{i}t)\prod_{j=1}^{k}\mathcal{Z}(t+\mathrm{i}\alpha_j)\mathcal{Z}(t+\mathrm{i}\beta_j)\,\mathrm{d}t$$

$$\sim \sum_{h,k\le T^\theta}\frac{a(h)b(k)(h,k)}{hk}\int_0^T \mathrm{d}t\left\{\frac{(-1)^k}{(k!)^2}\frac{1}{(2\pi\mathrm{i})^{2k}}\right.$$

$$\left.\oint\cdots\oint\frac{G(z_1,\ldots,z_k;z_{k+1},\ldots,z_{2k})\Delta(z_1,\ldots,z_{2k})^2}{\prod_{i=1}^{2k}\prod_{j=1}^{k}(z_i-\alpha_j)(z_i-\beta_j)}\,\mathrm{d}z_1\ldots\mathrm{d}z_{2k}\right\}$$

where we have written $H=h/(h,k)$ and $K=k/(h,k)$.

How big can $\theta$ be?

Certainly $\theta = 1$ is not allowed for all choices of $a(h)$ and $b(k)$ (e.g. $a(h) = 1$, $b(k) = 1$).

Possibly any $\theta < 1$ is okay for any choice of $a(h)$ and $b(k)$.

For certain choices of $a(h)$ and $b(k)$ (e.g. the Möbius function), maybe $\theta = \infty$ is okay (The Farmer conjecture).

Solving this problem essentially comes down to understanding

$$\int_0^T \left(\frac{t}{2\pi}\right)^{\frac{1}{2}(-\alpha_1-\alpha_2+\beta_1+\beta_2)} \sum_{mn\le(t/2\pi)^2} \frac{d_{\vec{\alpha}}(m)d_{\vec{\beta}}(n)}{\sqrt{mn}} \left(\frac{nK}{mH}\right)^{\mathrm{i}t} \, \mathrm{d}t$$

Diagonal terms: $nK = mH$

Off-diagonal terms: $nK \neq mH$

Estimate the off-diagonal terms using the $\Delta$-method of Duke, Friedlander and Iwaniec.

**Applications** (work in progress)

- Zeros on the critical line

$$\int_0^T |(M_1 + \zeta M_2)\zeta|^2 \ \mathrm{d}t$$

- Large gaps between zeros
- Explicit lower bound for the sixth moment

**Connections to random matrix conjectures**

- CFKRS full moments conjecture
- If $\theta = 1/2$, from the $k^{\mathrm{th}}$ moment obtain the $k+1^{\mathrm{st}}$
- If $\theta = 1$, from the $k^{\mathrm{th}}$ moment get the $k+1^{\mathrm{st}}$ and the $k+2^{\mathrm{nd}}$

**Moments and extreme values of the zeta function**

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2}+\mathrm{i}t)| = O\left(\exp\left(C\frac{\log T}{\log\log T}\right)\right)$$

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2}+\mathrm{i}t)| = \Omega\left(\exp\left(C'\sqrt{\frac{\log T}{\log\log T}}\right)\right).$$

Conjecture:

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2}+\mathrm{i}t)| = O\left(\exp\left((\frac{1}{\sqrt{2}}+\varepsilon)\sqrt{\log T\log\log T}\right)\right)$$

for all $\varepsilon > 0$ and for no $\varepsilon < 0$.

The maximum can be bounded above and below by moments:

$$\left(\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+\mathrm{i}t)|^{2k}\,\mathrm{d}t\right)^{1/2k} \le \max_{t\in[0,T]} |\zeta(\tfrac{1}{2}+\mathrm{i}t)|$$

$$\le 2(CT\log T)^{1/2\ell}\left(\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+\mathrm{i}t)|^{2\ell}\,\mathrm{d}t\right)^{1/2\ell}$$

For fixed $k$, Keating and Snaith conjectured:

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+\mathrm{i}t)|^{2k}\,\mathrm{d}t \sim \frac{G^2(k+1)}{G(2k+1)}\,a(k)\log^{k^2} T$$

$$\sim \exp\big(k^2\log\log T - k^2\log k + O(k^2\log\log k)\big)$$

The above implies this cannot hold for $k > 2\sqrt{2}\frac{\sqrt{\log T}}{\sqrt{\log\log T}}$.