THE SELBERG CLASS OF L-FUNCTIONS: NON-LINEAR TWISTS

Selberg (1950's; 1989) $\longrightarrow$ **Selberg class** $\mathcal{S}$:

(i) (ordinary Dirichlet series) $F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ absolutely convergent for $\sigma > 1$

(ii) (analytic continuation) $(s-1)^m F(s)$ entire function of finite order for some integer $m \geq 0$

(iii) (functional equation) $\Phi(s) = \omega \overline{\Phi}(1-s)$ where $\overline{f}(s) := \overline{f(\overline{s})}$,

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s)$$

$r \geq 0$, $Q > 0$, $\lambda_j > 0$, $\Re \mu_j \geq 0$, $|\omega| = 1$

(iv) (Ramanujan conj.) $a(n) \ll n^{\varepsilon} \quad \forall \varepsilon > 0$

(v) (Euler product) $\log F(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$ with $b(n) = 0$ unless $n = p^m$ with $m \geq 1$, and $b(n) \ll n^{\vartheta}$ for some $\vartheta < \frac{1}{2}$

extended Selberg class $\mathcal{S}^{\sharp}$: axioms *(i)* - *(iii)*

Examples: $L(s,\chi)$, $L_K(s,\chi)$, suitably normalized $L_f(s)$, Artin L-functions (mod. Artin conj.), automorphic L-functions (mod. Ramanujan conj.)

Axioms $\rightarrow$ standard analytic properties: Euler product $F(s) = \prod_p F_p(s)$ non-vanishing for $\sigma > 1$; critical strip and line, trivial and non-trivial zeros; $N_F(T) \sim c_F T \log T$; polynomial growth on vertical strips, Lindelöf μ-function,...

– **Structure of $\mathcal{S}$**

What is in $\mathcal{S}$? **Main Conjecture**:

$$\mathcal{S} = \text{class of automorphic } L\text{-functions}$$

(if true, very deep: **Langlands** ...)

degree of $F(s)$: $d_F = 2\sum_{j=1}^r \lambda_j$ (invariant)

($d_\zeta = 1$, $d_{L(\cdot,\chi)} = 1$, $d_{L_f} = 2$, $d_{\zeta_K} = [K:Q]$, ...)

$$\mathcal{S}_d = \{F \in \mathcal{S} : d_F = d\}$$

Conj. 1. (*general converse theorem*): for $d \in \mathbb{N}$

$$\mathcal{S}_d = \{\text{automorphic } L\text{-functions of degree } d\}$$

Conj. 2. (*degree conjecture*): for $d \notin \mathbb{N}$

$$\mathcal{S}_d = \emptyset$$

Remark. Conj.2 expected for $\mathcal{S}^\sharp$ as well, but **false** if

ordinary D-series $\rightarrow$ *general* D-series:

$D(\lambda, \mu, Q, \omega)$ = vector space of general D-series satisfying *(i)* - *(iii)*; using **Hecke**'s theory

Th. (K-P) $D(\lambda, \mu, Q, \omega)$ *has uncountable basis*

State of art: conjectures 1 and 2 true for

$$0 \leq d < 5/3$$

Precisely:

$d = 0$: $\mathcal{S}_0 = \{1\}$ (Conrey-Ghosh 1993)

$0 < d < 1$: $\mathcal{S}_d = \emptyset$ (Richert 1957, Bochner 1958, C-G 1993, Molteni 1999, K-P 2002,200?)

$d = 1$: $\mathcal{S}_1 = \{L(s + i\theta, \chi)\}$ (K-P 1999)

$1 < d < 5/3$: $\mathcal{S}_d = \emptyset$ (K-P 2002)

– Linear twists

Main tool for $d \geq 1$: **linear twists** ($e(x) = e^{2\pi i x}$)

$$F(s, \alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-n\alpha)$$

To study **analytic properties**: for $N, \alpha > 0$ by Mellin + functional equation

$$F_N(s, \alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-n\alpha) e^{-n/N}$$

$$= R_N(s, \alpha) + \omega Q^{1-2s} \sum_{n=1}^{\infty} \frac{\overline{a(n)}}{n^{1-s}} H_K\left(\frac{n}{Q^2(\frac{1}{N} + 2\pi i \alpha)}, s\right)$$

where

$$H_K(z, s) = \frac{1}{2\pi i} \int_{(-K-\frac{1}{2})} \prod_{j=1}^{r} \frac{\Gamma(\lambda_j(1-s) + \overline{\mu}_j - \lambda_j w)}{\Gamma(\lambda_j s + \mu_j + \lambda_j w)}$$

$$\times \Gamma(w) z^w dw,$$

the **hypergeometric functions**. For s **fixed** studied by **Braaksma** (1963); behaviour depends on value of

$$\mu = 2\sum_{j=1}^{r} \lambda_j - 1 = d_F - 1$$

$\mu = 0 \quad (d_F = 1)$ simpler case; $\mu > 0 \quad (d_F > 1)$ more complicated by "exponential part".

Development of **2-variables** theory; since want $N \to \infty$, main interest for $H_K(-iy, s)$, $y = \frac{n}{2\pi Q^2 \alpha}$.

Let **conductor** q_F and **shift** θ_F (invariants) be

$$q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j}, \quad \theta_F = \Im(2\sum_{j=1}^{r}(\mu_j - \tfrac{1}{2}));$$

critical value: $n_\alpha = q_F d_F^{-d_F} \alpha^{d_F}$; $a(n_\alpha) = 0$ if $n_\alpha \notin \mathbb{N}$

Case $d_F = 1$:

Th.1. (K-P) *Let $F \in \mathcal{S}_1^\sharp$ and $\alpha > 0$. Then $F(s,\alpha)$ is entire if $a(n_\alpha) = 0$, while if $a(n_\alpha) \neq 0$ then $F(s,\alpha)$ has at most simple poles at $s_k = 1 - k - i\theta_F \quad (k = 0, 1, ...)$, with non-vanishing residue at $s = s_0$*

Case $1 < d_F < 2$: let

$$\kappa = \frac{1}{d-1} \quad A = (d-1)q_F^{-\kappa} \quad s^* = \kappa(s+\frac{d}{2}-1+i\theta_F)$$

Th.2. (K-P) *Let* $1 < d < 2$, $F \in \mathcal{S}_d^\sharp$ *and* $\alpha > 0$. *Then*

$$F(s,\alpha) = e^{as+b} \sum_{n=1}^{\infty} \frac{\overline{a(n)}}{n^{s^*}} e(A(\frac{n}{\alpha})^\kappa) + G(s,\alpha)$$

where $G(s,\alpha)$ *is holomorphic for* $\sigma^* > \sigma_a(F) - \kappa$

Remark: $\sigma^* > \sigma$ for $\sigma > \frac{1}{2}$ and $1 < d < 2$, hence "*overconvergence*": suspicious !

– **Non-linear twists**

For $F \in \mathcal{S}_d^\sharp$ with $d > 0$, **non-linear twist** $(\alpha > 0)$

$$F(s,\alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-n^{1/d}\alpha)$$

Theorem 1 is special case of a general result for non-linear twists:

Th.3. (K-P) *Let* $d > 0$, $F \in \mathcal{S}_d^\sharp$ *and* $\alpha > 0$. *Then* $F(s, \alpha)$ *is entire if* $a(n_\alpha) = 0$, *while if* $a(n_\alpha) \neq 0$ *then* $F(s, \alpha)$ *has at most simple poles at* $s_k = \frac{d+1}{2d} - \frac{k}{d} - i\frac{\theta_F}{d}$ $(k = 0, 1, ...)$, *with non-vanishing residue at* $s = s_0$

Bounds on vertical strips, **uniform** for $F(s)$ in suitable families $\mathcal{F}$ (roughly, bounded degree and μ-coefficients) can also be obtained.

Applications of Theorem 3

i) **non-linear exponential sums**: $\phi(u)$ smooth with compact support, $F \in \mathcal{S}_d^\sharp$

$$S_F(x, \alpha) = \sum_{n=1}^{\infty} a(n) e(-n^{1/d}\alpha)\phi(\frac{n}{x}).$$

Then asymptotic expansion, uniform for $F \in \mathcal{F}$, of type

$$S_F(x, \alpha) = \sum_k c_k(F, \alpha) x^{s_k} + O(x^{-A})$$

(extending results by **Iwaniec-Luo-Sarnak** in degree 2, different method)

ii) Ω-**results**: $F \in \mathcal{S}_d^\sharp$ with $d \geq 1$

$$A_F(x) = \sum_{n \leq x} a(n).$$

Then

$$A_F(x) = \text{res}_{s=1} F(s)\frac{x^s}{s} + \Omega(x^{\frac{d-1}{2d}})$$

Remark. Exponent caused by pole of $F(s,\alpha)$ (with suitable α) at $s = s_0$; possibly same result obtainable by Voronoi-type arguments

iii) $S_d^\sharp = \emptyset$ **for** $0 < d < 1$: pole of $F(s,\alpha)$ at $s = s_0$ has real part > 1 if $0 < d < 1$, contradiction

iv) **charaterization of** $\zeta(s)$: let $F \in \mathcal{S}_d$ with $d \geq 1$. If

$$\sum_{n=1}^{\infty} \frac{a(n) - 1}{n^s}$$

converges for $\sigma > \frac{1}{5} - \delta$, then $F(s) = \zeta(s)$

Remark. Similarly for $L(s,\chi)$; uses $S_d = \emptyset$ for $1 < d < 5/3$