Magnetic Reconnection: Basic Concepts I

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1. Introduction

1.1. Aspects of Magnetic Reconnection

a) Energetics (acceleration of particles, heating, radiation, ..)

b) Dynamics (instabilities, time scales, efficiency, anomalous resistivity, ..)

c) Structure of the magnetic field (change of connectivity, flux conservation, helicity conservation, ..)

Many aspects are fairly well understood for 2D-reconnection.
Much less is known for the 3D case. In particular we have to answer:

- Where does reconnection occur?
- What types of reconnection exist in 3D?
- How does reconnection change the topology of the magnetic field?

1.2. Reconnection

- Magnetohydrodynamics: Reconnection of magnetic flux
  e.g. [Biskamp, D., *Magnetic Reconnection in Plasmas*, CUP 2000, Priest and Forbes, *Magnetic Reconnection*, CUP 2000]

- Hydrodynamics: Reconnection of vorticity
  e.g. [Kida, S., and M. Takaoka, *Vortex Reconnection*, Annu. Rev. Fluid Mech. **26**, 169 (1994)]

- Superfluids: Reconnection of quantized vortex elements
  e.g. [Koplik, J. and Levine, H., *Vortex reconnection in superfluid helium*, Phys. Rev. Letters **71**, 1375, (1993)]

- Cosmic Strings: Reconnection of topological defects
  e.g. [Shellard, E.P.S., *Cosmic String interactions*, Nucl. Phys. B 282, 624, (1987)]

- Liquid Crystals: Reconnection of topological defects
  e.g. [Chuang, I., Durrer, R., Turok, N., and Yurke, B., *Cosmology in the laboratory: Defect Dynamics in Liquid Crystals*, Science **251**, 1336, (1991)]

- Knot-theory surgery of framed knots
  e.g. [Kauffman, L.H. (1991)*Knots and Physics*, World Scientific, London]

- Enzymology Reconnection on strands of the DNA
  e.g. [Sumners D.(1990) *Untangling DNA* Math. Intell. **12** 71–80]

2. Topological Conservation Laws

2.1. Flux conservation

$$\partial_t \mathbf{B} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0$$

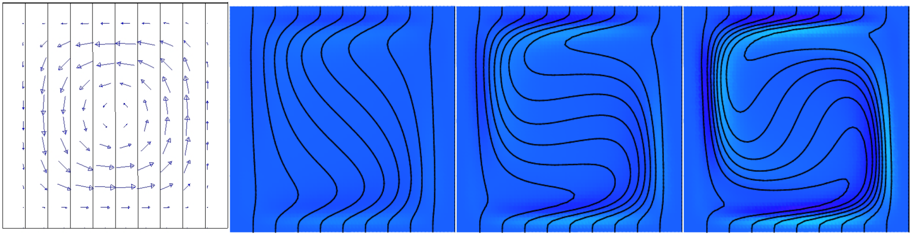
$$\Rightarrow \quad \int_{C^2(t)} \mathbf{B} \cdot d\mathbf{a} = const. \text{ for a comoving surface } \; C^2,$$

$\Rightarrow$ Conservation of flux

$\Rightarrow$ Conservation of field lines

$\Rightarrow$ Conservation of null points

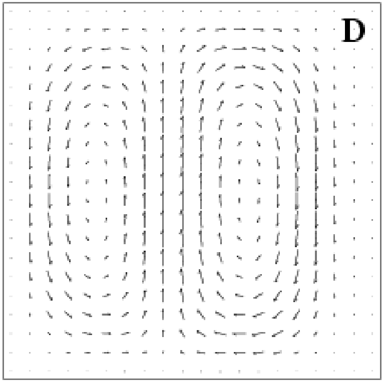
$\Rightarrow$ Conservation of knots and linkages of field lines



2.2. Integral conservation laws

A fluid in a domain $D \in I\!R^3$ with a velocity field $\mathbf{v}(\mathbf{x}, t)$ satisfying $\mathbf{v} \cdot \mathbf{n}|_{\partial V} = 0$. The flow $\mathbf{F}(\mathbf{x}, t)$ of $\mathbf{v}$ is defined by

$$\frac{\partial \mathbf{F}(\mathbf{x}, t)}{\partial t} = \mathbf{v}(\mathbf{F}(\mathbf{x}, t), t)$$

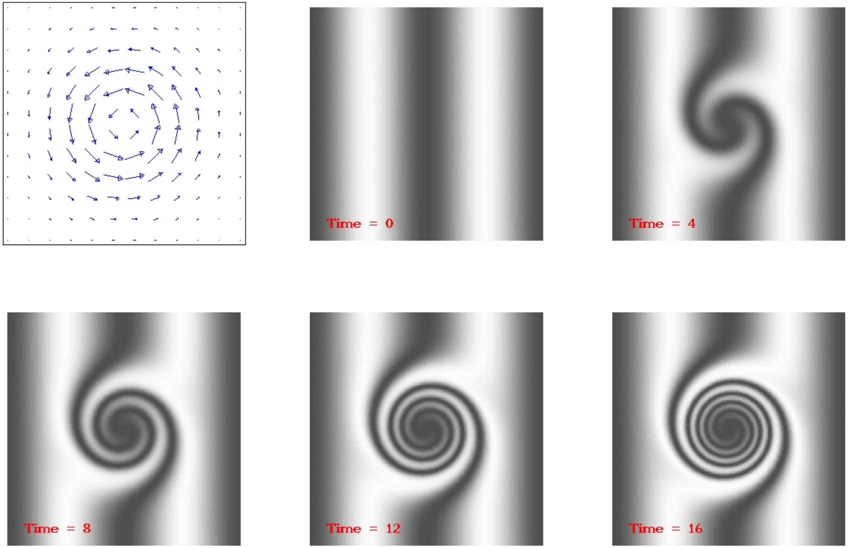


The advection of fields (ω^k) by the flow $\mathbf{F}(\mathbf{x}, t)$ of $\mathbf{v}$ implies the conservation of an integral over a k-dimensional comoving volume C^k.

$$\partial_t \omega^k + L_{\mathbf{v}} \omega^k = 0$$
$$\Rightarrow \int_{F(C^k)} \omega^k = const.$$

2.2.1. Scalar in $\mathbb{R}^3$

$$\partial_t \omega_\alpha^0 + L_{\mathbf{v}} \omega_\alpha^0 = 0 \quad \Leftrightarrow \quad \partial_t \, \alpha + \mathbf{v} \cdot \nabla \alpha = 0$$
$$\Rightarrow \alpha(F(\mathbf{x}, t), t) = const.$$



2.2.2. Vector (1-form) in $I\!R^3$

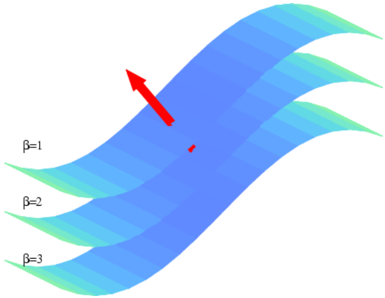
$$\partial_t \omega_A^1 + L_{\mathbf{v}}\, \omega_A^1 = 0 \quad \Leftrightarrow \quad \partial_t \mathbf{A} + \nabla(\mathbf{v}\cdot\mathbf{A}) - \mathbf{v} \times \nabla \times \mathbf{A} = 0$$
$$\Rightarrow \int_{C^1(t)} \mathbf{A}\cdot\mathbf{dl} = const.$$

Transport of a vector field represented locally as

$$\mathbf{A}(\mathbf{x}) = \alpha(\mathbf{x})\nabla\beta(\mathbf{x}) \quad (= \sum_{i=1}^{3} \alpha_i(\mathbf{x})\nabla\beta_i(\mathbf{x}))$$

(1-form $\omega_A^1 = \alpha\; d\beta$) by transporting $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$.

Example: $\mathbf{A} = \nabla\alpha$



2.2.3. Vector (2-form) in $I\!R^3$

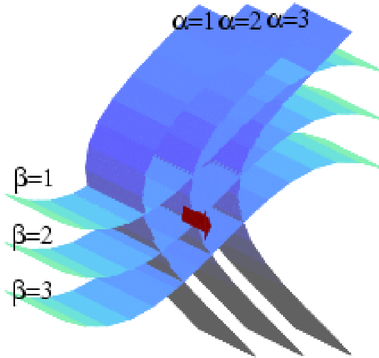
$$\partial_t \omega_B^2 + L_{\mathbf{v}}\, \omega_B^2 = 0 \quad \Leftrightarrow \quad \partial_t \mathbf{B} + \mathbf{v}\cdot\nabla\mathbf{B} - \mathbf{B}\cdot\nabla\mathbf{v} + \mathbf{B}\ \nabla\cdot\mathbf{v} = 0$$
$$\Leftrightarrow \quad \partial_t \mathbf{B} - \nabla\times\mathbf{v}\times\mathbf{B} = 0 \quad \text{for} \quad \nabla\cdot\mathbf{B} = 0$$
$$\Rightarrow \int_{C^2(t)} \mathbf{B}\cdot d\mathbf{a} = const.\ ,$$

Transport of a vector field represented locally as

$$\mathbf{B}(\mathbf{x}) = \nabla\alpha(\mathbf{x}) \times \nabla\beta(\mathbf{x}) \quad \left(= \sum_{i=1}^{3} \nabla\alpha_i(\mathbf{x}) \times \nabla\beta_i(\mathbf{x}) \right)$$
$$\omega_B^2 = d\alpha \wedge d\beta$$

by transporting $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$

Examples: **B** frozen-in vorticity (Kelvins Theorem) or frozen-in magnetic field (Alfvéns Theorem).



2.2.4. Scalar (3-form) in $I\!R^3$

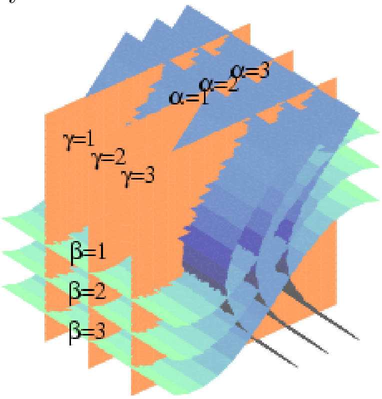
$$\partial_t \omega^3_\rho + L_{\mathbf{v}}\ \omega^3_\rho = 0 \quad \Leftrightarrow \quad \partial_t \rho + \nabla\cdot(\mathbf{v}\ \rho) = 0$$

$$\Rightarrow \int_{C^3(t)} \rho\ d^3x = const.$$

Transport of a density represented locally as

$$\begin{aligned} \rho(\mathbf{x}) &= (\nabla\alpha(\mathbf{x}) \times \nabla\beta(\mathbf{x})) \cdot \nabla\gamma(\mathbf{x}) \\ (\omega^3_\rho &= d\alpha \wedge d\beta \wedge d\gamma) \end{aligned}$$

by transporting $\alpha(\mathbf{x})$, $\beta(\mathbf{x})$ and $\gamma(\mathbf{x})$.
Example: ρ mass-density in hydrodynamics



2.2.5. Field lines in $\mathbb{R}^3$

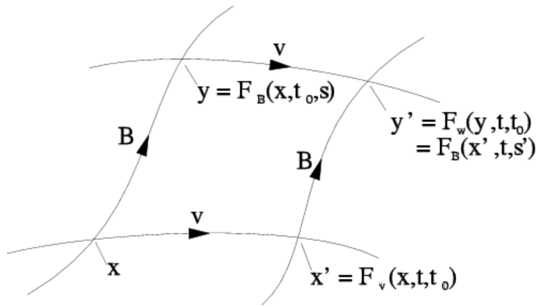
Transport of the field lines of a vector field

$$\mathbf{B}(\mathbf{F}_B(\mathbf{x}, s), s) := \frac{\partial \mathbf{F}_B(\mathbf{x}, s)}{\partial s}$$

by transporting the flow $F_B(\mathbf{x}, s)$:

$$\begin{aligned} & \mathbf{F}_B(\mathbf{F}(\mathbf{x}, t), t, s') = \mathbf{F}(\mathbf{F}_B(\mathbf{x}, t, s), t) \\ \Rightarrow \quad & \partial_t \mathbf{B} + \mathbf{v}(\mathbf{x}) \cdot \nabla \mathbf{B}(\mathbf{x}) - \mathbf{B}(\mathbf{x}) \cdot \nabla \mathbf{v}(\mathbf{x}) = \lambda \mathbf{B} \\ \Leftrightarrow \quad & \partial_t \mathbf{B} + L_{\mathbf{v}} \mathbf{B} = \lambda \mathbf{B} \end{aligned}$$

$\Rightarrow$ Conservation of magnetic field lines



Conservation of flux $\Rightarrow$ Conservation of magnetic field lines

$$\begin{aligned} & \partial_t \mathbf{B} + \mathbf{v}(\mathbf{x}) \cdot \nabla \mathbf{B}(\mathbf{x}) - \mathbf{B}(\mathbf{x}) \cdot \nabla \mathbf{v}(\mathbf{x}) + \mathbf{B} \ \nabla \cdot \mathbf{v} = 0 \\ \Rightarrow \quad & \partial_t \mathbf{B} + \mathbf{v}(\mathbf{x}) \cdot \nabla \mathbf{B}(\mathbf{x}) - \mathbf{B}(\mathbf{x}) \cdot \nabla \mathbf{v}(\mathbf{x}) = \lambda \mathbf{B} \quad \text{for } \lambda = -\nabla \cdot \mathbf{v} \end{aligned}$$

2.3. Conservation laws in space-time

$$L_V \omega^0_\alpha = 0 \quad \Leftrightarrow \quad V^t \partial_t \alpha + \mathbf{V}\cdot\nabla\alpha = 0$$

$$\Rightarrow \quad \int_{C^0} \omega^0_\alpha = \alpha = const.$$

$$L_V \omega^1_A = 0 \quad \Leftrightarrow \quad \begin{cases} \partial_t(V^t A^t) + \mathbf{V}\cdot\nabla A^t - \mathbf{A}\cdot\partial_t \mathbf{V} = 0 \\ V^t \partial_t \mathbf{A} + \nabla\left(\mathbf{V}\cdot\mathbf{A}\right) - \mathbf{V}\times\nabla\times\mathbf{A} - A^t \nabla V^t = 0 \end{cases}$$

$$\Rightarrow \quad \int_{C^1} \omega^1_A = \int_{C^1} A^t dt - \int_{C^1} \mathbf{A}\cdot d\mathbf{l} = const.$$

$$L_V \omega^2_{\mathbf{AB}} = 0 \quad \Leftrightarrow \quad \begin{cases} \partial_t(V^t \mathbf{A}) + \nabla\left(\mathbf{V}\cdot\mathbf{A}\right) - \mathbf{V}\times\nabla\times\mathbf{A} - \partial_t \mathbf{V}\times\mathbf{B} = 0 \\ V^t \partial_t \mathbf{B} - \nabla\times\left(\mathbf{V}\times\mathbf{B}\right) + \mathbf{V}\ \nabla\cdot\mathbf{B} + \nabla V^t \times\mathbf{A} = 0 \end{cases}$$

$$\Rightarrow \quad \int_{C^2} \omega^2_{\mathbf{AB}} = \int_{C^2} \mathbf{B}\cdot d\mathbf{a} - \int_{C^2} \mathbf{A}\cdot d\mathbf{l}\ dt = const.$$

$$L_V \omega^3_A = 0 \quad \Leftrightarrow \quad \begin{cases} V^t \partial_t A^t + \nabla\cdot(\mathbf{V} A^t) - \mathbf{A}\cdot\nabla V^t = 0 \\ \partial_t(V^t \mathbf{A}) - \nabla\times\left(\mathbf{V}\times\mathbf{A}\right) + \mathbf{V}\ \nabla\cdot\mathbf{A} - A^t \partial_t \mathbf{V} = 0 \end{cases}$$

$$\Rightarrow \quad \int_{C^3} \omega^3_A = \int_{C^3} A^t dV - \int_{C^3} \mathbf{A}\cdot d\mathbf{a}\ dt = const.$$

$$L_V \omega^4_\rho = 0 \quad \Leftrightarrow \quad \partial_t(\rho V^t) + \nabla\cdot(\rho\mathbf{V}) = 0$$

$$\Rightarrow \quad \int_{C^4} \omega^4_\rho = \int_{C^4} \rho\ dV^{(4)} = const.$$

2.4. Conservation in $\mathbb{R}^n$

$$\partial_s \omega^0_\alpha + L_V \omega^0_\alpha = 0$$
$$\Leftrightarrow \quad \partial_s \alpha + \mathbf{V} \cdot \nabla \alpha = 0$$
$$\Rightarrow \quad \int_{C^0} \omega^0_\alpha = \alpha = const.$$

$$\vdots$$

$$\vdots$$

$$\partial_s \omega^n_\rho + L_V \omega^n_\rho = 0$$
$$\Leftrightarrow \quad \partial_s \rho + \nabla \cdot (\rho \mathbf{V}) = 0$$
$$\Rightarrow \quad \int_{C^n} \omega^4_\rho = \int_{C^n} \rho \, dV^{(n)} = const.$$

Example: $\mathbf{V}$ Hamilton flow on $\mathbb{R}^{(2m)}$, $\omega^{(2k)} = dq^1 \wedge dp^1 \wedge ... \wedge dq^k \wedge dp^k$ yields Poincaré integral invariants.

2.5. Relations (Examples)

$$\mathbf{A} \text{ frozen-in as 1-form} \quad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A} \text{ is frozen-in as a 2-form}$$

$$\partial_t \mathbf{A} + \nabla(\mathbf{v}\cdot\mathbf{A}) - \mathbf{v} \times \nabla \times \mathbf{A} = 0 \qquad \partial_t \mathbf{B} - \nabla \times \mathbf{v} \times \mathbf{B} = 0$$

$$\mathbf{B} \text{ frozen-in as 2-form } (\nabla \cdot \mathbf{B} = 0) \quad \Rightarrow \quad \exists \text{ vector potential } \mathbf{A} \text{ frozen-in as a 1-form}$$

$$\partial_t \mathbf{B} - \nabla \times \mathbf{v} \times \mathbf{B} = 0 \qquad \partial_t \mathbf{A} - \mathbf{v} \times \nabla \times \mathbf{A} = -\nabla\Phi$$

Use gauge $\mathbf{A} \to \tilde{\mathbf{A}} = \mathbf{A} + \nabla\Psi$

with $\dfrac{d\Psi}{dt} = \Phi - \mathbf{v}\cdot(\mathbf{A} + \nabla\Psi)$

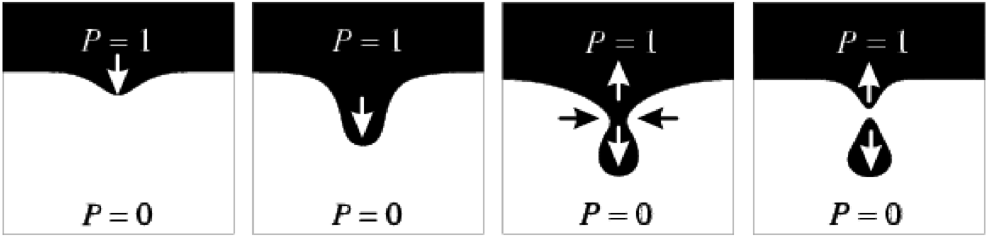
$$\Rightarrow \quad \partial_t \tilde{\mathbf{A}} + \nabla(\mathbf{v}\cdot\tilde{\mathbf{A}}) - \mathbf{v} \times \nabla \times \tilde{\mathbf{A}} = 0 \; .$$

$$\mathbf{A} \text{ frozen-in as 1-form} \quad \Rightarrow \quad h = \mathbf{A} \cdot \mathbf{B} \text{ is frozen-in as a 3-form}$$

$$\partial_t \mathbf{A} + \nabla(\mathbf{v}\cdot\mathbf{A}) - \mathbf{v} \times \nabla \times \mathbf{A} = 0 \qquad \partial_t h + \nabla\cdot(\mathbf{v}\; h) = 0$$

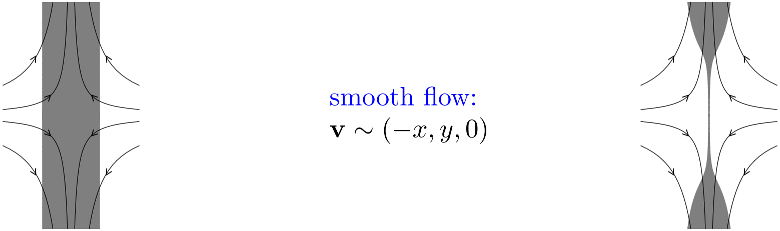
Remark: There are more relations of this type which can be derived applying the Lie-derivative to exterior products and derivatives of differential forms.

3. Example for reconnection: Detachment of a drop



$$\partial_t P(\mathbf{x}, t) + \mathbf{v} \cdot \nabla P(\mathbf{x}, t) = 0$$

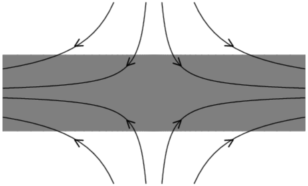
X-type structure of the flow near the point of detachment:



smooth flow:
$\mathbf{v} \sim (-x, y, 0)$

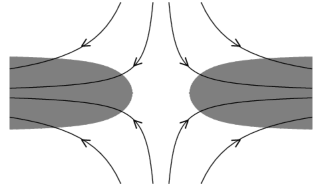
No detachment in a finite time for a smooth $\mathbf{v}(\mathbf{x}, t)$!
Time necessary to reach the X-point:

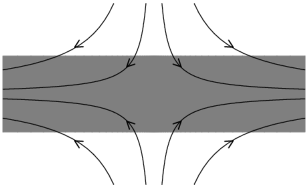
$$\int_{x_t}^{0} dt = \int_{x_t}^{0} 1/v_x \; dx = \infty$$



singular flow

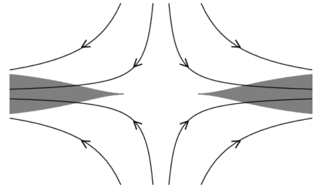
$\mathbf{w} \sim \left(-\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, 0\right)$





non-differentiable flow

$\mathbf{w} \sim (-\sqrt{x}, \sqrt{y}, 0)$



Detachment is a non-ideal evolution:

$$\partial_t P(\mathbf{x}, t) + \mathbf{v} \cdot \nabla P(\mathbf{x}, t) = r(\mathbf{x}, t)$$

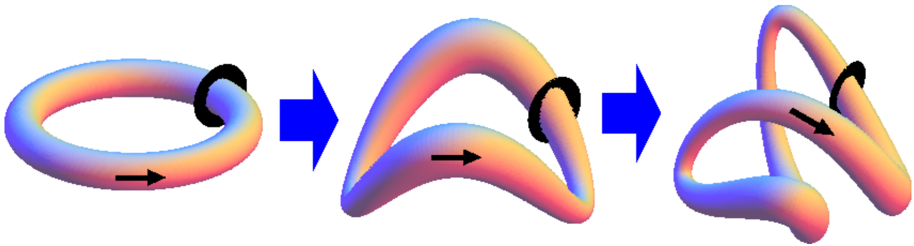
Non-idealness $r(\mathbf{x}, t)$
- is an effect of molecular forces
- can be neglected everywhere except for the detachment
- can not be represented in macroscopic fluid variables
- has macroscopic effects

The detachment is essentially independent of the profile and value of $r(\mathbf{x}, t)$

Non-ideal evolution $\partial_t P + \mathbf{v} \cdot \nabla P = r$	$\Rightarrow$	Quasi-ideal evolution ($\mathbf{w}$ singular) $\partial_t P + \mathbf{w} \cdot \nabla P = 0$

4. Conditions for magnetic reconnection to occur

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t}\mathbf{B} - \nabla \times \mathbf{v} \times \mathbf{B} = 0$$



$\Rightarrow$ Conservation of magnetic flux and field line topology: No reconnection.

Reconnection can only occur if the plasma is non-ideal.

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{N} \quad \text{with} \quad \mathbf{N} = \text{Non-ideal term. (e.g.}\mathbf{N} = \eta \mathbf{j})$$

But....

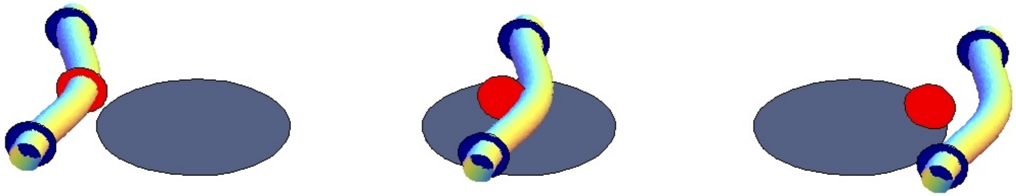
$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{N} \quad \text{with} \quad \mathbf{N} = \nabla\Phi + \mathbf{u} \times \mathbf{B} \tag{1}$$

$$\Leftrightarrow \quad \mathbf{E} + (\mathbf{v} - \mathbf{u}) \times \mathbf{B} = \nabla\Phi \tag{2}$$

$$\Leftrightarrow \quad \frac{\partial}{\partial t}\mathbf{B} - \nabla \times \mathbf{w} \times \mathbf{B} = 0 \quad \text{for} \quad \mathbf{w} = \mathbf{v} - \mathbf{u} \tag{3}$$

$\Rightarrow$ Conservation of magnetic flux and field line topology w.r.t $\mathbf{w}$: No reconnection.
$\mathbf{u}$: is a slippage velocity of plasma with respect to the field lines.
Eq. (1) is the most general form which leads to a flux conserving evolution.

Example of slippage



Example where the transport velocity of the magnetic flux $\mathbf{w}$ differs from the plasma velocity $\mathbf{v}$ in a non-ideal region (gray) while outside the non-ideal region the plasma is ideal ($\mathbf{v} = \mathbf{w}$). The blue and red cross sections are moving with the plasma velocity. The blue cross section always remains in the ideal domain while the red cross section crosses the non-ideal region.

Examples which satisfy $\partial_t \mathbf{B} - \nabla \times (\mathbf{w} \times \mathbf{B}) = 0$:

- Ideal plasma dynamics:

$$\begin{aligned} \mathbf{E} + \mathbf{v} \times \mathbf{B} &= 0 \\ \mathbf{E} + \mathbf{v} \times \mathbf{B} &= -\frac{1}{e\ n} \nabla P_e(n) \quad \Rightarrow \mathbf{w} = \mathbf{v} \end{aligned}$$

- Hall MHD:

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{1}{e\ n} \mathbf{j} \times \mathbf{B} \quad \Rightarrow \mathbf{w} = \mathbf{v}_e = \mathbf{v} - \frac{1}{e\ n} \mathbf{j}$$

- many other cases under constraints: e.g. 2D dynamics with $\mathbf{B} \neq 0$:

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = ... + \frac{m_e}{n\ e^2} \left(\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot (\mathbf{v}\mathbf{j} + \mathbf{j}\mathbf{v}) \right) + ...$$

$\mathbf{N}$ allows for reconnection if

$$\mathbf{N} \neq \nabla\Phi + \mathbf{u} \times \mathbf{B}$$

This condition assumes that the magnetic field is closed within the domain under consideration, otherwise the topology (and possibly dynamics) of the field outside the boundary has to be represented by corresponding boundary condition on $\mathbf{w}$ and Φ.

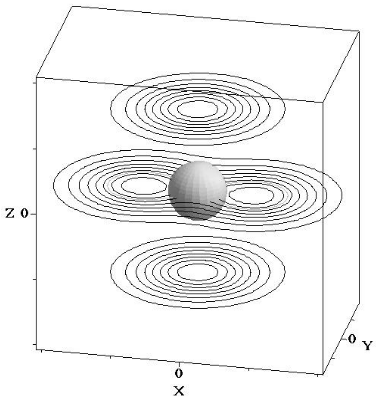
- $\mathbf{E} \cdot \mathbf{B} = \mathbf{N} \cdot \mathbf{B} \neq 0$ and $\mathbf{N} \cdot \mathbf{B} \neq \nabla\Phi \cdot \mathbf{B}$
- or $\mathbf{E} \cdot \mathbf{B} = 0$ and $\mathbf{N} \neq 0$ at $\mathbf{B} = 0$ and $\nabla \times \mathbf{N} \neq 0$

Remark: Note that these are conditions on the evolution of the electromagnetic field - they are not restricted to MHD.

$\Rightarrow$ Two cases of reconnection: vanishing or non-vanishing helicity source term.

5. Classification of three-dimensional reconnection

5.1. $\mathbf{E} \cdot \mathbf{B} = 0$-Reconnection



$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{N}$$

$$\mathbf{E} \cdot \mathbf{B} = 0 \Rightarrow \mathbf{N} \cdot \mathbf{B} = 0$$

$$\mathbf{w} := \mathbf{v} - \Delta\mathbf{v} = \mathbf{v} - \frac{\mathbf{B} \times \mathbf{N}}{B^2}.$$

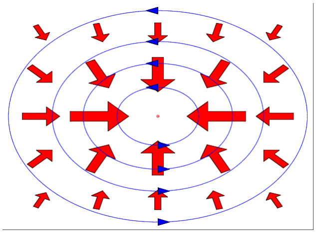
$$\mathbf{E} + \mathbf{w} \times \mathbf{B} = 0$$

A flux conserving flow $\mathbf{w}$ exists and it is smooth with exception of points where $B = 0$ but $N \neq 0$.

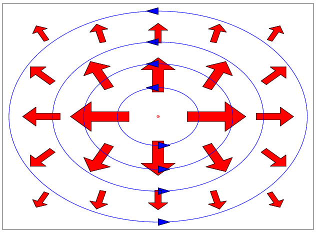
At a null point of $\mathbf{B}$ with $\mathbf{N} \neq 0$ the fields $\mathbf{B}$ and $\mathbf{w}$ are locally tangential to a plane perpendicular to $\mathbf{E}$.

5.2. Non-Ideal Evolution at an O-point

At the O-point $\mathbf{w}$ has the local structure:



a)

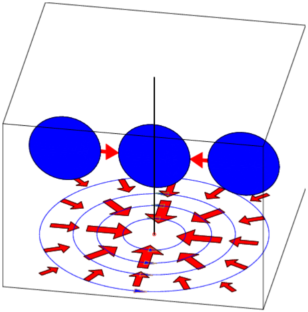


b)

Manifestation of loss (a) or creation (b) of flux at an O-point

Since for a generic null $\mathbf{B}$ is linear in $\mathbf{x}$, $\mathbf{w}$ is of the type $-\mathbf{x}/x^2$. It has a $1/x$ singularity at the origin. Thus a cross-section is transported in a finite time onto the null-line.

$$T = \int_0^T dt = \int_{x_0}^0 dt/dx\ dx = \int_{x_0}^0 1/w_x dx \sim x_0^2/2$$



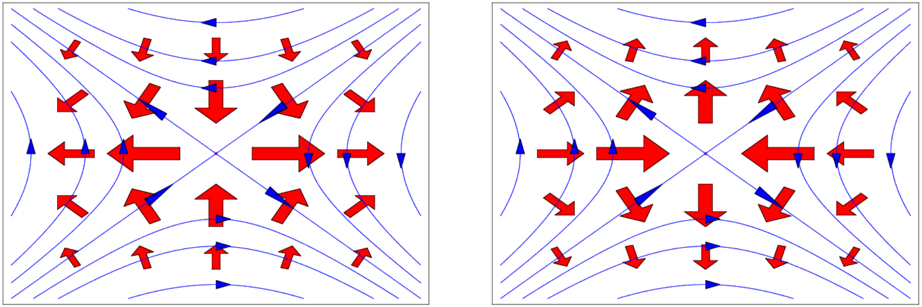
Rate of flux annihilation:

$$\frac{d\Phi_{rec}}{dt} = \frac{d}{dt}\int \mathbf{B}\cdot\mathbf{n} da = \int E_z\, dz \quad (= \int w_r B_{phi}\, dz)$$

Movie

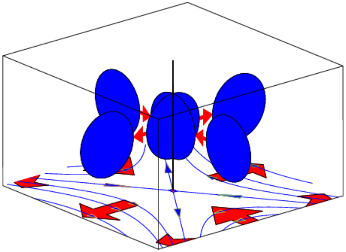
5.3. Non-Ideal Evolution at an X-point

Structure of $\mathbf{w}$ at an X-point:



Reconnection

Again the transporting flow has a $1/x$ singularity $\Rightarrow$ The flux (a cross section) is transported in a finite time onto the null-line. But this time the singularity is of X-type. Simultaneously the null line is the source of flux leaving the axis along the other direction. $\Rightarrow$ No loss of flux but a re-connection.



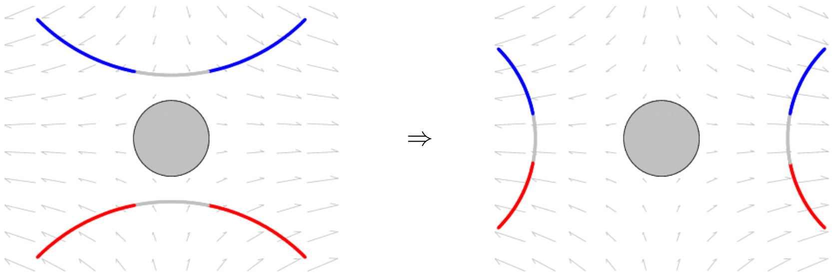
Rate of reconnected flux:

$$\frac{d\Phi_{rec}}{dt} = \frac{d}{dt}\int \mathbf{B}\cdot\mathbf{n}da = \int E_z\, dz \quad (= \int w_x B_y\, dz)$$

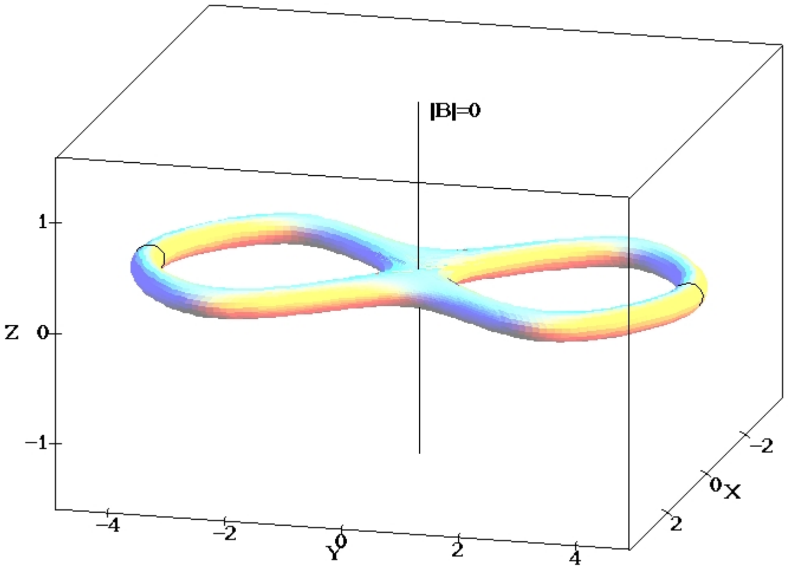
Movie

Properties of $\mathbf{E} \cdot \mathbf{B} = 0$-Reconnection:

- There exists a flux transporting velocity $\mathbf{w}$ which is singular at the X-point
- The process can be localised at an X-point of $\mathbf{B}$
- No dissipation (loss) of magnetic flux
- The rate of flux reconnection is given by $\int \mathbf{E} \cdot d\mathbf{l}$ along the X-line
- There is no production of magnetic helicity
- 1-1 correspondence of reconnecting field lines



Simple reconnection process

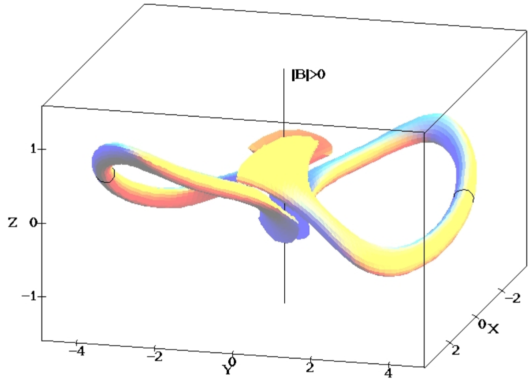


Movie

5.4. $\mathrm{E}\cdot\mathrm{B}\neq 0$-Reconnection

- Reconnection at magnetic null points
- Reconnection at $\mathbf{B}\neq 0$

Simple 3D reconnection process



Movie

5.5. Total Helicity

The (total) magnetic helicity is defined as

$$H(\mathbf{B}) := \int_V \underbrace{\mathbf{A}\cdot\mathbf{B}}_{\text{hel. density}} \; d^3x \qquad \text{for} \quad \mathbf{B}\cdot\mathbf{n}|_{\partial V} = 0 \ ,$$

where $\mathbf{A}$ is the vector potential for the magnetic field $\mathbf{B}$, which is tangent to the boundary ∂V.

Warning: The total helicity requires additional assumptions for multiple connected domains, e.g. a torus or a box with periodic boundary conditions.

In terms of the magnetic field only:

$$H(\mathbf{B}) = \frac{1}{4\pi} \int\int \mathbf{B}(\mathbf{x}') \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \cdot \mathbf{B}(\mathbf{x}) \; d^3x' \; d^3x$$

which shows that the helicity is of 2nd order in the magnetic field.

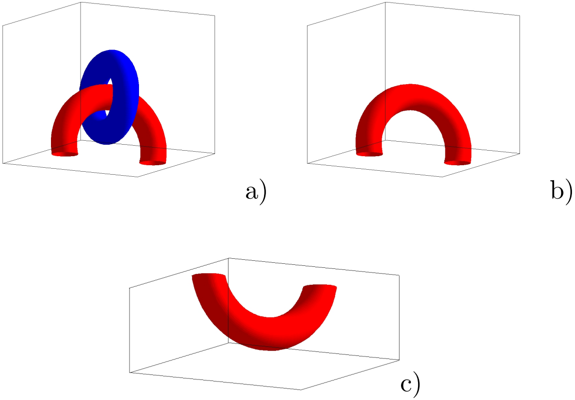
5.6. Relative Helicity

For cases where the boundary is not a magnetic surface, i.e where magnetic flux crosses the boundary, the total magnetic helicity is not well defined. In this case we can define a relative helicity between fields $\mathbf{B}_a$ and $\mathbf{B}_b$ satisfying the same boundary conditions.

Supplement the field using e.g. a vacuum field $\mathbf{B}_c$ to a closed configuration so that we can calculate the total magnetic helicities for the supplemented fields $\mathbf{B}_{a+c}$ and $\mathbf{B}_{b+c}$. Then

$$H_{rel} := H(B_{a+c}) - H(B_{b+c})$$

is well defined and does not depend on the extension $\mathbf{B}_c$.

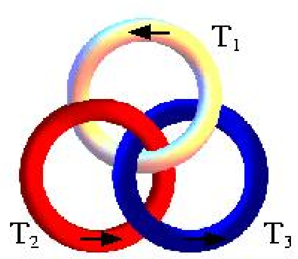


5.7. Interpretation

For systems of (untwisted) flux tubes the total magnetic helicity can be expressed as a sum over the mutual linking of flux tubes [Moffatt 1969]:

$$H(B) = 2 \sum_{i<j} lk(T_i, T_j)\Phi_i\Phi_j \ ,$$

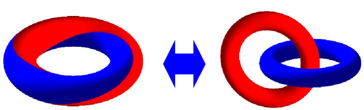
where $lk(T_i, T_j)$ is the linking number of the tube T_i and T_j with magnetic fluxes Φ_i and Φ_j.



c)

This interpretation was generalized by [Arnold 1986] for the generic case where field lines are not closed using asymptotic linking numbers.

Note: Twist is a linkage of sub-flux tubes:



5.8. Evolution of helicity

The homogeneous Maxwell's equation yield a balance equation for the helicity density:

$$\frac{\partial}{\partial t} \underbrace{\mathbf{A}\cdot\mathbf{B}}_{\text{hel. density}} + \nabla\cdot(\underbrace{\Phi\mathbf{B} + \mathbf{E}\times\mathbf{A}}_{\text{hel. current}}) = \underbrace{-2\,\mathbf{E}\cdot\mathbf{B}}_{\text{hel. source}}$$

Remarks:

- There is no freedom to add certain terms either to the current or to the source since the covariant formulation uniquely determines the helicity current.
- The helicity density and the helicity current are not gauge invariant, but the source term is gauge invariant.

Integrating over are volume yields an expression for the total helicity

$$\frac{d}{dt}\int_V \mathbf{A}\cdot\mathbf{B}\; d^3x = -2\int_V \mathbf{E}\cdot\mathbf{B}\; d^3x \;,$$

if the helicity current across the boundary vanishes, this is the case ...

- If the boundary is a magnetic surface $\mathbf{B}\cdot\mathbf{n}|_{\partial D} = 0$ and $\mathbf{E}\times\mathbf{n}|_{\partial D} = 0$ or ...
- If $\mathbf{E}+\mathbf{v}\times\mathbf{B} = 0$ and the boundary is a comoving magnetic surface or...
- If $\mathbf{A}\times\mathbf{n}|_{\partial D} = 0$, a gauge which is always possible for simply connected domains (Proof)

For an ideal evolution ($\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$) the balance equation reads

$$\frac{\partial}{\partial t} h + \nabla \cdot (\mathbf{v}\ h + (\Phi - \mathbf{v} \cdot \mathbf{A})\ \mathbf{B})\ = 0 \qquad h = \mathbf{A} \cdot \mathbf{B}$$

Using a certain gauge the $(\Phi - \mathbf{v} \cdot \mathbf{A})\ \mathbf{B}$ term in the helicity current can be made to vanish. This leads to

$$\frac{\partial}{\partial t} h + \nabla \cdot (\mathbf{v} h)\ = 0$$

especially in this case the total helicity is conserved for arbitrary comoving volumes:

$$\frac{d}{dt} \int_V \mathbf{A} \cdot \mathbf{B}\ d^3x = 0$$

5.9. Helicity conservation

5.10. General considerations

For a non-ideal evolution $\mathbf{E}+\mathbf{v}\times\mathbf{B}=\mathbf{N}$ the change of the total helicity in a magnetically closed volume is given by

$$\frac{d}{dt}\int_V \mathbf{A}\cdot\mathbf{B}\; d^3x = -2\int_V \mathbf{N}\cdot\mathbf{B}\; d^3x,$$

if $\mathbf{N}$ vanishes on the boundary The total helicity is strictly conserved for ideal MHD or more general for $\mathbf{N}\cdot\mathbf{B}=0$ e.g. in case of 2-d reconnection.

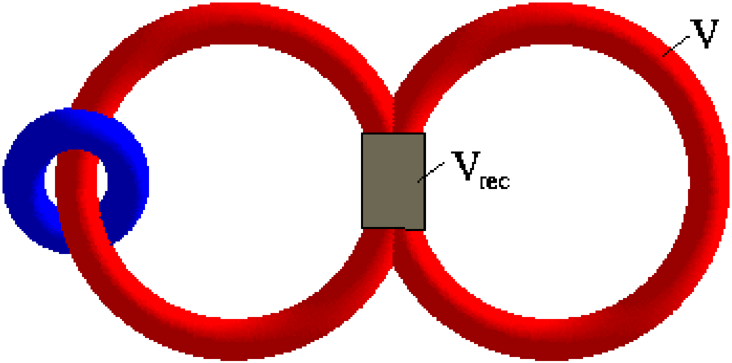
The total helicity is approximately conserved on the time scale of energy dissipation for a resistive plasma [Berger 1984].

$$\left|\frac{\Delta H}{H}\right| \leq \sqrt{\frac{\Delta t}{\tau_d}} \qquad \text{with} \quad \tau_d = L^2/\eta \quad \text{and} \quad L = \left|\frac{A}{B}\right| = \left|\frac{\mathbf{A}\cdot\mathbf{B}}{B^2}\right|$$

$\Rightarrow$ If $\Delta t \ll \tau_d$ then $\Delta H/H \ll 1$.

5.11. Production of helicity in reconnection

Astrophysical plasmas differ from many technical plasmas in the size of the reconnection region V_{rec}, where the dissipation dominates the evolution, compared to the volume V of magnetic flux connected to V_{rec}.



Especially $d \ll L$ for astrophysical plasnas, where d is the diameter of V_{rec} and L of V.

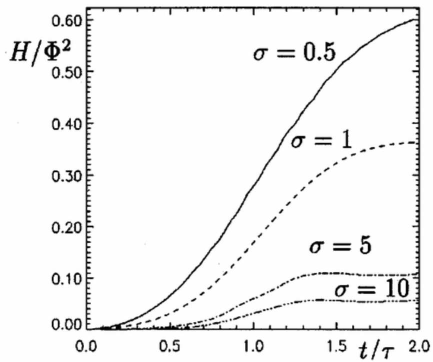
$$\tau_{hel} = \left| \frac{\int_V \mathbf{A} \cdot \mathbf{B} \; d^3x}{\int_V \mathbf{E} \cdot \mathbf{B} d^3x} \right| \sim \frac{B \; V \; L}{E \; V_{rec}} \qquad \text{with} \quad L = \left| \frac{A}{B} \right| \; ,$$

$$\tau_{diss} = \left| \frac{\int_V B^2/(8\pi) d^3x}{\int_V \mathbf{E} \cdot \mathbf{J} d^3x} \right| \sim \frac{B \; V \; d}{E \; V_{rec}} \qquad \text{with} \quad d = \left| \frac{B}{J} \right|_{V_{rec}}$$

$$\Rightarrow \frac{\tau_{hel}}{\tau_{diss}} \sim \frac{L}{d} \gg 1$$

5.12. Numerical verification of helicity conservation in 3-d reconnection

Numerical experiment of a 3-d reconnection process.

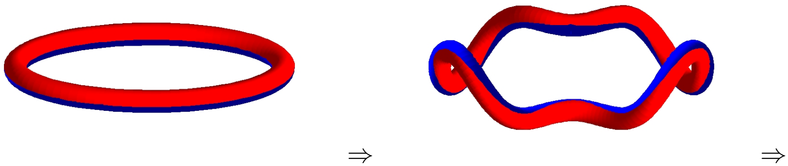


Production of helicity normalized to the total reconnected flux for different values of the inverse width $\sigma = 1/d$, which characterizes the thickness of the reconnection region compared to the length scale of the magnetic flux $L = 1$ [Hornig & Rastätter, 1997].

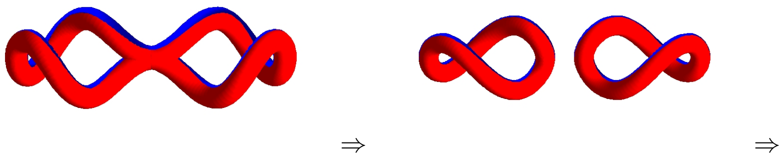
In plasma of high magnetic Reynolds numbers the helicity production in a single reconnection event is small compared to the potential helicity of the field(e.g. a corresponding constant-α force-free field).

⇒ How is the helicity created which we observe on the solar surface ?

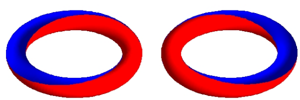
A principal method:



An initially untwisted flux tube with vanishing total helicity is twisted and ...



...reconnected into two flux tubes with negative and positive total helicity.



Reconnection does not produce much helicity but can separate helicity!

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