Workshop: Matrix Ensembles and L-functions

12-16 July, 2004

Bounds at s=1 for an axiomatic class of L-functions

G. Molteni

I wish to thank the organizers for the opportunity they give me to be here and tell you my **little** contribute to the theory of L-functions.

The class: axioms

We define the general L-function $L(s,\mathcal{A})$ by five axioms.

1. (**Polynomial Euler product**)

   $$L(s,\mathcal{A}) = \prod_p \prod_{j=1}^{\mathbf{d}} (1-\alpha_j(p)p^{-s})^{-1} = \sum_{n=1}^{\infty} a(n)n^{-s} \quad \text{ab. conv. for } \sigma = \Re s > 1.$$

2. (**Continuation**)
   $\exists m = m(\mathcal{A}) \in \mathbb{N}$ such that $(s-1)^m L(s,\mathcal{A})$ is entire of order 1.

3. (**Functional equation**)
   $\exists L(s,\mathcal{A}^*)$ with 1.-2., and there exist $Q_{\mathcal{A}} > 0$, $\lambda_1,\ldots,\lambda_r > 0$, $\mu_1,\ldots,\mu_r \in \mathbb{C}$ with $\Re\mu_j \geq \kappa$ where $\kappa \in \mathbb{R}$ and $|\omega| = 1$ such that

   $$\Phi(s,\mathcal{A}) := Q_{\mathcal{A}}^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) L(s,\mathcal{A})$$
   $$\Phi(s,\mathcal{A}^*) := Q_{\mathcal{A}}^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \bar{\mu}_j) L(s,\mathcal{A}^*),$$
   $$\Phi(1-s,\mathcal{A}) = \omega \Phi(s,\mathcal{A}^*).$$

4. (**Growth of** $\alpha_j(p)$)

   individual: $\alpha_j(p) \ll p^{1-\rho}$ for some $\rho > 0$, $\forall j, p$,
   collective: let $s_j(p) := \sum_{1\leq i_1 < i_2 < \cdots < i_j \leq \mathbf{d}} \alpha_{i_1}(p)\cdots\alpha_{i_j}(p)$ jth symm. pol.,

   $$|s_j(p)|^{1/j} \ll p^{1/2} \quad \text{for } j = 2,\ldots,\mathbf{d}.$$

5. (**Convolution**)

   let $L(s,\widetilde{\mathcal{A}\otimes\bar{\mathcal{A}}}) := \prod_p \prod_{i,j=1}^{\mathbf{d}} (1-\alpha_i(p)\bar{\alpha}_j(p)p^{-s})^{-1}$,

   there exists $P(s,\mathcal{A}\otimes\bar{\mathcal{A}})$, a quotient of two **finite** Euler products, holomorphic for $\sigma > 1-\rho$, $\rho > 0$ (ramified primes), such that

   $L(s,\mathcal{A}\otimes\bar{\mathcal{A}}) := P(s,\mathcal{A}\otimes\bar{\mathcal{A}}) L(s,\widetilde{\mathcal{A}\otimes\bar{\mathcal{A}}})$ satisfies 1., 2., 3. .

Remarks.

- *Axiom 1 holds for "concrete" L-functions with "algebraic origin".*
- *In Axiom 3 the uniform lower bound κ for parameters μ_j can be negative.*
- *Axiom 4 holds when $\alpha_j(p) \ll p^{1/2}$, hence it is weaker than the usual Ramanujan-Petersson condition $|\alpha_j(p)| \leq 1$.*
- *Axiom 4 does not assume anything about $s_1(p)$, hence is weaker than $\alpha_j(p) \ll p^{1/2}$.*
- *Axiom 4 is not required for $L(s, \mathcal{A} \otimes \bar{\mathcal{A}})$.*
- *The family $L(s, \pi)$ attached to cuspidal automorphic representations of* $\mathrm{GL}(\mathbf{d})$ *satisfies 1.-5.* **unconditionally**.

Definition. Main parameter: $\mathcal{R}_{\mathcal{A}} := (1 + Q_{\mathcal{A}}) \prod_{j=1}^{r} (1 + |\mu_j|)$.

Remarks.

- $\mathcal{R}_{\mathcal{A}} \asymp q$ *for Dirichlet $L(s, \chi)$ with χ a primitive character modulo q,*
- $\mathcal{R}_{\mathcal{A}} \asymp kN$ *for holomorphic cusp forms with weight k, level N,*
- $\mathcal{R}_{\mathcal{A}} \asymp \lambda N$ *for Maass cusp forms with eigenvalue λ, level N,*
- $\mathcal{R}_{\mathcal{A}}$ *captures, in a rough form, the algebraic information about $L(s, \mathcal{A})$ and for cuspidal automorphic* $\mathrm{GL}(\mathbf{d})$ *L-functions is of the order of the* analytic conductor *of Iwaniec-Sarnak.*

Problem. *getting upper bounds for* $f(s) := (s-1)^{m_{\mathcal{A}}} L(s,\mathcal{A})$ *and for its derivatives of type*

$$f^{(j)}(s) \ll_\epsilon \mathcal{R}_{\mathcal{A}}^{\epsilon} \quad \text{when} \quad |s-1| \ll 1/\log \mathcal{R}_{\mathcal{A}} \ .$$

Motivations.

- *Upper bounds of such type (and even better) can be easily proved assuming* $|\alpha_j(p)| \le 1$*, but in our class this estimate is not required.*
- *All the known (to me) technics proving* **lower bounds** *of Siegel type, i.e.,*

  $$L(s,\mathcal{A}) \gg_\epsilon \mathcal{R}_{\mathcal{A}}^{-\epsilon} \ ,$$

  need such upper bounds.

The results

Theorem 1. *For $L(s,\mathcal{A})$ belonging to the class,*

$$\sum_{n\le x}\frac{|a(n)|}{n} \ll_\epsilon (\mathcal{R}x)^\epsilon \ , \quad \forall \epsilon>0 \ .$$

By standard technics it follows that

Theorem 2. *Define $f(s) := (s-1)^{m_\mathcal{A}}L(s,\mathcal{A})$, then*

$$f^{(j)}(s) \ll_{j,\epsilon} \mathcal{R}^\epsilon \quad \textit{for} \quad |s-1| \ll 1/\log\mathcal{R} \ .$$

Actually, more can be proved

Theorem 3. *Define $f(s) = (s-1)^{m_\mathcal{A}}L(s,\mathcal{A})$ and let $\theta\in\mathbb{R}$, then*

$$f^{(j)}(s) \ll_{j,\epsilon} \mathcal{R}^\epsilon(1+|\theta|)^\epsilon \quad \textit{for} \quad |s-1-i\theta| \ll 1/\log(\mathcal{R}(1+|\theta|)) \ .$$

In particular

Corollary. *Let π be an automorphic cuspidal representation of $\mathrm{GL}(\mathbf{d})$ and $L(s,\pi)$ be the associate L-function. Let $f(s) := (s-1)^m L(s,\pi)$, where m is the order of pole of $L(s,\pi)$ at $s=1$, then*

$$|f^{(j)}(s)| \le c\ \mathcal{R}^\epsilon(1+|\theta|)^\epsilon \quad \textit{for} \quad |s-1-i\theta| \ll 1/\log(\mathcal{R}(1+|\theta|)) \ ,$$

for a suitable $c = c(\epsilon, j, m, \mathbf{d}) > 0$, for any $\theta\in\mathbb{R}$.

Remark.

Modifying Axiom 5 in such a way that $L(s,\mathcal{A}\otimes\mathcal{B})$ be included and strengthening Axiom 4 to

$$|s_j(p)|^{1/j} \ll p^{1/4}, \quad \textit{for} \quad j=2,\ldots,\mathbf{d} \ ,$$

similar upper bounds can be proved for $L(1,\mathcal{A}\otimes\mathcal{B})$.

How to prove Th. 1

The sequence $a(n)$ coming from an Euler product of **polynomial** type satisfies good algebraic properties which **replace** for our purposes the **Ramanujan-type upper bound**. This fact has been shown firstly by Iwaniec ('90) to get an analogous estimate for Maass forms ($\mathbf{d} = 2$) and then by Hoffstein and Lockhart ('94) for the symmetric square L-function of a Maass form ($\mathbf{d} = 3$). Our proof is based on the following proposition and generalise their technic to every degree $\mathbf{d}$.

Proposition. *Let* $r(l)$ *be the* **completely multiplicative** *arithmetic function with*

$$r(p) := \sqrt{2} \sum_{j=2}^{\mathbf{d}} |s_j(p)|^{1/j} \ ,$$

then

$$|a(m)\,a(n)| \leq \sum_{\substack{l|(m,n)^{\mathbf{d}} \\ l|mn}}^{*} r(l)\,|a(\frac{mn}{l})| \ ,$$

where $\sum^*$ *means that the sum is for only squarefull divisors* l*, i.e., either* $l = 1$ *or* $p|l$ *implies* $p^2|l$ *for every prime* p*.*

The proof can be done by elementary tools but is quite intricate.

Note that for $\mathbf{d} = 1$ or 2 the exact identity is simple:

$(\mathbf{d} = 1)$ $\quad a(m)\,a(n) = a(mn),$

$(\mathbf{d} = 2)$ $\quad a(m)\,a(n) = \sum_{\substack{l|(m,n)^2 \\ l|mn}} R_2(l) a(\frac{mn}{l}), \quad R_2(p^u) = \begin{cases} s_2(p)^{u/2} & \text{even } u \\ 0 & \text{odd } u, \end{cases}$

but already for $\mathbf{d} = 3$

$$a(m)\,a(n) = \sum_{\substack{l|(m,n)^3 \\ l|mn}} R_3(l)\ a(\frac{mn}{l})$$

where $R_3(l)$ is a complicated multiplicative function which is zero for non-squarefull l.

Proof (sketch).

Let $L(s,\mathcal{A})=\sum_{n=1}a(n)n^{-s}$, $\quad L(s,\mathcal{A}\otimes\bar{\mathcal{A}})=\sum_{n=1}A(n)n^{-s}$,

$$S(x):=\sum_{n\le x}\frac{|a(n)|}{n}\ , \qquad \tilde{S}(x):=\sum_{n\le x}\frac{A(n)}{n}\ .$$

From Axiom 5 a uniform bound of type $\ll(\mathcal{R}(1+|t|))^c$ for a convenient $c>0$ holds in vertical strips for $L(s,\mathcal{A}\otimes\bar{\mathcal{A}})$, so that from the integral representation

$$\sum_{n\le x}\frac{A(n)}{n}\le\sum_{n\le 2x}\frac{A(n)}{n}\Big(2-\frac{n}{x}\Big)^r=\frac{2^r r!}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{L(s+1,\mathcal{A}\otimes\bar{\mathcal{A}})(2x)^s}{s(s+1)\dots(s+r)}\,ds$$

we get a preliminary bound

$$\tilde{S}(x)\ll_\epsilon\mathcal{R}^c x^\epsilon\ , \qquad \forall\epsilon>0\ , \quad \text{for some } c>0\ .$$

From $\ |a(n)|\le 1+|a(n)|^2\le 1+A(n)\ $ we get

$$S(x)\ll_\epsilon\mathcal{R}^c x^\epsilon\ , \qquad \forall\epsilon>0\ .$$

From previous proposition and Axiom 4

$$\begin{aligned}
S^2(x)=S(x)S(x)&=\sum_{n,m\le x}\frac{|a(m)\,a(n)|}{mn}\\
&\le\sum_{m,n\le x}\frac{1}{mn}\sum_{\substack{l|(m,n)^{\mathbf{d}}\\ l|mn}}^{*}r(l)\,|a(\frac{mn}{l})|\\
&\le\log x\Big(\sum_{l\le x^2}^{*}\frac{r(l)}{l}\Big)\Big(\sum_{k\le x^2}\frac{|a(k)|}{k}\Big)\\
&\le\log x\Big(\prod_{p\le x^2}\big(1+\frac{r^2(p)}{p^2}+\cdots\big)\Big)S(x^2)\le\log x\Big(\prod_{p\le x^2}\big(1+\frac{c}{p}\big)\Big)S(x^2)\\
&\le S(x^2)\log^{c'}x\ll_\epsilon x^\epsilon S(x^2)
\end{aligned}$$

Iterating

$$S(x)^{2^m}\ll_{\epsilon,m}x^{2^m\epsilon}S(x^{2^m})\ll_{\epsilon,m}x^{2^m\epsilon}\mathcal{R}^c\quad\forall m\ ,$$

hence

$$S(x)\ll_{\epsilon,m}x^\epsilon\mathcal{R}^{c/2^m}\ , \quad \forall m\ .$$

Choose $m=m(\epsilon)$ large enough.

What about lower bounds?

Lower bounds of Siegel type can be proved for $L(s, \mathcal{A})$ whenever there exists a second L-function, $L(s, \mathcal{B}_\mathcal{A})$ say, such that

$$L(s, \mathcal{A})^m L(s, \mathcal{B}_\mathcal{A}) = \sum_{n=1}^{\infty} \lambda(n) n^{-s} \qquad \lambda(n) \geq 0 \ , \quad \lambda(1) = 1 \ ,$$

with the exponent m equal to the order of pole at $s = 1$ of $L(s, \mathcal{A})^m L(s, \mathcal{B}_\mathcal{A})$ (much more: no Siegel zero if m is strictly greater than the order.)

In the class we have introduced there is not a general way to find $L(s, \mathcal{B}_\mathcal{A})$ for an arbitrary given $L(s, \mathcal{A})$, hence there is not a general lower bound. Nevertheless, some new results can be proved now, for example:

Let f, g be two **fixed and different** Maass cusp forms, let χ be a real primitive and non-principal character of modulus q, then $\forall \epsilon > 0$:

- $L_\chi(1, f \otimes \operatorname{sym}^2 f) \gg_{f,\epsilon} q^{-\epsilon}$.
- Since $L_\chi(s, f \otimes \operatorname{sym}^2 f) = L_\chi(s, f) L_\chi(s, \operatorname{sym}^3 f)$, the upper-bound for $L_\chi(s, f)$ and previous result give

$$L_\chi(1, \operatorname{sym}^3 f) \gg_{f,\epsilon} q^{-\epsilon} \ .$$

- $L_\chi(1, f \otimes \operatorname{sym}^2 g) \gg_{f,g,\epsilon} q^{-\epsilon}$.
- $L_\chi(1, \operatorname{sym}^2 f \otimes \operatorname{sym}^2 g) \gg_{f,g,\epsilon} q^{-\epsilon}$.

For the first result we use the following positive convolution

$$F(s) = L\big(s, (1 + \chi) \otimes (1 + \chi') \otimes (f + \operatorname{sym}^2 f) \otimes (f + \operatorname{sym}^2 f)\big)$$

which has a **double** pole at $s = 1$ coming from $L(s, f \otimes f)$ and $L(s, \operatorname{sym}^2 f \otimes \operatorname{sym}^2 f)$ and is divisible by $L_\chi(1, f \otimes \operatorname{sym}^2 f)^2$.

The other results come from similar products.