$$0 < \gamma_1 \le \gamma_2 \le \cdots$$

The average of $\gamma_{n+1} - \gamma_n$ is $\sim 2\pi/\log \gamma_n$

How are the differences $\gamma_{n+1} - \gamma_n$ distributed?

$$\sum_{n\le x} \frac{\Lambda(n)}{n^s} = \frac{-1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\zeta'}{\zeta}(s+w)x^w \,\frac{dw}{w}$$

$$= \frac{x^{1-s}}{1-s} - \frac{\zeta'}{\zeta}(s) - \sum_{\rho} \frac{x^{\rho-s}}{\rho - s}$$

$$+\sum_{k=1}^{\infty} \frac{x^{-2k-s}}{2k+s}$$

$\sigma > 1:$

$$\sum_{\rho} \frac{x^{\rho-s}}{\rho-s} = \sum_{n>x} \frac{\Lambda(n)}{n^s} \; + \; \cdots$$

$$\Big|\sum_{\rho}\frac{x^{\rho-s}}{\rho-s}\Big|^2=\sum_{\rho,\rho'}\frac{x^{1-2\sigma+i(\gamma-\gamma')}}{(s-\rho)(\overline{s-\rho'})}$$

Integrate this with respect to t

$$\int_0^T\Big|\sum_{n=1}^{\infty}a_n n^{-it}\Big|^2\,dt=\sum_{n=1}^{\infty}|a_n|^2(T+O(n))$$

$$F(\alpha,T):=\frac{2\pi}{T\log T}\sum_{\substack{0<\gamma\le T\\0<\gamma'\le T}}T^{i\alpha(\gamma-\gamma')}w(\gamma-\gamma')$$

$$w(u)=\frac{4}{4+u^2}\qquad (w(0)=1)$$

$$F(\alpha,T)\in\mathbb{R}$$

$$F(-\alpha,T)=F(\alpha,T)$$

$$F(\alpha, T) \geq 0$$

If $0 \leq \alpha < 1$ **then**

$$F(\alpha, T) = (1 + o(1))T^{-2\alpha} \log T + \alpha + o(1)$$

Strong Pair Correlation Conjecture:

$$F(\alpha, T) \sim 1 \qquad (1 \leq \alpha \leq A)$$

$$F(\alpha, T) \leq F(0, T) \sim \log T$$

If $r \in L^1(\mathbb{R})$ **then**

$$\sum_{\substack{0<\gamma\leq T \\ 0<\gamma'\leq T}} r\Big((\gamma - \gamma')\frac{\log T}{2\pi}\Big) w(\gamma - \gamma')$$

$$= \frac{T}{2\pi}(\log T) \int_{-\infty}^{\infty} F(\alpha)\widehat{r}(\alpha)\, d\alpha$$

Weak Pair Correlation Conjecture:

If $0 < \alpha < \beta$ **then**

$$\sum_{\substack{0<\gamma\le T\\ 0<\gamma'\le T\\ \alpha\le(\gamma-\gamma')(\log T)/(2\pi)\le\beta}} 1$$

$$\sim \frac{T}{2\pi}(\log T)\int_\alpha^\beta 1-\Big(\frac{\sin \pi u}{\pi u}\Big)^2\,du$$

$$\sum_{0<\gamma\le T} m(\rho) = \sum_{\substack{0<\gamma\le T\\ \text{distinct } \gamma}} m(\rho)^2 \ \sim\ \frac{T}{2\pi}\log T$$

$$N(T) = \sum_{0<\gamma\le T} 1 = \sum_{\substack{0<\gamma\le T\\ \text{distinct } \gamma}} m(\rho) \ \sim\ \frac{T}{2\pi}\log T$$

Goldston–Montgomery (1987)

Assuming RH, the following are equivalent:

(a) Strong Pair Correlation Conjecture

(b) $\displaystyle\int_0^X (\psi(x+h)-\psi(x)-h)^2\,dx \sim hX\log\frac{X}{h}$

uniformly for $1 \le h \le X^{1-\varepsilon}$

(c) $\displaystyle\int_1^X (\psi((1+\delta)x)-\psi(x)-\delta x)^2 \sim \frac{1}{2}\delta X^2\log\frac{1}{\delta}$

uniformly for $X^{-1} \le \delta \le X^{-\varepsilon}$

Cramér's model

For $n \ge 3$ **let** X_n **be independent Bernoulli variables with**

$$\mathbf{P}(X_n = 1) = \frac{1}{\log n}, \qquad \mathbf{P}(X_n = 0) = 1 - \frac{1}{\log n}$$

$$\int_0^X (\psi(x+h)-\psi(x)-h)^2\,dx \sim hX\log X$$

Montgomery–Soundararajan (1996–2004)

$$\Lambda_0(n) = \Lambda(n) - 1$$

$$\sum_{n=1}^{N} (\psi(n+H) - \psi(n) - H)^K$$

$$= \sum_{n=1}^{N} \Big(\sum_{h=1}^{H} \Lambda_0(n+h) \Big)^K$$

$$= \sum_{\substack{h_1,\ldots,h_K \\ 1 \le h_i \le H}} \sum_{n=1}^{N} \prod_{i=1}^{K} \Lambda_0(n+h_i)$$

$$\sum_{n \le x} \prod_{i=1}^{k} \Lambda(n+d_i) \;=\; \mathfrak{S}(\mathcal{D})x + E_k(x;\mathcal{D})$$

$$\begin{aligned}
\mathfrak{S}(\mathcal{D}) &= \sum_{\substack{q_1,\dots,q_k \\ 1\le q_i<\infty}} \left(\prod_{i=1}^{k} \frac{\mu(q_i)}{\phi(q_i)}\right) \\
&\qquad \times \sum_{\substack{a_1,\dots,a_k \\ 1\le a_i\le q_i \\ (a_i,q_i)=1 \\ \sum a_i/q_i\in\mathbb{Z}}} e\Big(\sum_{i=1}^{k} \frac{a_i d_i}{q_i}\Big) \\
&= \prod_p \Big(1-\frac{1}{p}\Big)^{-k}\Big(1-\frac{\nu_p(\mathcal{D})}{p}\Big)
\end{aligned}$$

$$\begin{aligned}
\mathfrak{S}_0(\mathcal{D}) = \sum_{\substack{q_1,\dots,q_k \\ 1<q_i<\infty}} \left(\prod_{i=1}^{k} \frac{\mu(q_i)}{\phi(q_i)}\right) \\
\times \sum_{\substack{a_1,\dots,a_k \\ 1\le a_i\le q_i \\ (a_i,q_i)=1 \\ \sum a_i/q_i\in\mathbb{Z}}} e\Big(\sum_{i=1}^{k} \frac{a_i d_i}{q_i}\Big)
\end{aligned}$$

$$\sum_{n\le x}\prod_{i=1}^{k}\Lambda_0(n+d_i) \;=\; \mathfrak{S}_0(\mathcal{D})x + E_k'(x;\mathcal{D})$$

Gallagher (1981): $\lambda \asymp 1$, $E_k(x;\mathcal{D}) = o(x) \Longrightarrow$

$$\int_1^X (\pi(x+\lambda\log x)-\pi(x))^k\,dx \sim m_k(\lambda)X$$

$$\sum_{\substack{p\le x\\ p'-p>c\log p}} 1 \sim e^{-c}\pi(x)$$

$$\sum_{\substack{d_1,\dots,d_k\\ 1\le d_i\le h\\ d_i\,\text{distinct}}} \mathfrak{S}(\mathcal{D}) \;\sim\; h^k$$

$$\sum_{\substack{d_1,\dots,d_k\\ 1\le d_i\le h\\ d_i\,\text{distinct}}} \mathfrak{S}_0(\mathcal{D}) \;= \mu_k(-h\log h + Ah)^{k/2}$$

$$+\,O_k\big(h^{k/2-1/(7k)+\varepsilon}\big)$$

$$A = 2 - C_0 - \log 2\pi$$

$$\sum_{n=1}^{q} \Bigg(\sum_{\substack{m=1 \\ (m+n,q)=1}}^{h} 1 - h\phi(q)/q \Bigg)^k = q \Big(\frac{\phi(q)}{q} \Big)^k V_k(q;h)$$

$$V_k(q;h) = \sum_{\substack{d_1,\dots,d_k \\ 1 \le d_i \le h}} \sum_{\substack{q_1,\dots,q_k \\ 1<q_i|q}} \Big(\prod_{i=1}^{k} \frac{\mu(q_i)}{\phi(q_i)} \Big)$$

$$\times \sum_{\substack{a_1,\dots,a_k \\ 1 \le a_i \le q_i \\ (a_i,q_i)=1 \\ \sum a_i/q_i \in \mathbb{Z}}} e\Big(\sum_{i=1}^{k} \frac{a_i d_i}{q_i} \Big)$$

$$\ll_k \Big(\frac{hq}{\phi(q)} \Big)^{k/2} \big(1 + O(h^{-1/(7k)} (q/\phi(q))^{2^k + k/2}) \big)$$

$$= \mu_k \Bigg(\sum_{\substack{d|q \\ d>1}} \frac{\mu(d)^2}{\phi(d)^2} \sum_{\substack{a=1 \\ (a,d)=1}}^{d} |E(a/d)|^2 \Bigg)^{k/2} + \cdots$$

$$E(\alpha) = \sum_{m=1}^{h} e(m\alpha)$$

If

$$E_k(x; \mathcal{D}) \ll N^{1/2+\varepsilon}$$

$(1 \leq k \leq K, 0 \leq x \leq N, 1 \leq d_i \leq H, d_i$ **distinct**$)$

then

$$\sum_{n=1}^{N} (\psi(n+H) - \psi(n) - H)^K$$

$$= \mu_K H^{K/2} \int_1^N \left(\log \frac{x}{H} + B \right)^{K/2} dx$$

$$+ O\Big(N (\log N)^{K/2} H^{K/2} \Big(\frac{H}{\log N} \Big)^{-1/(8K)} \Big)$$

$$+ O\Big(H^K N^{1/2+\varepsilon} \Big)$$

$B = 1 - C_0 - \log 2\pi$

Conjecture:

$$\sum_{n=1}^{N} (\psi(n+H) - \psi(n) - H)^K$$

$$= (\mu_K + o(1)) N \Big(H \log \frac{N}{H} \Big)^{K/2}$$

uniformly for $N^\delta \leq H \leq N^{1-\delta}$

$$\psi(x+h) - \psi(x) - h = -\sum_\rho \frac{(x+h)^\rho - x^\rho}{\rho} + \cdots$$

$$\frac{(x+h)^\rho - x^\rho}{\rho} \begin{cases} \sim h x^{\rho-1} & (|\gamma| \leq x/h) \\ \ll x^{1/2}/|\gamma| & (|\gamma| > x/h) \end{cases}$$

Chan: Conjecture is equivalent to

$$\int_1^X \Big[\sum_{0<\gamma\leq T} \cos(\gamma \log x) \Big]^k dx = \big(\mu_k + o(1)\big) X \Big(\frac{T}{4\pi} \log T \Big)^{k/2}$$

Rains (1997):

If $A \in U(N)$ **has eigenvalues** $e(\theta_1), \ldots, e(\theta_N)$ **and** $|M| \geq N$**,** $M \in \mathbb{Z}$ **then** $(M\theta_1, \ldots, M\theta_N)$ **is uniformly distributed in** $\mathbb{T}^N$**.**

Hence the distribution of

$$\mathrm{Re\ Trace}\, A^M = \sum_{n=1}^{N} \cos(2\pi M\theta_n)$$

is exactly the same as that of

$$\sum_{n=1}^{N} \cos 2\pi X_n$$

(X_n **independent, u. d. on** $[0,1]$**)**