Symmetry beyond Root Numbers:
A GL(6) Example

Eduardo Dueñez

(Joint with Steven J. Miller, Ohio State)

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Families of L-functions

- $\mathcal{F}$: (finite) family of L-functions satisfying GRH.

- Critical zeros of $f \in \mathcal{F}$: $\{\frac{1}{2} + i\gamma_f^{(j)}\}_{j \in \boldsymbol{Z}}$.
  (In order)

- Assume existence of constants c_f (analytic conductors), $f \in \mathcal{F}$, giving correct "unfolding" of low-lying zeros (near central point $s = \frac{1}{2}$):

$$\tilde{\gamma}_f^{(j)} = \frac{\log c_f}{2\pi} \gamma_f^{(j)}.$$

  – The low-lying $\tilde{\gamma}_f^{(j)}$ have unit mean-spacing.

Central 1-level Density of a Family $\mathcal{F}$

- $D_{1,\mathcal{F}}$ is a positive measure on $\boldsymbol{R}$:

$$D_{1,\mathcal{F}} = \left\langle \sum_j \delta_{\tilde{\gamma}_f^{(j)}} \right\rangle_{f\in\mathcal{F}} = \frac{1}{|\mathcal{F}|} \sum_{f\in\mathcal{F}} \sum_j \delta_{\tilde{\gamma}_f^{(j)}}$$

- For a test function g:

$$D_{1,\mathcal{F}}(g) = \frac{1}{|\mathcal{F}|} \sum_{f\in\mathcal{F}} \sum_j g(\tilde{\gamma}_f^{(j)}).$$

- For a reasonable "family of families" $\{\mathcal{F}_n\}$ with $|\mathcal{F}_n| \to \infty$ the following limit exists:

$$\lim_{n\to\infty} D_{1,\mathcal{F}_n} = D_{1,G(\mathcal{F})},$$

and depends only on $G(\mathcal{F})$, the (hypothetical) underlying symmetry group: O(N), SO(even), SO(odd) or symplectic.

Folklore Conjecture

SO(even/odd) families come from splitting a full orthogonal family according to root number $+/-1$ (sign of the functional equation).

- In particular, a "lone-standing" family with root number $+1$ should be symplectic.

Counterexample to Folklore Conjecture

- ϕ: **fixed** even Hecke-Maass cusp form.

- Symmetric-square family $\{L(s, \text{Sym}^2 f)\}$, f holomorphic modular Hecke eigenform of level 1, level k going to infinity.

- Twist symmetric-square family by ϕ (Rankin-Selberg convolution) to obtain

$$\mathcal{L}_k = \{L(s, \phi \times \text{Sym}^2 f) : f \in H_k\}.$$

Theorem (D-S. J. Miller). The central 1- and 2-level densities for this family agree with SO(even).

Symplectic and SO(even) Central 1-level Densities

- Sine Kernels:

$$S(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)}$$
$$S_{\pm}(x,y) = S(x,y) \pm S(x,-y)$$

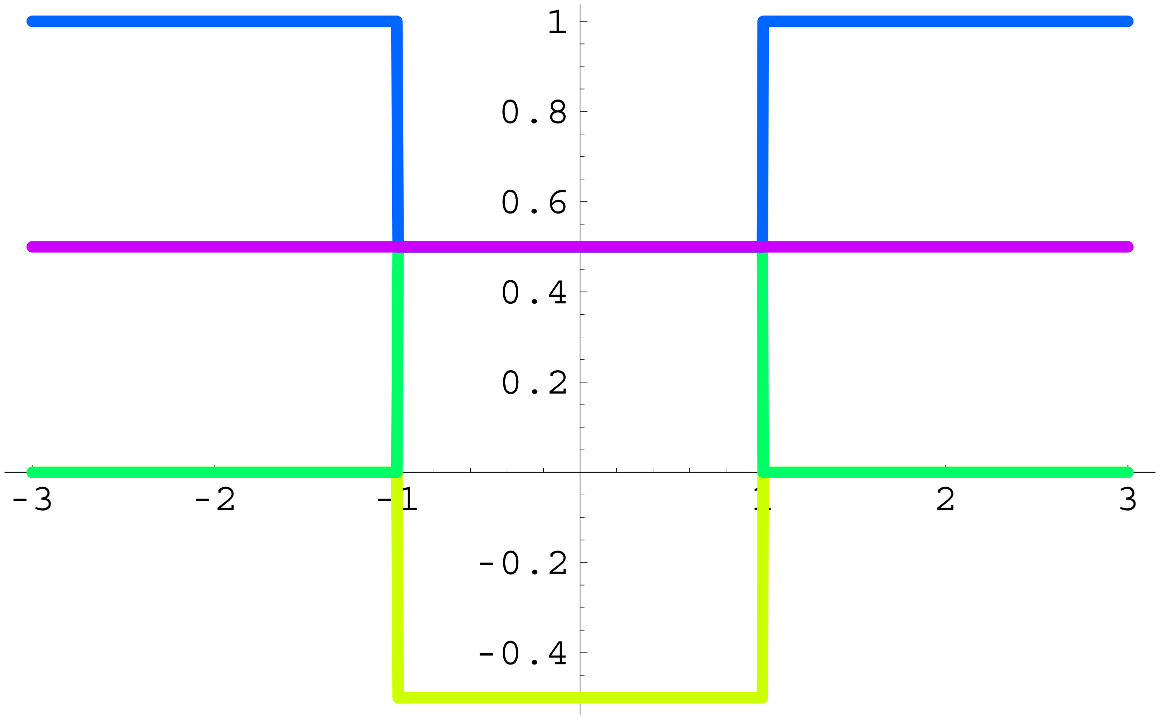
- 1-level densities: $dD_{1,G}(x) = W_{1,G}(x)dx$.

$$W_{1,SO(\text{even})}(x) = S_{+}(x) = 1 + \frac{\sin 2\pi x}{2\pi x}$$
$$W_{1,USp}(x) = S_{-}(x) = 1 - \frac{\sin 2\pi x}{2\pi x}$$

- Fourier transforms:

$$\widehat{W}_{1,SO(\text{even})}(\xi) = \delta(\xi) + \frac{1}{2}\chi_{(-1,1)}(\xi),$$
$$\widehat{W}_{1,USp}(\xi) = \delta(\xi) - \frac{1}{2}\chi_{(-1,1)}(\xi).$$

Fourier Transform of Central 1-level Density



Fourier tranform of the central 1-level density for USp (yellow), SO(even) (green), SO(odd) (blue) and O (magenta).

$$\mathcal{F} = \{L(s, \phi \times \mathrm{Sym}^2 f)\}$$

This family is intimately related by work of Luo-Sarnak to the quantum fluctuations of observables on the modular surface.

- Automorphic representation associated (by Kim-Shahidi) to $L(s, \phi \times \mathrm{Sym}^2 f)$ has local parameters $\alpha_p^{\pm 2}\beta_p^{\pm 1}$ and $\beta_p^{\pm 1}$ for p prime. Call them $\delta_p(j)$, $1 \le j \le 6$.
  - $\alpha_p^{\pm 1}$: Satake parameters of f. By Deligne $|\alpha_p| = 1$.
  - $\beta_p^{\pm 1}$: Satake parameters of ϕ. By Kim-Sarnak $|\beta_p| \le p^{7/64}$.

- Local parameters at archimedean place are $k-1 \pm it_\phi$, $k \pm it_\phi$ and $1 \pm it_\phi$ if f of weight k and $\triangle\phi + (1/4 + t_\phi^2)\phi = 0$.

- Completed L-function is entire and has **even** functional equation as $s \mapsto 1 - s$.

- Assume RH for $L(s, \phi \times \mathrm{Sym}^2 f)$.
  For $R > 0$ unfold the critical zeros: $\tilde{\gamma}_j = \frac{\log R}{2\pi}\gamma_j$.

Explicit Formula (Smooth Version)

Let g be an even Schwartz function on $\boldsymbol{R}$. Then

$$\sum_j g\left(\frac{\gamma_j}{2\pi}\log R\right) = \frac{A}{\log R} - 2\sum_p \sum_{\nu=1}^{\infty} \widehat{g}\left(\frac{\nu \log p}{\log R}\right) \frac{a_{\phi,\mathrm{Sym}^2 f}(p^\nu)\log p}{p^{\nu/2}\log R},$$

where

$$a_{\phi,\mathrm{Sym}^2 f}(p^\nu) = \sum_{j=1}^{6} \delta_p(j)^\nu,$$

are the coefficients of the logarithmic derivative $L'/L(s, \phi \times \mathrm{Sym}^2 f)$ and $A = \widehat{g}(0)(\log k^4 + O(1))$ arises from an integral involving the Gamma factors.

- Choose conductor $R = k^4$: replace $A/\log R$ term by $\widehat{g}(0)$ at price of error $O(1/\log R)$.

Coefficients in Terms of Local Parameters

$$\begin{aligned}
a_{\phi,\mathrm{Sym}^2 f}(p) =& (\alpha_p^2 + 1 + \alpha_p^{-2})(\beta_p + \beta_p^{-1}) \\
=& \lambda_f(p^2)\lambda_\phi(p); \\
a_{\phi,\mathrm{Sym}^2 f}(p^2) =& (\alpha_p^4 + 1 + \alpha_p^{-4})(\beta_p^2 + \beta_p^{-2}) \\
=& (\lambda_f(p^4) - \lambda_f(p^2) + 1)\times \\
& \times (\lambda_\phi(p^2) - 1).
\end{aligned}$$

- The last '-1' above is the culprit for symmetry change from symplectic to SO(even)!

1-Level Density

$$\frac{1}{|H_k|}\sum_{f\in H_k}\frac{\zeta(2)}{L(1,\mathrm{Sym}^2 f)}\sum_j g\left(\gamma_f^{(j)}\frac{\log c_{f_\phi}}{2\pi}\right)$$
$$= \widehat{g}(0) - \frac{2}{|H_k|}\sum_{f\in H_k}\frac{\zeta(2)}{L(1,\mathrm{Sym}^2 f)}\times$$
$$\times\sum_{\nu=1}^{\infty}\sum_{p=2}^{R^\sigma}\frac{a_{\phi,\mathrm{Sym}^2 f}(p^\nu)\log p}{p^{\nu/2}\log R}\widehat{g}\left(\nu\frac{\log p}{\log R}\right)+O\left(\frac{1}{\log R}\right).$$

- Sum over $p \le R^\sigma$: Support($\widehat{g}$) $\subset [-\sigma,\sigma]$.

- Note weights $w_f := \zeta(2)/L(1,\mathrm{Sym}^2 f)$.
  (Easily removed since ≈ 1.)

- Using K-S (even K-Sh), terms with $\nu \ge 3$ contribute $O(1/\log R)$.

Orthogonality Relation for the Fourier Coefficients

For m, n coprime with a bounded number of factors (I-L-S, from Petersson formula):

$$\Delta_k(m,n) := \sum_{f \in H_k} \psi_f(m)\psi_f(n) \qquad \left(\psi_f(n) = \sqrt{\frac{\Gamma(k-1)}{(4\pi)^{k-1}\|f\|^2}}\lambda_f(n)\right)$$

$$= \delta(m,n) + \begin{cases} O\left(\frac{(mn)^{1/4}\log mn}{k^{5/6}}\right); \\ O\left(\frac{\sqrt{mn}}{2^k}\right) \qquad\qquad (12\pi\sqrt{mn} \le k). \end{cases}$$

Contribution from $\nu = 1$.

$$T_1 = -2 \left\langle \sum_{p=2}^{R^\sigma} \frac{a_{\phi,\mathrm{Sym}^2 f}(p) \log p}{\sqrt{p} \log R} \widehat{g}\left(\frac{\log p}{\log R}\right) \right\rangle_{f \in H_k},$$

but $a_{\phi,\mathrm{Sym}^2 f}(p) = \lambda_\phi(p)\lambda_f(p)$ and $\lambda_f(1) = 1$, so

$$= -2 \sum_{p=2}^{R^\sigma} \frac{\lambda_\phi(p) \log p}{\sqrt{p} \log R} \widehat{g}\left(\frac{\log p}{\log R}\right) \left\langle \lambda_f(1)\lambda_f(p^2) \right\rangle.$$

The last average is essentially $\Delta_k(1, p^2)$ (non-diagonal term in Petersson formula).

Contribution from $\nu = 2$

$$
\begin{aligned}
T_2 &= -2 \left\langle \sum_{p=2}^{R^\sigma} \frac{a_{\phi,\mathrm{Sym}^2 f}(p^2)\log p}{p \log R} \widehat{g}\left(2\frac{\log p}{\log R}\right) \right\rangle \\
&= -2 \sum_{p=2}^{R^\sigma} \frac{\lambda_\phi(p^2)\log p}{p \log R} \widehat{g}\left(2\frac{\log p}{\log R}\right) \times \\
&\times \left(\lambda_\phi(p^2) - 1\right) \left\langle \lambda_f(p)^4 - \lambda_f(p^2) + 1 \right\rangle .
\end{aligned}
$$

- Diagonal contributions are:

$$\lambda_\phi(p^2) \cdot \langle +1 \rangle \approx \lambda_\phi(p^2)$$

which contributes $O(1/\log R)$ by RH for $\mathrm{Sym}^2\phi$; and

$$(-1)\langle +1 \rangle \approx -1$$

(true main term: by PNT the sum over primes $\asymp -\frac{1}{4}g(0)$.)

1-level Density: conclusion

- For family $\{L(s, \phi \times \mathrm{Sym}^2 f)\}$:

$$\lim_{k\to\infty} \left\langle D_{1,\mathcal{F}_k}(g)\right\rangle = \widehat{g}\,(0) + \frac{1}{2}g(0) = \int \left(\delta(\xi) + \frac{1}{2}\right)\widehat{g}\,(\xi)\,d\xi = \int W_{1,\mathrm{SO(even)}}(x)g(x)dx.$$

- Evidence of SO(even) symmetry.

- Using procedure above get support up to $\sigma = 1/8$.

- Slight refinement using (unconditional) Ramanujan on average gives $\sigma = 5/24$.

- Small support means we cannot *sensu stricti* exclude SO(odd) or orthogonal.

2-Level Density

g_1, g_2 even Schwartz functions (eventually with $\widehat{g}_1, \widehat{g}_2$ compactly supported), $g(x,y) = g_1(x)g_2(y)$.

$$D_{2,\mathcal{F}}(g) = \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \sum_{j_1 \neq \pm j_2} g\left(\frac{\log R}{2\pi}\gamma_f^{(j_1)}, \frac{\log R}{2\pi}\gamma_f^{(j_2)}\right).$$

The SO(even) 2-level density is

$$D_{2,SO(\text{even})}(g) = \left[\widehat{g_1}(0) + \frac{1}{2}g_1(0)\right] \cdot \left[\widehat{g_2}(0) + \frac{1}{2}g_2(0)\right] - 2\int |u|\widehat{g_1}\widehat{g_2}(u)du - 2\widehat{g_1 g_2}(0) - g_1(0)g_2(0).$$

- 2-level density is powerful enough to distinguish all symmetry types for arbitrarily small support.

2-Level Density (cont'd)

For the family $\{L(s, \phi \times \mathrm{Sym}^2 f)\}$,

$$D_{2,\mathcal{F}}(g) = \left\langle \sum_{j_1,j_2} g_1\left(\frac{\log R}{2\pi}\gamma_f^{(j_1)}\right) g_2\left(\frac{\log R}{2\pi}\gamma_f^{(j_2)}\right)\right\rangle_f - 2D_{1,\mathcal{F}}(g_1 g_2).$$

- Explicit formula: rewrite sums over zeros.
- Terms with exponent $\nu \geq 3$ do not contribute.

2-Level Density (cont'd)

To account for the $\nu = 2$ contribution at the outset, set

$$b_{\phi,\mathrm{Sym}^2 f}(p) = a_{\phi,\mathrm{Sym}^2 f}(p)$$
$$b_{\phi,\mathrm{Sym}^2 f}(p^2) = a_{\phi,\mathrm{Sym}^2 f}(p^2) + 1.$$

Then we need to consider the averages of

$$\prod_{i=1}^{2} \left[\left(\widehat{g}_i(0) + \frac{1}{2} g_i(0) \right) - 2 \sum_{\nu_i=1}^{2} \sum_{p_i} \frac{b_{\phi,\mathrm{Sym}^2 f}(p^{\nu_i}) \log p}{p^{\nu_i/2} \log R} \widehat{g}_i \left(\nu_i \frac{\log p_i}{\log R} \right) \right].$$

In fact, all reduces to understanding the averages of products of the double sums above for each pair of values (ν_1, ν_2); $\nu_1, \nu_2 = 1, 2$.

- $(1,1)$-term contributes

$$2\int |u|\widehat{g}_1\widehat{g}_2(u)du + O(1/\log R)$$

by RH for $\text{Sym}^2\phi$ and Petersson formula.

- $(1,2)$- and $(2,1)$-terms contribute $O(1/\log R)$ (any non-trivial bound towards Ramanujan suffices).

- $(2,2)$-term contributes $O(1/\log R)$ (by bounds towards Ramanujan for $L(s,\phi)$, e. g., B-D-H-I.)

- Get agreement with 2-level density of SO(even) for small support.

Basic References

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