Automorphic Summation Formulae and Moments of $\zeta(s)$

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This lecture is based on:

- Beineke and Bump, Moments of the Riemann zeta function and Eisenstein series I, *J. Number Theory* **105** (2004), 150–174.

- Beineke and Bump, A summation formula for divisor functions associated to lattices, Preprint (2004).

Available at `http://math.stanford.edu/~bump/` .

- Expert Bessel-functioneer Moshe Baruch consulted on this work.

Also we will discuss

- Conrey, Farmer, Keating, Rubinstein, Snaith, *Integral moments of L-functions* [CFKRS]: `http://arxiv.org/ps/math.NT/0206018`.

Moment Conjectures

Conjectures of [CFKRS]. Let $\alpha = (\alpha_1, \cdots, \alpha_{2n})$, $|\alpha_i|$ small,

$$Z(s,\alpha) = \zeta(s+\alpha_1)\cdots\zeta(s+\alpha_n)\zeta(1-s-\alpha_{n+1})\cdots\zeta(1-s-\alpha_{2n}).$$

We are interested in

$$\int_{-\infty}^{\infty} Z\left(\frac{1}{2}+it,\alpha\right) g(t)\,dt$$

with a suitably smooth test function $g(t)$ and the α_i near zero.

Conjecture [CFKRS]: *This should match*

$$\int_{-\infty}^{\infty} M\left(\frac{1}{2}+it,\alpha\right) g(t)\,dt,$$

where M will now be described.

The factor M is a sum of $\binom{2n}{n}$ terms indexed by

$$\Xi \subset S_{2n} \qquad (S_n = \text{symmetric group}).$$

The set Ξ of representatives of $S_{2n}/(S_n \times S_n)$ consists of w with

$$w(1) < \cdots < w(n), \qquad w(n+1) < \cdots < w(2n).$$

Let $R(\alpha)=A(\alpha)N(\alpha)$ where

$$N(\alpha)=\prod_{\substack{1\leqslant j\leqslant n\\ n+1\leqslant k\leqslant 2n}}\zeta(1+\alpha_j-\alpha_k).$$

We won't define the "arithmetic part" $A(\alpha)$. It is an Euler product, convergent if $\mathrm{re}(\alpha_i)$ are small. If $n\geqslant 3$ it does not have meromorphic continuation to all α_i (Estermann).

$$M\left(\frac{1}{2}+it,\alpha\right)=\left(\frac{t}{2\pi}\right)^{-\alpha_1-\dots-\alpha_n+\alpha_{n+1}+\dots+\alpha_{2n}}\times$$

$$\sum_{\sigma\in\Xi}R(w\alpha)\left(\frac{t}{2\pi}\right)^{-\alpha_{w(1)}-\dots-\alpha_{w(n)}+\alpha_{w(n+1)}+\dots+\alpha_{w(2n)}}.$$

To summarize, the $2n$-th moment

$$\int_{-\infty}^{\infty}\prod_{j=1}^{n}\zeta\left(\frac{1}{2}+it+\alpha_j\right)\zeta\left(\frac{1}{2}-it-\alpha_{n+j}\right)g(t)dt$$

is conjecturally a sum of $\binom{2n}{n}$ terms, each involving a product $N(w\alpha)$ of n^2 zeta functions.

- The conjectures are proved if $n\leqslant 2$ (Ingham).
- The case of real interest is when all $\alpha_i=0$. However the structure of the asymptotics is made more transparent by taking α_i distinct.
- If $n=2$, Motohashi gave an *exact* formula for the 4-th moment involving L-values at 1/2 of automorphic forms for $\mathrm{SL}_2(\mathbb{Z})$.

Eisenstein Series on GL_{2n}.

Let $\chi(s) = 2^s\,\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)$. Thus

$$\zeta(s) = \chi(s)\,\zeta(1-s).$$

We'll define $E_\alpha =$ Eisenstein series on GL_{2n} with L-function

$$\zeta(s+\alpha_1)\cdots\zeta(s+\alpha_n)\zeta(s+\alpha_{n+1})\cdots\zeta(s+\alpha_{2n}) = \chi(s+\alpha_{n+1})\cdots\chi(s+\alpha_{2n})\;Z(s,\alpha).$$

$B =$ group of upper triangular mats in GL_{2n}
$\mathbb{A} =$ adele ring of $\mathbb{Q}$
$\chi_\alpha =$ character of $B_{\mathbb{A}}$

$$\chi_\alpha(b) = \prod_{i=1}^{2k} |y_i|^{\alpha_i}, \qquad \delta(b) = \prod_{i=1}^{2k} |y_i|^{k+1-2n},$$

$$\text{when } b = \begin{pmatrix} y_1 & * & \cdots & * \\ & y_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & y_n \end{pmatrix}$$

$$K = \prod_{\text{places } v \text{ of } \mathbb{Q}} K_v, \qquad K_v = \begin{cases} O(2n) & \text{if } v=\infty; \\ \mathrm{GL}_{2n}(\mathbb{Z}_v) & \text{if } v = p. \end{cases}$$

We have $\mathrm{GL}_{2n}(\mathbb{A}) = B_{\mathbb{A}}K$ (Iwasawa). Define f on $\mathrm{GL}_{2n}(\mathbb{A})$:

$$f(bk) = (\delta^{1/2}\chi_\alpha)(b), \qquad b \in B_{\mathbb{A}}, k \in K.$$

$$E_\alpha(g) = \left\{ \prod_{1 \leqslant i < j \leqslant 2n} \zeta^*(\alpha_i - \alpha_j + 1) \right\} \sum_{B_{\mathbb{Q}} \backslash G_{\mathbb{Q}}} f_\alpha(\gamma g),$$

$$\zeta^*(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

$$N(\alpha)^* = \prod_{\substack{1 \leqslant i \leqslant k \\ k+1 \leqslant j \leqslant 2k}} \zeta^*(\alpha_i - \alpha_j + 1)$$

like N before but with Γ factors

$E_\alpha^M =$ corresponding Eisenstein series on $M = \mathrm{GL}_n \times \mathrm{GL}_n$.

$$E_\alpha^M(g) = \prod_{\substack{1 \leqslant i < j \leqslant n \\ \text{or} \\ n+1 \leqslant i < j \leqslant 2n}} \zeta^*(\alpha_i - \alpha_j + 1) \sum_{\gamma \in (B \cap M)_{\mathbb{Q}} \backslash M_{\mathbb{Q}}} f_\alpha(\gamma g).$$

$$\int_{\mathrm{Mat}_n(\mathbb{Q}) \backslash \mathrm{Mat}_n(\mathbb{A})} E_\alpha\left(\begin{pmatrix} I & X \\ & I \end{pmatrix} g \right) dX = \sum_{w \in \Xi} N(w\alpha)^* E^M_{w^{-1}\alpha}(g).$$

The terms in constant term expansion of E_α correspond exactly to the terms of in moment conjectures. (The arithmetic factor is missing however.)

We see that:

- The L-function of E_α coincides with $Z(s,\alpha)$ after using the functional eq'n of on half the ζ's

- The constant term along the n, n-parabolic shows the same structure as the conjectural moments of $Z(s,\alpha)$: A sum of $\binom{2n}{n}$ terms indexed by $w \in \Xi$, involving a factor $N(w\alpha)$ which is a product of n^2 (constant) ζ's.

The challenge is to relate the moment conjecture more precisely to these Eisenstein series.

Even when $n=1$, for the second moment of ζ, the connection is not obvious. The ingredients turn out to be:

- The Hecke construction of the L-function on GL_2, applied to the Eisenstein series;

- Oppenheim's generalization of the Voronoi summation formula.

The generalization to $n>1$ should involve:

- The Friedberg-Jacquet construction of the standard L-function, based on Shalika models.

- A Voronoi summation formula for Shalika models.

Oppenheim Summation Formula

When $n = 1$, the "missing link" between the GL_2 Eisenstein series and the second moment is Oppenheim's generalization of the Voronoi summation formula. Let $a \in \mathbb{C}$,

$$\tau_a(n) = \sum_{d|n} \left(\frac{d}{n/d}\right)^a.$$

Then $\tau_{\sigma-1/2}(n)$ is the n-th Fourier coefficient of the Eisenstein series

$$E_\sigma(g) = \sum_{B_{\mathbb{Q}} \backslash \mathrm{GL}(2,\mathbb{Q})} f_s(\gamma g),$$

where

$$f_s(bg) = \delta(b)^s f(g), \qquad \delta\begin{pmatrix} y_1 & * \\ & y_2 \end{pmatrix} = \left|\frac{y_1}{y_2}\right|^s.$$

Let ϕ have smooth support in $\mathbb{R}_+$. The ***Hankel transform***

$$\begin{aligned} \tilde{\phi}_s(y) = \int_0^\infty \phi(x)[-2\pi\cos(s\pi)\, J_{1-2s}(4\pi\sqrt{yx}) \\ -2\pi\sin(s\pi) Y_{1-2s}(4\pi\sqrt{yx}) + \\ 4\sin(s\pi)\, K_{1-2s}(4\pi\sqrt{yx})]\, dx. \end{aligned}$$

It is of rapid decay as $y \longrightarrow \infty$.

Theorem (Oppenheim) *We have*

$$\sum_{n=1}^{\infty} \tau_{\sigma-1/2}(n)\phi(n) =$$

$$\zeta(2\sigma)\int_0^{\infty} \phi(x)x^{\sigma-1/2}\,dx + \zeta(2-2\sigma)\int_0^{\infty} \phi(x)x^{1/2-\sigma}\,dx$$

$$+\sum_{n=1}^{\infty} \tau_{\sigma-1/2}(n)\tilde{\phi}_s(n).$$

- Iif $s = 1/2$ this is the Voronoi summation formula.
- The two factors $\zeta(2\sigma)$ and $\zeta(2-2\sigma)$ appear also in the asymptotics of the second moment of ζ, which is a special case, due to Ingham of [CFKRS].

$$\int_0^{T} |\zeta(\sigma+it)|^2\,dt =$$

$$\zeta(2\sigma)T + \frac{(2\pi)^{2\sigma-1}}{2-2\sigma}\zeta(2-2\sigma)T^{2-2\sigma} + O(T^{1-\sigma}\log(T))$$

- Atkinson (1939) and Atkinson (1949) gave two methods of relating the second moment to the "Dirichlet divisor problem." The relationship is expressed as follows:

$$\int_0^{T} |\zeta(\sigma+it)|^2\,dt \sim 2\pi \sum_{n<T/2\pi} \tau_{\sigma-\frac{1}{2}}(n)n^{\frac{1}{2}-\sigma}$$

Using method of Atkinson (1949) K. Matsumoto, Meurman and Ivic improved the error in Ingham's est.

Method of Atkinson (1939)

$$|\zeta(\sigma+it)|^2 = \chi(\sigma-it)\zeta(\sigma+it)\zeta(1-\sigma+it)$$
$$= \chi(\sigma-it)\sum_{n=1}^{\infty} \tau_{\sigma-1/2}(n) n^{-\frac{1}{2}-it}.$$

(Pretend this converges.) Stirling:

$$n^{-\frac{1}{2}-it}\chi(\sigma-it) \cong$$
$$n^{-\frac{1}{2}}\left|\frac{t}{2\pi}\right|^{\frac{1}{2}-\sigma}\exp\left(i\left(t\log\left|\frac{t}{2\pi n}\right| - \frac{\pi}{4} - t\right)\right).$$

We have

$$\frac{d}{dt}\left(t\log\left(\frac{t}{2\pi n}\right) - \frac{\pi}{4} - t\right) = \log\left(\frac{t}{2\pi n}\right).$$

Stationary phase: in an oscillatory integral, main contribution is where oscillations cease ($t = 2\pi n$). So in

$$\int_0^T |\zeta(\sigma+it)|^2 dt = \int_0^T \sum_{n=1}^{\infty} \tau_{\sigma-1/2}(n)\chi(\sigma-it) n^{-\frac{1}{2}-it} dt$$

we may ignore $n > T/2\pi$. This leads to

$$\int_0^T |\zeta(\sigma+it)|^2\, dt \sim 2\pi \sum_{n<T/2\pi} \tau_{\sigma-\frac{1}{2}}(n) n^{\frac{1}{2}-\sigma}$$

and the right side may be estimated by Oppenheim.

Shalika Models

Let (Π, V) be a rep'n of $\mathrm{GL}_{2n}(F)$, F local. A nonzero linear functional $\lambda: V \longrightarrow \mathbb{C}$ is called a **Shalika functional** if

$$\lambda\left(\Pi\begin{pmatrix} I & X \\ & I \end{pmatrix}\Pi\begin{pmatrix} g & \\ & g \end{pmatrix}v\right) = \psi(\mathrm{tr}\, X)\lambda(v)$$

where $\psi: F \longrightarrow \mathbb{C}$ is a nontrivial additive character.

Theorem 1. (Jacquet and Rallis) *If π has a Shalika functional it is unique (up to scalar).*

If Π admits a Shalika functional let $\mathcal{H}$ be the space of functions of the form

$$H_v(g) = \lambda(\Pi(g)v), \qquad v \in V.$$

- $\mathcal{H}$ is a space of functions on $\mathrm{GL}_{2n}(F)$ on which $\mathrm{GL}_{2n}(F)$ acts by right translation.
- This space is called the ***Shalika model***.
- It is expected that Π has a Shalika model if and only if it is a functorial lift from SO_{2n+1}.
- The Shalika model is the basis of a construction of the standard L-function due to Friedberg and Jacquet.

The Degenerate Principal Series

Let π be a principal series representation of $\mathrm{GL}_{2n}(F)$, F local. Let P be the parabolic subgroup with Levi factor $\mathrm{GL}_n \times \mathrm{GL}_n$. We form the induced representation $\Pi_s = \mathrm{Ind}_P^{\mathrm{GL}_{2n}}(\pi \otimes \hat{\pi} \otimes \delta^s)$ where $\delta\colon P_F \longrightarrow \mathbb{C}$ is the modular quasicharacter

$$\delta\begin{pmatrix} g_1 & * \\ & g_2 \end{pmatrix} = \left|\frac{\det(g_1)}{\det(g_2)}\right|^{ns}.$$

- This representation will admit a Shalika model.
- The L-function of Π_s is

$$L(\sigma, \Pi_s) = L(\sigma + s, \pi) L(\sigma + 1 - s, \hat{\pi}).$$

We will take π to be the trivial representation. Then Π_s is in the ***degenerate principal series***.

- The degenerate principal series are too simple for applications to the moment problem. The application would be to the second moment of

$$L(s, \pi_{\mathrm{trivial}}) = \zeta(s)\zeta(s+1)\cdots\zeta(s+n-1)$$

  and only one ζ can be in the critical strip.
- Still we hope this situation is a prototype of a theory for general π with momentous applications.

The Bessel Distribution

Let F be a local field. If $\Pi = \Pi_s$ is the degenerate principal series there is a ***Bessel distribution*** J_s on $\mathrm{GL}_n(F)$ describing the action of

$$w_0 = \begin{pmatrix} & -I \\ I & \end{pmatrix}$$

on the ***Shalika-Kirillov model*** $\mathcal{K}$. Specifically, $\mathcal{K}$ is the space of functions

$$Q_f(g) = \lambda\left(\Pi\begin{pmatrix} g & \\ & I \end{pmatrix} f\right), \qquad \lambda = \text{Shalika functional}.$$

We have

$$Q_{\Pi(w_0)f}(h) = \int_{\mathrm{GL}_n(F)} Q_f(g)\, J_s(hg)\, dg. \tag{1}$$

- We assume $\mathrm{re}(s) > 1 - \frac{1}{2n}$.
- At present J_s is known to be a distribution and (1) is known if Q_f has compact support in $\mathrm{GL}_n(F)$. This is enough for the summation formula.
- It is expected that J_s is a locally integrable function and (1) is true in some sense for all of $\mathcal{K}$.

Theorem. *If* $\mathrm{re}(s) > 1 - \frac{1}{2n}$*, then for* $\phi \in C_c^\infty(\mathrm{GL}_n(F))$ *the integral*

$$\langle J_s, \phi\rangle = \int_{\mathrm{Mat}_n(F)} \int_{\mathrm{Mat}_n(F)} |\det(g)|^{ns}\phi(g)\psi(-\mathrm{tr}(g\xi))\,dg \\ \psi(-\mathrm{tr}\,\xi^{-1})|\det(\xi)|^{2n(s-1)}d\xi$$

is convergent and continuous, hence defines a distribution.

- The function J_s is a Bessel distribution as we defined it – convolution with J_s describes the action of w_0 on compactly supported vectors in the Shalika-Kirillov model.

- Formally, $J_s(g) =$

$$|\det(g)|^{ns}\int_{\mathrm{Mat}_n(F)} |\det(\xi)|^{2n(s-1)}\,\psi(-\mathrm{tr}\,(g\xi+\xi^{-1}))\,d\xi.$$

- This integral is not absolutely convergent but may perhaps be studied by the method of Baruch and Mao, *Bessel identities in the Waldspurger correspondence: Archimedean theory* (2002).

- Bessel functions for representations of $\mathrm{GL}_2(\mathbb{R})$ have been studied by Gelfand and Kazhdan, Gelfand, Graev and Piatetski-Shapiro, Vilenkin, Cogdell and PS, and by Baruch and Mao. (Bessel functions for the symmetric space were studied by Herz and Bengtson.)

The Hankel Transform

Take $F = \mathbb{R}$.

$$\tilde{\phi}_s(g) = \int_{\mathrm{Mat}_n(\mathbb{R})} \phi(h)\, |\det(h)|^{-n/2}\, J_s({}^t g\, h)\, dh.$$

- We say that $f\colon \mathrm{Mat}_n(\mathbb{R}) \longrightarrow \mathbb{C}$ is ***of rapid decay*** if $|p(X) f(X)|$ is bounded for all polynomials $p(X)$.
- For many authors rapid decay = Schwartz class. We do not ask that the derivatives decay or even exist.

Theorem. *If* $\mathrm{re}(s) > 1 - \frac{1}{2n}$ *then* $\tilde{\phi_s}$ *extends to a function of rapid decay on* $\mathrm{Mat}_n(\mathbb{R})$.

- The transform $\tilde{\phi_s}$ is not of Schwartz class.
- What is true is that if N is given, there exists a constant $c(N)$ such that if $\mathrm{re}(s) > c(N)$, then all partial derivatives of order $\leqslant N$ are of rapid decay.
- Assume that $n = 2$. There are also two *degenerate Hankel transforms* $\tilde{\phi}_s^0(g)$, $\tilde{\phi}_s^1(g)$ and another integral transform $\tilde{\phi}_s^2(g)$. These are only defined for g of rank one. They also satisfy suitable properties.

Divisor Functions

Let $L \subseteq \mathbb{Z}^n$ be a lattice, that is, a subgroup of finite index. Let $a \in \mathbb{C}$, and define

$$\tau_a(L) = \sum_{\substack{\text{lattice } L' \\ L \subseteq L' \subseteq \mathbb{Z}^n}} \left(\frac{[\mathbb{Z}^n : L']}{[L' : L]} \right)^a .$$

We have

$$\sum_L \tau_a(L) [\mathbb{Z}^n : L]^{-s} = \prod_{k=0}^{n-1} \zeta(s+a-k)\zeta(s-a-k).$$

Let $\mathrm{Mat}_n^*(\mathbb{Z}) = \mathrm{Mat}_n(\mathbb{Z}) \cap \mathrm{GL}_n(\mathbb{R})$. This is a parameter space for lattices:

$$\text{If } A \in \mathrm{Mat}_n^*(\mathbb{Z}), \text{ let } L_A = \text{row lattice}$$

$A \longrightarrow L_A$ is a bijection

$$\mathrm{GL}_n(\mathbb{Z}) \backslash \mathrm{Mat}_n^*(\mathbb{Z}) \longrightarrow \{\text{Lattices } L \subseteq \mathbb{Z}^n\}.$$

If $A \in \mathrm{Mat}_n^*(\mathbb{Z})$, we will denote $\tau_a(A) = \tau_a(L_A)$. If $\Phi \in C_c^\infty(\mathrm{GL}_n(\mathbb{Z}) \backslash \mathrm{GL}_n(\mathbb{R}))$ we describe a summation formula for

$$\sum_{A \in \mathrm{GL}_n(\mathbb{Z}) \backslash \mathrm{Mat}_n^*(\mathbb{Z})} \Phi(A).$$

Divisor Functions and Shalika Models

Let F be a nonarchimedean local field with ring $\mathfrak{o}$ of integers. A *lattice* in $\mathfrak{o}^n$ is a compact open subgroup L. Define

$$\tau_a(L) = \sum_{\substack{\text{lattice } L' \\ L \subseteq L' \subseteq \mathfrak{o}^n}} \left(\frac{[\mathfrak{o}^n : L']}{[L' : L]} \right)^a.$$

Let H be the unique $\mathrm{GL}_{2n}(\mathfrak{o})$-fixed vector in the Shalika model of the degenerate principal series Π_s.

Theorem (Fumihiro Sato) *Let*

$$h(g) = |\det(g)|^{-n/2} H \begin{pmatrix} g & \\ & I \end{pmatrix}, \qquad g \in \mathrm{GL}_n(F).$$

Then

$$h(g) = \begin{cases} \tau_{n(s-1/2)}(g) & \text{if } g \in \mathrm{Mat}_n(\mathfrak{o}), \\ 0 & \text{otherwise.} \end{cases}$$

- This explains the appearance of the divisor functions in the summation formula.

- The spherical vector in the Shalika model of the *nondegenerate principal series* has been computed by Y. Sakellaridis.

The Summation Formula

The summation formula for

$$\sum_{A \in \mathrm{GL}_n(\mathbb{Z}) \backslash \mathrm{Mat}_n^*(\mathbb{Z})} \Phi(A), \qquad \Phi \in C_c^\infty(\mathrm{GL}_n(\mathbb{Z}) \backslash \mathrm{GL}_n(\mathbb{R}))$$

with n general has as its "main terms"

$$\begin{aligned} &\zeta(2ns)\zeta(2ns-1)\cdots\zeta(2ns-n+1) \times \\ &\int_{\mathrm{GL}_n(\mathbb{Z}) \backslash \mathrm{Mat}_n(\mathbb{R})} |\det(g)|^{n(s-1/2)}\, \Phi(g)\, dg \\ &+ \zeta(2n-2ns)\zeta(2n-1-2ns)\cdots\zeta(n+1-2ns) \times \\ &\int_{\mathrm{GL}_n(\mathbb{Z}) \backslash \mathrm{Mat}_n(\mathbb{R})} |\det(g)|^{n(s-1/2)}\, \Phi(g)\, dg. \end{aligned}$$

Another term is

$$\sum_{A \in \mathrm{GL}_n(\mathbb{Z}) \backslash \mathrm{Mat}_n^*(\mathbb{Z})} \tilde{\Phi}_s(A),$$

where the "automorphic Hankel transform" is as follows. Find ϕ such that $\Phi(g) = \sum_{\gamma \in \mathrm{GL}_n(\mathbb{Z})} \phi(\gamma g)$ and let

$$\tilde{\Phi}_s(g) = \sum_{\gamma \in \mathrm{GL}_n(\mathbb{Z})} \tilde{\phi}_s(\gamma g) \qquad \text{(independent of choice of } \phi).$$

- The complexity of the formula grows with n and to state a precise version we take $n = 2$.

Theorem. *Let* $\phi \in C_c^\infty(\mathrm{GL}_2(\mathbb{R}))$ *and assume* $\mathrm{re}(s) > 1 - \frac{1}{2n}$. *Then*

$$
\begin{aligned}
\sum_{\substack{A \in \mathrm{Mat}_n(\mathbb{Z}) \\ \det(A) \neq 0}} \phi(A)\, \tau_{2(s-1/2)}(A) = {} & \sum_{\substack{A \in \mathrm{Mat}_n(\mathbb{Z}) \\ \det(A) \neq 0}} \tilde{\phi}_s(A)\, \tau_{2(s-1/2)}(A) + \\
& + \zeta(4s)\zeta(4s-1) \int_{\mathrm{Mat}_2(\mathbb{R})} |\det(g)|^{2s-1}\, \phi(g)\, dg \\
& + \zeta(4-4s)\zeta(3-4s) \int_{\mathrm{Mat}_2(\mathbb{R})} |\det(g)|^{1-2s}\, \phi(g)\, dg \\
& + \zeta(4s-2) \sum_{\substack{A \in \mathrm{Mat}_2(\mathbb{Z}) \\ \mathrm{rank}(A)=1}} \tilde{\phi}_s^0(A) \sum_{d \mid \gcd(A)} d^{3-4s} \\
& + \zeta(4s-1) \sum_{\substack{A \in \mathrm{Mat}_2(\mathbb{Z}) \\ \mathrm{rank}(A)=1}} \tilde{\phi}_s^1(A) \sum_{d \mid \gcd(A)} d^{4s-1} \\
& + \frac{1}{2}\, \zeta(2-4s) \sum_{\substack{A \in \mathrm{Mat}_2(\mathbb{Z}) \\ \mathrm{rank}(A)=1}} \tilde{\phi}_s^2(A).
\end{aligned}
$$

- It is hoped that for the *nondegenerate principal series* there is a corresponding summation formula which will have $\binom{2n}{n}$ "main terms" corresponding to the conjectured asymptotics of the $2n$-th moments of zeta.

- The last term is really a $\mathrm{GL}_2 \times \mathrm{GL}_2$ Eisenstein series (with data not a pure tensor).

PROOF.

Let $f = \prod_v f_v$ in the $\otimes_v \Pi_{v,s}$ where $\Pi_{v,s} =$ degenerate principal series for $\mathrm{GL}_{2n}(\mathbb{Q}_v)$. If $v = p$ is nonarchimedean then $f_p = f_p^\circ$ is the standard $\mathrm{GL}_{2n}(\mathfrak{o})$ fixed vector in $\Pi_{v,s}$, while

$$f_\infty\left(\begin{pmatrix} Y_1 & * \\ & Y_2 \end{pmatrix} w_0 \begin{pmatrix} I & X \\ & I \end{pmatrix}\right) =$$

$$\left|\frac{\det(Y_1)}{\det(Y_2)}\right|^{ns} \int_{\mathrm{Mat}_n(\mathbb{R})} |\det(U)|^{n(s-1/2)}\,\phi(U)\,\psi(\mathrm{tr}(UX))\,dU,$$

$f_\infty(g) = 0$ off this Bruhat cell. If $\mathrm{re}(s)$ is sufficiently large let

$$E_s(g) = \sum_{P_\mathbb{Q}\backslash G_\mathbb{Q}} f(\gamma g).$$

Make a Fourier expansion:

$$\sum_{A \in \mathrm{Mat}_n(\mathbb{Q})} \int_{\mathrm{Mat}_n(\mathbb{Q})\backslash \mathrm{Mat}_n(\mathbb{A})} E_s\left(\begin{pmatrix} I & X \\ & I \end{pmatrix} g\right) \psi(\mathrm{tr}\, AX)\,dX.$$

With our data, only $A \in \mathrm{Mat}_n(\mathbb{Z})$ really contribute. There are various cases indexed by the Bruhat decomposition. If A is invertible, the term is a Shalika coefficient. The left side of the summation formula is $E_s(I)$ while the right side is $E_s(w_0)$. Since E_s is automorphic, these are equal. Done if $\mathrm{re}(s) \gg 0$ but both sides are analytic for $\mathrm{re}(s) > 1 - \frac{1}{2n}$.

QED