ON THE MOMENTS OF HECKE SERIES AT CENTRAL POINTS

Aleksandar Ivić

University of Belgrade

Let $\{\lambda_j = \kappa_j^2 + \frac14\} \cup \{0\}$ be the eigenvalues of the hyperbolic Laplacian

$$\Delta = -y^2\left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right)$$

acting over the Γ-automorphic functions which are square integrable with respect to the hyperbolic measure ($\Gamma = \mathrm{PSL}(2,\mathbb{Z})$). Let $\{\psi_j\}_{j=1}^\infty$ be a maximal orthonormal system such that $\Delta\psi_j = \lambda_j\psi_j$ for each $j \ge 1$ and $T(n)\psi_j = t_j(n)\psi_j$ for each integer $n \in \mathbb{N}$, where $T(n)$ is the Hecke operator, and ($s = \sigma + it$)

$$H_j(s) \;=\; \sum_{n=1}^\infty t_j(n)n^{-s} \qquad (\sigma > 1),$$

is the Hecke series associated with the Maass wave form $\psi_j(z)$. It continues analytically to an entire function over $\mathbb{C}$ and t satisfies the functional equation

$$H_j(s) = 2^{2s-1}\pi^{2s-2}\Gamma(1-s+i\kappa_j)\Gamma(1-s-i\kappa_j)\times$$

$$\times(\varepsilon_j\cosh(\pi\kappa_j) - \cos(\pi s))H_j(1-s),$$

where ε_j $(= \pm1)$ is the parity sign of $\psi_j(z)$. By the Phragmén–Lindelöf principle (convexity) it implies

$$H_j(\tfrac12) \;\ll_\varepsilon\; \kappa_j^{\frac12+\varepsilon},$$

where here and later ε denotes arbitrarily small, positive constants, not necessarily the same ones at each occurrence. Note that from the work of Katok–Sarnak (1993), it is known that $H_j(\frac12) \ge 0$.

Connection of spectral theory and the fourth moment of $|\zeta(\frac{1}{2}+it)|$: Let $D > 0$ be an arbitrary constant. For $T^{1/2}\log^{-D}T \le G \le T/\log T$ we have

$$J_2(T,G) = O(\log^{3D+9}T) + \frac{\pi}{\sqrt{2T}}\sum_{j=1}^\infty \alpha_j H_j^3(\tfrac{1}{2})\frac{1}{\sqrt{\kappa_j}}\sin\bigl(\kappa_j \log\frac{\kappa_j}{4eT}\bigr)\exp(-\tfrac{1}{4}(\frac{G\kappa_j}{T})^2),$$

proved by Motohashi (1989), where ($k \in \mathbb{N}$ is fixed)

$$J_k(T,G) := \frac{1}{\sqrt{\pi}G}\int_{-\infty}^{\infty}|\zeta(\tfrac{1}{2}+iT+it)|^{2k}\exp(-(t/G)^2)\,\mathrm{d}t.$$

The asymptotic formula for the fourth moment of $|\zeta(\frac{1}{2}+it)|$ is:

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,\mathrm{d}t \;=\; TP_4(\log T) \;+\; E_2(T),$$

$$P_4(x) \;=\; \sum_{j=0}^4 a_j x^j,$$

with $a_4 = 1/(2\pi^2)$ and explicit a_j. Explicit formula for $J_2(T,G)$ leads to best known results on $E_2(T)$ (Ivić-Motohashi), e.g.

$$E_2(T) \ll T^{2/3}\log^C T, \quad E_2(T) = \Omega_\pm(T^{1/2}),$$
$$\int_1^T E_2(t)\,\mathrm{d}t = O(T^{3/2}), \quad \int_1^T E_2(t)\,\mathrm{d}t = \Omega_\pm(T^{3/2}),$$

and

$$\int_1^T E_2^2(t)\,\mathrm{d}t \ll T^2\log^C T, \quad \int_1^T E_2^2(t)\,\mathrm{d}t \gg T^2.$$

Problem. Evaluate $\sum_{\kappa_j \le T}\alpha_j H_j^k(\frac{1}{2})$ for $k \in \mathbb{N}$ fixed,

$$\alpha_j = |\rho_j(1)|^2(\cosh \pi\kappa_j)^{-1},$$

and $\rho_j(1)$ is the first Fourier coefficient of $\psi_j(z)$, $\kappa_j^{-\varepsilon} \ll_\varepsilon \alpha_j \ll_\varepsilon \kappa_j^\varepsilon$.

Conjecture. $H_j(\frac{1}{2}) \ll_\varepsilon \kappa_j^\varepsilon$, in fact one has the analogue of the Lindelöf conjecture $\zeta(\frac{1}{2}+it) \ll_\varepsilon |t|^\varepsilon$, the conjecture

$$H_j(\tfrac{1}{2}+it) \ll_\varepsilon (\kappa_j + |t|)^\varepsilon.$$

Known (A.I. 1999). $H_j(\frac{1}{2}) \ll_\varepsilon \kappa_j^{\frac{1}{3}+\varepsilon}$, breaks the convexity bound $H_j(\frac{1}{2}) \ll_\varepsilon \kappa_j^{\frac{1}{2}+\varepsilon}$, follows from the short interval result that

$$\sum_{T-1\le\kappa_j\le T+1} \alpha_j H_j^3(\tfrac{1}{2}) \ll_\varepsilon T^{1+\varepsilon}$$

and $H_j(\frac{1}{2}) \ge 0$. Also follows from (M. Jutila 1999)

$$\sum_{K\le\kappa_j\le K+K^{1/3}} \alpha_j H_j^4(\tfrac{1}{2}) \ll_\varepsilon K^{\frac{4}{3}+\varepsilon}$$

and (M. Jutila 2003)

$$\sum_{\kappa_j\le K} \alpha_j H_j^{12}(\tfrac{1}{2}) \ll_\varepsilon K^{4+\varepsilon}.$$

The general conjecture for moments of $H_j(\frac{1}{2})$ can be obtained by Random Matrix Theory. For a family $\mathcal{F}$ of L-functions with functional equations $s \to 1-s$ and appropriate gamma-factors, $L_f(\frac{1}{2})$ is the 'critical value' or 'central value'. The conjecture is that, as $Q \to \infty$,

$$(*) \qquad \frac{1}{Q^*} \sum_{f\in\mathcal{F},c(f)\le Q} L_f^k(\tfrac{1}{2}) \sim \frac{a_k g_k}{\Gamma(1+B(k))}(\log Q^A)^{B(k)},$$

where the family $\mathcal{F}$ is partially ordered by "conductor" $c(f)$, with Q^* the number of elements with $c(f) \le Q$. The family $H_j(s)$ of Hecke series belongs to the symmetry type described by Katz–Sarnak as O, for "orthogonal". Here $A = 1, B(k) = \frac{1}{2}k(k-1)$.

In the notation of Random Matrix theory g_k is the so-called *geometric* part. In our case it is

$$g_k = \left(\tfrac{1}{2}k(k-1)\right)!\, 2^{k(k+1)/2-1} \prod_{j=1}^{k-1} \frac{j!}{(2j)!},$$

so that $g_1 = 1, g_2 = 2, g_3 = 8, g_4 = 128$. The constant a_k is the *arithmetic* part. It equals

$$a_k = \prod_p \left(1 - \frac{1}{p}\right)^{k(k-1)/2} \times \sum_{j=0}^{\infty} \binom{k+j-1}{j} \binom{k+j-2}{j} \frac{1}{(j+1)p^j}.$$

We have $a_1 = a_2 = a_3 = 1, a_4 = 1/\zeta(2) = \frac{6}{\pi^2}$, and in general a_k can be expressed in terms of hypergeometric functions, but for $k > 4$ it seems that a_k is not expressible in simple closed form (J.B. Conrey).

In the context of moments of $H_j(\frac{1}{2})$ Random Matrix theory says that one should have, as $K \to \infty$,

$$\sum_{\kappa_j \le K} \alpha_j H_j^k(\tfrac{1}{2}) = K^2 P_{\frac{1}{2}(k^2-k)}(\log K) + o(K^2) \tag{1}$$

for $k \in \mathbb{N}$ fixed. The leading coefficient of $P_{\frac{1}{2}(k^2-k)}(x)$ equals

$$d_k = \frac{a_k g_k}{\pi^2 \left(\frac{k(k-1)}{2}\right)!},$$

since (N.V. Kuznetsov 1980)

$$\sum_{\kappa_j \le K} \alpha_j = K^2 \pi^{-2} + O(K \log K). \tag{2}$$

The theory gives a conjecture for moments without the normalizing factor α_j of the same shape, only more complicated.

Asymptotic formulas for $k = 1, 2, 3, 4$ of moments of $H_j(\frac{1}{2})$ will be discussed (the known cases!). In all cases d_k, the value predicted by Random Matrix Theory, coincides with the actual value of the coefficient.

Conjecture. For $k \in \mathbb{N}$ fixed and any given $\varepsilon > 0$,

$$\sum_{\kappa_j \le T} \alpha_j H_j^k(\tfrac{1}{2}) + \frac{2}{\pi}\int_0^T \frac{|\zeta(\frac{1}{2}+it)|^{2k}}{|\zeta(1+2it)|^2}\,\mathrm{d}t$$
$$= T^2 P_{\frac{1}{2}(k^2-k)}(\log T) + O_{\varepsilon,k}(T^{1+c_k+\varepsilon}),$$

where $P_{\frac{1}{2}(k^2-k)}(z)$ is a polynomial of degree $\frac{1}{2}(k^2-k)$ in z whose coefficients depend on k, and $0 \le c_k < 1$; perhaps even $c_k = 0$ is true.

THEOREM 1 (A.I. & M. Jutila 2003). *We have*

$$\sum_{\kappa_j \le T} \alpha_j H_j(\tfrac{1}{2}) + \frac{2}{\pi}\int_0^T \frac{|\zeta(\frac{1}{2}+it)|^{2}}{|\zeta(1+2it)|^2}\,\mathrm{d}t = \left(\frac{T}{\pi}\right)^2 + O(T\sqrt{\log T}). \tag{3}$$

The evaluation of the integral in (3) is given by

THEOREM 2 (A.I. & M. Jutila 2003). *There exist constants $A\,(>0)$ and B which are effectively computable such that, for any given $\varepsilon > 0$,*

$$\int_0^T \frac{|\zeta(\frac{1}{2}+it)|^{2}}{|\zeta(1+2it)|^2}\,\mathrm{d}t = T(A\log T + B) + O_\varepsilon(T^{\frac{33}{35}+\varepsilon}). \tag{4}$$

Corollary. *If A is the constant appearing in* (4), *then*

$$\sum_{\kappa_j \le T} \alpha_j H_j(\tfrac{1}{2}) = \left(\frac{T}{\pi}\right)^2 - \frac{2A}{\pi} T\log T + O(T(\log T)^{1/2}). \tag{5}$$

The formula (5) shows that there are actually two main terms in the asymptotic formula for the sum of $\alpha_j H_j(\frac{1}{2})$. Note that the error terms in (4), (5) may be improved to $O_\varepsilon(T(\log T)^\varepsilon)$, by use of special smooth functions instead of Gaussian weight function, namely for $G = C\sqrt{\log K}$,

$$f(r,K) = \frac{(r^2+\frac{1}{4})}{(r^2+1000)}\Big\{\exp\Big(-\Big(\frac{r-K}{G}\Big)^2\Big) + \exp\Big(-\Big(\frac{r+K}{G}\Big)^2\Big)\Big\}.$$

Our method of proof improves (2) to

$$\sum_{\kappa_j \le T} \alpha_j = \Big(\frac{T}{\pi}\Big)^2 + O_\varepsilon(T(\log T)^\varepsilon).$$

In the general problem of evaluating $\sum_{\kappa_j\le T} \alpha_j H_j^k(\frac{1}{2})$ one encounters the integrals

$$I_k(T) := \int_0^T \frac{|\zeta(\frac{1}{2}+it)|^{2k}}{|\zeta(1+2it)|^2}\,\mathrm{d}t \qquad (k \in \mathbb{N}),$$

where k is fixed. By general convexity results

$$I_k(T) \gg_k T(\log T)^{k^2}. \tag{6}$$

One expects the lower bound in (6) to be of the correct order of magnitude this, but this seems at present impossible to prove for $k \ge 3$. Even for $k = 2$ an upper bound for $I_2(T)$ corresponding to the lower bound in (6) seems difficult to obtain and represents an open problem. One may obtain a slightly weaker bound, namely $I_2(T) \ll T(\log T)^4(\log\log T)^2$.

THEOREM 3. (Y. Motohashi 1992). *With* $\gamma = -\Gamma'(1)$ *we have*

$$\sum_{\kappa_j\le T} \alpha_j H_j^2(\tfrac{1}{2}) = 2\pi^{-2}T^2(\log T + \gamma - \tfrac{1}{2} - \log(2\pi)) + O(T\log^6 T).$$

THEOREM 4 (A.I. 2002). *We have*

$$\sum_{\kappa_j \le T} \alpha_j H_j^3(\tfrac{1}{2}) = T^2 P_3(\log T) + O(T^{5/4}\log^{37/4} T),$$

where $P_3(x)$ *is a polynomial of degree three with leading coefficient* $4/(3\pi^2)$, *whose remaining coefficients may be explicitly evaluated.*

THEOREM 5 (A.I. 2002). *We have*

$$\sum_{\kappa_j \le T} \alpha_j H_j^4(\tfrac{1}{2}) = T^2 P_6(\log T) + O(T^{3/2}\log^{25/2} T),$$

where $P_6(x)$ *is a polynomial of degree six with leading coefficient* $16/(15\pi^4)$, *whose remaining coefficients may be explicitly evaluated.*

The constants 5/4 and 3/2 in the above theorems correspond to (sharpest known) exponents in the bounds

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^6 \,\mathrm{d}t \ll T^{5/4}\log^C T,$$

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^8 \,\mathrm{d}t \ll T^{3/2}\log^C T,$$

which come from the fourth moment and (Heath-Brown 1979)

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{12} \,\mathrm{d}t \ll T^{2}\log^C T.$$

Proofs of Theorem 4 and Theorem 5 yield $c_3 = 1/7, c_4 = 1/3$. If $c_k = o(k)$ as $k \to \infty$, then $\zeta(\frac{1}{2}+it) \ll_\varepsilon |t|^\varepsilon$ (LH) is true and $H_j(\frac{1}{2}) \ll_\varepsilon \kappa_j^\varepsilon$.

Essential ingredients for proof of Theorem 1:

Lemma 1. (The first Bruggeman-Kuznetsov trace formula). *Let* $f(r)$ *be an even, regular function for* $|\Im\mathrm{m}\, r| \le \frac{1}{2}$ *such that* $f(r) \ll (1+|r|)^{-2-\delta}$

for some $\delta > 0$. *Then*

$$\sum_{j=1}^{\infty} \alpha_j t_j(m) t_j(n) f(\kappa_j) + \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{\sigma_{2ir}(m)\sigma_{2ir}(n) f(r)\,\mathrm{d}r}{(mn)^{ir}|\zeta(1+2ir)|^2}$$
$$= \frac{1}{\pi^2}\delta_{m,n} \int_{-\infty}^{\infty} r\tanh(\pi r) f(r)\,\mathrm{d}r$$
$$+ \sum_{\ell=1}^{\infty} \frac{1}{\ell} S(m,n;\ell) f_+\left(\frac{4\pi\sqrt{mn}}{\ell}\right),$$

where $\delta_{m,n} = 1$ *if* $m = n$ *and zero otherwise* $(m, n > 0)$, $\sigma_a(d) = \sum_{d|n} d^a$, $S(m,n;\ell)$ *is the Kloosterman sum and*

$$f_+(x) = \frac{2i}{\pi} \int_{-\infty}^{\infty} \frac{r}{\cosh(\pi r)} J_{2ir}(x) f(r)\,\mathrm{d}r.$$

The J-Bessel function is defined as

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)} \qquad (|\arg z| < \pi).$$

Lemma 2. *Let* $\kappa_j = (1+o(1))K$, $r = (1+o(1))K \quad (r \in \mathbb{R})$ *as* $K \to \infty, Y = (1+\delta)\frac{K^2}{4\pi^2}$, *with* $\delta > 0$ *a given constant. Then, for any fixed positive constant* $A > 0$, *there exists a constant* $C = C(A,\delta) > 0$ *such that, for* $h = C\log K$, *we have*

$$H_j(\tfrac{1}{2}) = \sum_{n \le (1+\delta)Y} t_j(n) n^{-1/2} \mathrm{e}^{-(n/Y)^h} + O(K^{-A}),$$

and

$$\zeta(\tfrac{1}{2}+ir)\zeta(\tfrac{1}{2}-ir) = \sum_{n \le (1+\delta)Y} \sigma_{2ir}(n) n^{-\frac{1}{2}-ir} \mathrm{e}^{-(n/Y)^h}$$
$$+ O(K^{-A}).$$

Lemma 3. *For $C\sqrt{\log K} \le G \le K$ and a sufficiently large constant $C > 0$ we have*

$$\sum_{K\le\kappa_j\le K+G} \alpha_j H_j(\tfrac12) \ll GK.$$

Essential ingredients for proof of Theorem 3 and 4:

The transformation formulas of Y. Motohashi (1992) for

$$\sum_{j=1}^\infty \alpha_j H_j^k(\tfrac12) h_0(\kappa_j) \qquad (k = 3, 4),$$

$$h_0(r) = (r^2 + \tfrac14)\left\{\mathrm{e}^{-(r-k)^2/G^2} + \mathrm{e}^{-(r+k)^2/G^2}\right\}.$$

For $k = 3$ start from $H_j(u)H_j(v)H_j(\frac12)$ in the region of absolute convergence, replace $H_j(\frac12)$ by suitable sums of $t_j(f)f^{-1/2}$. Use ($\sigma_a(n) = \sum_{d|n} d^a$)

$$H_j(s)H_j(s-\alpha) = \zeta(2s-\alpha)\sum_{n=1}^\infty \sigma_\alpha(n)t_j(n)n^{-s},$$

which is the analytic equivalent of the multiplicativity of the arithmetic function $t_j(n)$, namely

$$t_j(m)t_j(n) \;=\; \sum_{d|(m,n)} t_j\left(\frac{mn}{d^2}\right).$$

After this there is summation of $t_j(m)t_j(f)$ in both cases $k = 3, 4$, which is effected by applying the Bruggeman-Kuznetsov trace formula. It is here that delicate questions of analytic continuation arise before one can take $(u, v) = (\frac12, \frac12)$. For $k = 4$ can use M. Jutila's approach (1999): combine Motohashi's formula for

$$\sum_{j=1}^\infty \alpha_j H_j^2(\tfrac12) t_j(f) h(\kappa_j) \quad (f \in \mathbb{N})$$

with the approximate functional equation

$$
\begin{aligned}
H_j^2(\tfrac12) = & \sum_{mn\le 3K^2} \frac{t_j(m)t_j(n)}{\sqrt{mn}} \exp(-(mn/K^2)^\lambda)\\
& - \sum_{mn\le 3K^2} \frac{t_j(m)t_j(n)}{\sqrt{mn}} R_j(mnK^2) + O(1),
\end{aligned}
$$

for $|\kappa_j - K| \le G\log K$ with $\log^2 K < G < K^{1-\delta}$ for $0<\delta<1$, $\lambda = C\log K$ with sufficiently large $C>0$. The function R_j comes from the squaring of the functional equation for $H_j(\frac12 + w)$, namely

$$
\begin{aligned}
R_j(x) = & \frac{1}{2\pi^4 i\lambda} \int_{-\lambda^{-1}-i\lambda^2}^{-\lambda^{-1}+i\lambda^2} (16\pi^4 x)^w \Gamma^2(\tfrac12 - w + i\kappa_j)\\
& \times \Gamma^2(\tfrac12 - w - i\kappa_j)(\cosh(\pi\kappa_j) + \sin(\pi w))^2 \Gamma(w/\lambda)\,\mathrm{d}w.
\end{aligned}
$$

Second method for $k=4$: correct and simplify N.V. Kuznetsov's claim (1999)

$$
\sum_{\kappa_j\le T} \alpha_j H_j^4(\tfrac12) = T^2 P_6(\log T) + O_\varepsilon(T^{4/3+\varepsilon}),
$$

with wrong leading coefficient $2^9/(15\pi^3)$ of P_6, unproved exponent $4/3+\varepsilon$, unproved transformation formula for sums of $H_j^4(\frac12)$. The basic formula is

$$
\begin{aligned}
& \sum_{j\ge1} \alpha_j H_j^4(\tfrac12) \exp\left(-\left(\frac{\kappa_j - T}{Q}\right)^2\right)\\
& + \frac{2}{\pi}\int_0^\infty \frac{|\zeta(\frac12+ir)|^8}{|\zeta(1+2ir)|^2} \exp\left(-\left(\frac{r-T}{Q}\right)^2\right)\mathrm{d}r\\
& = \sum_{k=0}^{6} a_k \hat\psi^{(6-k)}(1) + \sum_{j\ge1} \alpha_j H_j^4(\tfrac12)\tilde h(\kappa_j)\\
& + \frac{1}{\pi}\int_0^\infty \frac{|\zeta(\frac12+ir)|^8}{|\zeta(1+2ir)|^2} \tilde h(r)\,\mathrm{d}r + O(Q\log^6 T).
\end{aligned}
$$

Here $Q = CT^{1/3}$, $\sum_{k=0}^{6} \cdots$ will produce the main term, $\tilde{h}(r)$ is the oscillatory integral transform of the smooth function $h(r)$. In the relevant range

$$\tilde{h}(r) \ll Qr^{-1/2}\exp(-CQ^2r^2T^{-2}) \quad (C > 0).$$

We integrate this formula from $T = T_0$ to $T = 2T_0$. Basic line of attack on oscillatory terms: expand by Taylor's formula, take advantage of the Gaussian weight via

$$\int_{-\infty}^{\infty} y^j \mathrm{e}^{Ay-y^2}\,\mathrm{d}y \;=\; P_j(A)\mathrm{e}^{\frac{1}{4}A^2}$$

for $j = 0, 1, 2, \ldots,$ $P_0(A) = \sqrt{\pi}$, where $P_j(z)$ is a polynomial in z of degree j.

For $k = 3$ oscillatory terms reduce to (integrating from $K = K_0$ to $K = 2K_0$)

$$\begin{aligned} &GK_0^{5/2} \sum_{f \le 3K_0} f^{-\frac{1}{2}} \sum_{m \le fG^{-2}\log^2 K_0} (\frac{m}{f})^{\frac{1}{4}} d(m)d(m+f) \\ &\times \left(\sqrt{\frac{m}{f}} + \sqrt{1+\frac{m}{f}}\right)^{-2iK_0} \mathrm{e}^{-CG^2mf^{-1}} \\ &\times \log\left(\sqrt{\frac{m}{f}} + \sqrt{1+\frac{m}{f}}\right)^{-1} \end{aligned}$$

with $K_0^{\varepsilon} \le G \le K_0^{1/3}$. Divide the sum over m into subsums of length $\asymp M$. For large M use Cauchy-Schwarz and the exponent pair $(\frac{1}{2}, \frac{1}{2})$ for the resulting exponential sum. For small M set $n = m + f$ and transform the sum over n by the Voronoi formula and finally apply the saddle point method and choose $G = K_0^{1/7}$. Note that in Theorem 3, apart from the contribution of the integral with six zeta values, the remaining terms are of the order $K_0^{8/7+\varepsilon}$ with the choice $G = K_0^{1/7}$, and more refined exponential sum techniques could yield even smaller values of G.

In the proof based on Jutila's approach one obtains

$$\begin{aligned}
&\sum_{\kappa_j \le K} \alpha_j H_j^4(\tfrac12) + O\left(\log^2 K \int_0^K |\zeta(\tfrac12+it)|^8 \,\mathrm{d}t\right) \\
&= K^2 P_6(\log K) + O_\varepsilon(K^{4/3+\varepsilon}).
\end{aligned}$$

The main term (i.e., $K^2P_6(\log K)$ is derived analogously as was done in the proof of Theorem 3. It will be of the form

$$4\pi^{-3/2}K^3G(\mathcal{D}_1^*(K,G)+\mathcal{D}_2^*(K,G)),$$

where

$$\begin{aligned}
\mathcal{D}_1^*(K,G) = \frac{1}{2\pi i\lambda}\int_{(1)} &\Big\{(\log K+\gamma-\log(2\pi))\frac{\zeta^4(w+1)}{\zeta(2w+2)} \\
&+\frac12\left(\frac{\zeta^4(w+1)}{\zeta(2w+2)}\right)'\Big\}\zeta(2w+1)K^{2w}\Gamma(w/\lambda)\,\mathrm{d}w,
\end{aligned}$$

and analogously for $\mathcal{D}_2^*(K,G)$. The integrand has a pole of order six at $w=0$. We shift the line of integration to $\Re e\, w = -1$, developing the integrand into power series to calculate the residue. The coefficient of $\log^6 K$ is found to be $4/(15\pi^2)$, and clearly the coefficients of lower powers of the logarithm can be also evaluated explicitly. The coefficient of $\log^6 K$ coming from $\mathcal{D}_2^*(K,G)$ will be the same, and we see that the leading coefficient of $P_6(x)$ is $16/(15\pi^4)$, as claimed.

The relevant sum to be estimated in the error term is

$$\begin{aligned}
&GK^{5/2}\sum_{f\ll K^2} v(f)d(f)f^{-3/4} \\
&\times \sum_{m\le fG^{-2}\log^2 K} m^{-1/4}d(m)d(m+f) \\
&\times \left(\sqrt{\frac{m}{f}}+\sqrt{1+\frac{m}{f}}\right)^{-2iK} \\
&\times \exp\left(-G^2\log^2\left(\sqrt{\frac{m}{f}}+\sqrt{1+\frac{m}{f}}\right)\right) \\
&\times \left(\log\left(\sqrt{\frac{m}{f}}+\sqrt{1+\frac{m}{f}}\right)\right)^{-1},
\end{aligned}$$

where v is a smooth weight function supported in $[F, 2F]$ with $F \ll K_0^2$, and $K_0 \leq K \leq 2K_0$. The optimal choice is $G = K_0^{1/3}$, and we obtain another proof of Theorem 4.