Introduction to function fields

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… le mathématicien qui étudie ces problèmes a l'impression de déchiffrer une inscription trilingue. Dans la première colonne se trouve la théorie riemannienne des fonctions algébriques au sens classique. La troisième colonne, c'est la théorie arithmétique des nombres algébriques. La colonne du milieu est celle dont la découverte est la plus récente; elle contient la théorie des fonctions algébriques sur un corps de Galois … nous n'avons bien entendu que des fragments …. il y a des grandes différences de sens d'une colonne à l'autre, mais rien ne nous en avertit à l'avance …

"De la métaphysique aux mathématiques"
A. Weil, 1960

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Finite fields

For p a prime, $q = p^a$, there is a field $\mathbf{F}_q$ with q elements (unique up to isomorphism).

$$\mathbf{F}_{q_1} \hookrightarrow \mathbf{F}_{q_2} \Leftrightarrow q_2 = q_1^f$$

$$\text{Gal}(\mathbf{F}_{q^f}/\mathbf{F}_q) \cong \mathbf{Z}/f\mathbf{Z}, \text{ generated by } x \mapsto x^q.$$

$$\mathbf{F}_q = \{x \in \mathbf{F}_{q^f} \,|\, x^q - x = 0\}$$

Function fields

We consider finitely generated fields of transcendence 1 over $\mathbf{F}_p$.

Examples:

$$\text{(1)} \qquad F = \mathbf{F}_q(x)$$

$$\text{(2)} \qquad F = \mathbf{F}_q(x, y) \text{ with } y^2 = x^3 - 1$$

$$\text{(general case)} \qquad F = \text{Frac}\left(\frac{\mathbf{F}_q[x, y]}{(g(x, y))}\right)$$

$$= \frac{\mathbf{F}_q(x)[y]}{(g(x, y))}$$

where $g \in \mathbf{F}_q[x, y]$ is an irreducible polynomial. (It's convenient to assume g is geometrically irreducible.)

Curves over finite fields

A "function field" as defined above is the field of rational functions (meromorphic functions) on a curve over $\mathbf{F}_q$.

More precisely, given F, there is a unique non-singular projective curve $\mathcal{C}$ whose function field $\mathbf{F}_q(\mathcal{C})$ is F. Conversely, given such a curve $\mathcal{C}$, $F = \mathbf{F}_q(\mathcal{C})$ is a function field in our sense. Non-constant maps of curves correspond to inclusions of function fields.

A plane model of the curve is given by the equation $g(x, y) = 0$. This is an affine curve and needs to have finitely many "points at infinity" added to get a projective curve. It will in general also have singularities which can be resolved by "blowing up."

The resulting smooth projective curve can always be imbedded in $\mathbf{P}^3$ and sometimes in $\mathbf{P}^2$.

The curve $\mathcal{C}$ can also be presented as a branched covering of the projective line $\mathcal{C} \to \mathbf{P}^1$ as one does with a Riemann surface.

Points

We write $\mathcal{C}(k)$ for the set of points of $\mathcal{C}$ with coordinates in an extension field k of $\mathbf{F}_q$. These are the "k-valued points" of $\mathcal{C}$.

The set $\mathcal{C}(\mathbf{F}_q)$ is too small to be useful—it might even be empty. The set $\mathcal{C}(\overline{\mathbf{F}}_q)$ is always infinite and in some sense captures the curve $\mathcal{C}$ uniquely.

There is another, more arithmetic, notion of point, the "closed points" or "prime divisors" of $\mathcal{C}$. To motivate it, consider divisors on $\mathcal{C}$, i.e., finite, formal, integer linear combinations of ($\overline{\mathbf{F}}_q$-valued) points on $\mathcal{C}$:

$$D = \sum a_x[x]$$

We say D is effective if all the $a_x \geq 0$ and that D is $\mathbf{F}_q$-rational if for every $\sigma \in \mathrm{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q)$,

$$D^\sigma = \sum a_x[\sigma(x)] = D.$$

A "prime divisor" is an effective $\mathbf{F}_q$-rational divisor which cannot be written as the sum of two non-zero effective $\mathbf{F}_q$-rational divisors. Therefore, a prime divisor is just the sum of the points in an orbit of Gal$(\overline{\mathbf{F}}_q/\mathbf{F}_q)$ acting on $\mathcal{C}(\overline{\mathbf{F}}_q)$, each point with coefficient 1.

The degree of a prime divisor is just the number of points in the orbit.

A "closed point" is the modern (scheme-theoretic) version of a prime divisor.

For example, if $F = \mathbf{F}_q(x)$ so that $\mathcal{C} = \mathbf{P}^1$, then the prime divisors/closed points are: the point at infinity and one prime divisor for each irreducible monic polynomial $f \in \mathbf{F}_q[x]$. The $\overline{\mathbf{F}}_q$-valued points in the divisor corresponding to f are just the zeros of f.

Analogies

$\mathbf{Q}$	general K
$\mathbf{Z}$	$\mathcal{O}_K$
(p) with p prime	$\mathfrak{p}$
$\mathbf{Z}_{(p)}$ numbers integral at p	$\mathcal{O}_{(\mathfrak{p})}$ numbers integral at $\mathfrak{p}$
$\mathbf{Z}/p \cong \mathbf{F}_p$	$\mathcal{O}_K/\mathfrak{p} \cong \mathbf{F}_{p^d}$ $d = \deg \mathfrak{p}$
$\mathbf{N}p = p$	$\mathbf{N}\mathfrak{p} = p^d$

$\mathbf{F}_q(x)$	general F
$\mathbf{F}_q[x]$	$R = \mathbf{F}_q[x,y]/(g(x,y)$
(f) with f irreducible	$\mathfrak{p}$
$\mathbf{F}_q[x]_{(f)}$ functions defined at zeroes of f	$R_{(\mathfrak{p})}$ functions defined at zeroes of $\mathfrak{p}$
$\mathbf{F}_q[t]/(f) \cong \mathbf{F}_{q^d}$ $d = \deg f$	$R/\mathfrak{p} \cong \mathbf{F}_{q^d}$ $d = \deg \mathfrak{p}$
$\mathbf{N}f = q^{\deg f}$	$\mathbf{N}\mathfrak{p} = q^d$

Frobenius elements

Let K/F be a finite degree d Galois extension of function fields over $\mathbf{F}_q$, corresponding to a finite Galois branched cover of curves $\pi : \mathcal{C}_1 \to \mathcal{C}_2$ and let $G = \text{Gal}(K/F)$.

Let $x \in \mathcal{C}_2(\mathbf{F}_{q^n})$ be a point over which the cover π is not branched. Then the inverse image of x consists of d distinct points, but these points need not be $\mathbf{F}_{q^n}$-rational. In general they will have coordinates in a larger finite field. Thus they will be permuted by the map $y_i \mapsto y_i^{q^n}$.

A basic fact is that there is an element of G which achieves this permutation; (slight cheat here ...) The element is well-defined up to conjugation and will be denoted Fr_x.

Example: Let $F = \mathbf{F}_q(x)$ and $K = \mathbf{F}_q(x, y)$ with

$$y^2 = x^3 - 1$$

corresponding to $\mathcal{C}_1 \to \mathbf{P}^1$. Then Gal$(K/F)$ has order 2.

If $x \in \mathbf{P}^1(\mathbf{F}_{q^n})$ is a finite point (i.e., just an element of $\mathbf{F}_{q^n}$) with $x^3 - 1 \neq 0$, then the two points of $\mathcal{C}_1$ over it are the pairs (x, y) with y solving $y^2 = x^3 - 1$.

If $x^3 - 1$ is a square in $\mathbf{F}_{q^n}$ then these two points are $\mathbf{F}_{q^n}$-rational and Fr_x is trivial. If $x^3 - 1$ is not a square in $\mathbf{F}_{q^n}$, then Fr_x is the non-trivial element of Gal(K/F).

Classical Equidistribution

A classical result (Cebotarev density) says that the Fr_x become equidistributed in the space of conjugacy classes of G as $n \to \infty$. More precisely, if $Z \subset G$ is a conjugation invariant subset, then

$$\lim_{n\to\infty} \frac{\#\{x \in \mathcal{C}_2(\mathbf{F}_{q^n}) | Fr_x \in Z\}}{\#\mathcal{C}_2(\mathbf{F}_{q^n})} = \frac{\#Z}{\#G}$$

More precisely,

$$\left| \frac{\#\{x \in \mathcal{C}_2(\mathbf{F}_{q^n}) | Fr_x \in Z\}}{\#\mathcal{C}_2(\mathbf{F}_{q^n})} - \frac{\#Z}{\#G} \right| \leq Cq^{-n/2}$$

as $n \to \infty$.

The constant C can be made explicit in terms of Z and the characters of G.

In our example, this says something pretty concrete: As x varies through $\mathbf{F}_{q^n}$, $x^3 - 1$ is a square about one half the time, with an explicit estimate for the error which tends to zero as $n \to \infty$.

Zeta functions

In analogy with the zeta-function of a number field:

$$\zeta(F,s) = \zeta(\mathcal{C},s) = \prod_{\mathfrak{p}} \left(1 - (\mathbf{N}\mathfrak{p})^{-s}\right)^{-1}$$
$$= \prod_{\mathfrak{p}} \left(1 - q^{-s\,\mathrm{deg}\,\mathfrak{p}}\right)^{-1}$$

Where $\mathcal{C}$ is the curve corresponding to F.

Let C_m be the number of closed points of $\mathcal{C}$ of degree m and let N_n be the number of $\mathbf{F}_{q^n}$-valued points of $\mathcal{C}$. Then

$$N_n = \sum_{m|n} mC_m.$$

Using this,

$$\zeta(\mathcal{C},s) = \exp\left(\sum_{n=1}^{\infty} \frac{N_n}{n} q^{-ns}\right)$$

which makes the diophantine interest of ζ quite visible.

Examples of zetas

If $F = \mathbf{F}_q(x)$, so $\mathcal{C} = \mathbf{P}^1$, then

$$\zeta(\mathcal{C}, s) = \frac{1}{(1 - q^{-s})(1 - q^{1-s})}$$

If $F = \mathbf{F}_q(x, y)$, with $y^2 = x^3 - 1$, so $\mathcal{C}$ is an elliptic curve, then

$$\zeta(\mathcal{C}, s) = \frac{1 - aq^{-s} + q^{1-2s}}{(1 - q^{-s})(1 - q^{1-s})}$$

where $q + 1 - a$ is the number of $\mathbf{F}_q$-rational points on $\mathcal{C}$.

For a general F, corresponding to a curve of genus g

$$\zeta(\mathcal{C}, s) = \frac{P(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})}$$

where P is a polynomial of degree $2g$ with integer coefficients.

P has the form

$$P(T) = \prod_{i=1}^{2g} (1 - \alpha_i T)$$

where $\{\alpha_i\} = \{\overline{\alpha_i}\}$ and $|\alpha_i| = q^{1/2}$ for all i.

Properties of zetas

In all these examples,

- $\zeta(\mathcal{C}, s)$ is defined by an Euler product
- it is meromorphic in s, with poles at $s = 0$, $s = 1$ and holomorphic elsewhere
- it satisfies a functional equation for $s \leftrightarrow 1 - s$
- its zeroes are on the line $\Re s = 1/2$

These properties extend to a vast array of other zeta- and L-functions.

For example, if E is an elliptic curve *over* F (and not over $\mathbf{F}_q$), then $L(E, s)$ is a polynomial in q^{-s} of known degree, it satisfies a functional equation for $s \leftrightarrow 2 - s$ and its order of vanishing at $s = 1$ is conjectured to be equal to the rank of the finitely generated abelian group $E(F)$. (BSD conjecture)

Cohomology

The properties of $\zeta(\mathcal{C}, s)$ can be established using nothing more than the Riemann-Roch theorem. The general picture is much more involved and is best understood in terms of cohomology groups.

If X is a smooth proper variety of dimension d over $\mathbf{F}_q$ then there are cohomology groups

$$H^i(X) = H^i(X \times \overline{\mathbf{F}}_q, \mathbf{Q}_\ell) \qquad \text{for } i = 0, \ldots, 2d$$

and an operator Fr on them such that

$$\zeta(X, s) = \prod_{i=0}^{2d} \det\left(1 - Fr\, q^{-s} | H^i(X)\right)^{(-1)^{i+1}}$$

The functional equation is equivalent to a duality between $H^i(X)$ and $H^{2d-i}(X)$ and the Riemann hypothesis is that the eigenvalues of Fr on $H^i(X)$ have absolute value $q^{i/2}$.

Curve case

When $X = \mathcal{C}$ is a curve of genus g, then $H^0(\mathcal{C})$ is $\mathbf{Q}_\ell$ with Fr acting trivially, $H^2(\mathcal{C})$ with Fr acting by multiplication by q, and $H^1(\mathcal{C})$ is $2g$-dimensional with

$$P(T) = \det(1 - TFr|H^1(\mathcal{C}))$$

This gives an interpretation of quadratic Dirichlet L-functions: let $F = \mathbf{F}_q(x)$ and let $\mathcal{C}$ correspond to a quadratic extension $K = F(\sqrt{f})$, so that we have a 2-sheeted covering $\mathcal{C} \to \mathbf{P}^1$. Then we have

$$\zeta(\mathcal{C}, s) = \zeta(\mathbf{P}^1, s) L(\chi_f, s)$$

where χ_f is the quadratic character of F corresponding to K and where

$$\begin{aligned} L(\chi_f, s) &= \prod_{\mathfrak{p}} \left(1 - \chi_f(\mathfrak{p})(\mathbf{N}\mathfrak{p})^{-s}\right)^{-1} \\ &= \prod_{\mathfrak{p}} \left(1 - \chi_f(\mathfrak{p}) q^{-s \deg \mathfrak{p}}\right)^{-1} \end{aligned}$$

This means that $L(\chi_f, s)$ is just the numerator of the zeta-function of $\mathcal{C}$. In particular,

$$L(\chi_f, s) = \det\left(1 - q^{-s} Fr|H^1(\mathcal{C})\right)$$

Families

We want to study families of ζ-functions or L-functions *parameterized by an algebraic variety*

Example 1: Quadratic Dirichlet L-functions. Let $F = \mathbf{F}_q(x)$ and consider quadratic extensions $K = F(\sqrt{f})$ where $f \in \mathbf{F}_q[x]$ is monic of even degree. (So K is "real".) For each such f we have a quadratic Dirichlet character χ_f and an L-series

$$L(\chi_f, s) = \prod_{\mathfrak{p}} \left(1 - \chi_f(\mathfrak{p})(\mathbf{N}\mathfrak{p})^{-s}\right)^{-1}$$

In analogy with the classical situation, we could order these L-functions by the conductor of χ_f, i.e., by the degree of f. This makes sense but is hard to handle geometrically. So we change the problem a little.

Note that monic polynomials f of degree d are parameterized by an affine space:

$$f = x^d + a_1 x^{d-1} + \cdots + a_d \leftrightarrow (a_1, \ldots, a_d)$$
$$\{\text{monic, degree } d \text{ polys } f\} \leftrightarrow (\mathbf{F}_q)^d = \mathbf{A}^d(\mathbf{F}_q)$$

Let us *fix the degree* d *but allow coefficients in* $\mathbf{F}_{q^n}$ *for variable* n. We get infinitely many f (and so infinitely many $L(\chi_f, s)$) all parameterized by the points of a single variety $\mathbf{A}^d$.

In fact, we should pass to the open subvariety $X \subset \mathbf{A}^d$ corresponding to polynomials f with distinct roots ("square free" f).

Katz-Sarnak

Now we ask: what is the behavior of $L(\chi_f, s)$ as f varies through $X(\mathbf{F}_{q^n})$, $n \to \infty$? Katz and Sarnak give a definitive answer:

1. For each $f \in X(\mathbf{F}_{q^n})$ there is a symplectic matrix $\theta_f \in USp(2N)$ ($N = (d-2)/2$), well defined up to conjugation, such that
$$L(\chi_f, s) = \det\left(I - q^{n(1/2-s)}\theta_f\right)$$

2. The θ_f become equidistributed in the conjugacy classes of $USp(2N)$ as $n \to \infty$. More precisely, for any continuous conjugation-invariant function h on $USp(2N)$,
$$\lim_{n\to\infty} \frac{1}{\#X(\mathbf{F}_{q^n})} \sum_{f \in X(\mathbf{F}_{q^n})} h(\theta_f) = \int_{USp(2N)} h \, d\mu_{Haar}$$

Even more precisely, if we use Peter-Weyl to write

$$h = \sum_{\lambda} a_\lambda Tr(\rho_\lambda) \qquad a_\lambda \in \mathbf{C}$$

then we have the estimate

$$\left| \frac{1}{\#X(\mathbf{F}_{q^n})} \sum_{f \in X(\mathbf{F}_{q^n})} h(\theta_f) - \int_{USp(2N)} h \, d\mu_{Haar} \right| \leq$$

$$C q^{-n/2} \sum_{\lambda} |a_\lambda| \, \mathrm{dim}(\rho_\lambda)$$

where $C = C(d, q)$.

Note that we get a precise level at finite N, i.e., before passing to the limit as $N \to \infty$.

Cartoon sketch of proof

First we build a variety Y with a map $\pi : Y \to X$ such that the fiber $\pi^{-1}(f)$ over $f \in X(\mathbf{F}_{q^n})$ is the curve $\mathcal{C}_f$ corresponding to $K = \mathbf{F}_{q^n}F(\sqrt{f})$.

Then the general machinery of ℓ-adic cohomology gives us a sheaf $\mathcal{F}$ on X together with actions of the Frobenius elements Fr_f such that

$$L(\chi_f, s) = \det\left(1 - q^{-ns}Fr_f|\mathcal{F}\right)$$

(So $\mathcal{F}$ combines all the cohomologies $H^1(\mathcal{C}_f)$ for varying f.) The action of Fr_f on $\mathcal{F}$ gives us an ℓ-adic matrix related to $L(\chi_f, s)$.

One then uses $\overline{\mathbf{Q}}_\ell \tilde{\to} \mathbf{C}$, the Weyl unitarian trick, and the Peter-Weyl theorem to show that there is a unique conjugacy class of matrix $\theta_f \in USp(2N)$ whose characteristic polynomial gives $L(\chi_f, s)$.

Why $USp(2N)$ (as opposed to say $U(2N)$)? The sheaf $\mathcal{F}$ is related to $H^1(\mathcal{C}_f)$ and this group admits a skew-symmetric pairing

$$H^1(\mathcal{C}_f) \times H^1(\mathcal{C}_f) \to H^2(\mathcal{C}_f) = \mathbf{Q}_\ell$$

which is respected by Frobenius. This means that Frobenius acts by symplectic similitudes and there is a twist to make the action honestly symplectic. This proves the first part of the Katz-Sarnak statement.

For the second part, there are two ingredients, a monodromy calculation and Deligne equidistribution.

As part of his second proof of the Weil conjectures (“Weil II”), Deligne showed that the Frobenii are equidistributed in whatever group they fill out (more precisely, in the smallest algebraic group that contains them).

We saw above that there is a pairing that forces the Frobenii into the symplectic group, and Katz showed that in fact they are dense in the symplectic group.

Another family, orthogonal this time

Example 2: Katz and Sarnak considered many other families. Here is one more example, treated later by Katz, which has orthogonal symmetry.

Let $F = \mathbf{F}_q(t)$ for simplicity and fix an elliptic curve

$$E: \qquad y^2 = x^3 + ax + b$$

where $a, b \in F$ and $j(E) \notin \mathbf{F}_q$. We consider quadratic twists

$$E_g: \qquad gy^2 = x^3 + ax + b$$

In analogy with the classical set-up, we could let g run through $\mathbf{F}_q[t]$ and order the twists by the degree of g. Again this would be hard to treat geometrically.

Instead we let g run through polynomials of fixed (large) degree d, but with coefficients in $\mathbf{F}_{q^n}$. We also impose that the zeroes of g be distinct and prime to the conductor of E. The twists are then parameterized by a variety $X \subset \mathbf{A}^d$.

Each twist E_g has an L function $L(E_g, s)$ which is a polynomial in q^{-ns}. The BSD conjecture asserts that $\text{ord}_{s=1} L(E_g, s) = \text{Rank}\, E_g(\mathbf{F}_{q^n} F)$.

There are various cases (O(even), O(odd), SO(even) SO(odd)) depending on the input data. For simplicity, assume we fall into the O(even) case. Then the result is:

1. For each $g \in X(\mathbf{F}_{q^n})$ there is an orthogonal matrix $\theta_g \in O(2N)$, well defined up to conjugation, such that
$$L(E_g, s) = \det\left(I - q^{n(1-s)}\theta_g\right)$$

2. The θ_gs become equidistributed in the conjugacy classes of $O(2N)$ as $n \to \infty$.

Here N is about d, the degree of the twisting polynomials.

The reason we have orthogonal symmetry here is that the relevant cohomology group is an H^2 equipped with a symmetric pairing $H^2 \times H^2 \to H^4 \cong \mathbf{Q}_\ell$.

Two consequnces

Corollary (Katz): An analogue of Goldfeld's conjecture is true:

$$\lim_{n\to\infty} \frac{\#\{g \in X(\mathbf{F}_{q^n}) | \epsilon_g = +1 \text{ and } \mathrm{ord}_{s=1} L(E_g, s) \geq 2\}}{\#\{g \in X(\mathbf{F}_{q^n}) | \epsilon_g = +1\}} = 0$$

and

$$\lim_{n\to\infty} \frac{\#\{g \in X(\mathbf{F}_{q^n}) | \epsilon_g = -1 \text{ and } \mathrm{ord}_{s=1} L(E_g, s) \geq 3\}}{\#\{g \in X(\mathbf{F}_{q^n}) | \epsilon_g = -1\}} = 0$$

Prediction (CKRS): For sufficiently large d and sufficiently large n,

$$\#\{g \in X(\mathbf{F}_{q^n}) | \epsilon_g = +1 \text{ and } \mathrm{ord}_{s=1} L(E_g, s) \geq 2\} \geq Cq^{nd(\frac{3}{4}-\epsilon)}$$

A possibly relevant fact

Theorem: For any prime p, let E be the elliptic curve defined over $F = \mathbf{F}_p(t)$ by the Weierstrass equation

$$y^2 + xy = x^3 - t^d$$

where $d = p^n + 1$ and n is a positive integer. Then $j(E) \notin \mathbf{F}_p$, the conjecture of Birch and Swinnerton-Dyer holds for E over F, and the rank of $E(F)$ is at least $(p^n - 1)/2n$.

(In fact the exact rank is known.) These curves have conductor N of degree about p^n and so their L-function vanishes at the critical point to order about

$$C \frac{\deg N}{\log \deg N}$$

This suggests the following

Conjecture: For a positive integer N, let $r(N)$ be the maximum, over all elliptic curves E over $\mathbf{Q}$ with conductor N, of the rank of $E(\mathbf{Q})$; if there are no such curves, we set $r(N) = 0$. Then

$$\limsup_N \frac{r(N)}{\log N / \log \log N} > 0$$

Another possibly relevant fact

Chowla's conjecture asserts that $L(\chi, 1/2) \neq 0$ for all quadratic Dirichlet characters χ.

A function field analogue would be $L(\chi, 1/2) \neq 0$ where χ is a quadratic Dirichlet character of $\mathbf{F}_q(t)$, or equivalently, $\zeta(\mathcal{C}, 1/2) \neq 0$ for all hyperelliptic curves over $\mathbf{F}_q$.

This analogue is true for $\mathcal{C}$ of genus 0 (trivial) and genus 1 (integrality of numerator of ζ). By the Weil bound, it is also ok for $\mathcal{C}$ of genus 2 and $p = 2$ or 3.

It fails in many other cases, perhaps (?) all other cases! For example, if $\mathcal{C}$ is defined by the equation

$$y^2 = x^q - x$$

over $\mathbf{F}_q$ and $q \equiv 1 \mod 4$, then $\#\mathcal{C}(\mathbf{F}_q) = \#\mathcal{C}(\mathbf{F}_{q^2}) = q + 1$ and so

$$\zeta(\mathcal{C}, s) = \frac{(1 - q^{1-2s})^{(q-1)/2}}{(1 - q^{-s})(1 - q^{1-s})}$$

and this vanishes to order $(q-1)/2$ at $s = 1/2$.

There are other examples in genus 2 for small p.