What Can Rational Investors Do About Excessive Volatility?

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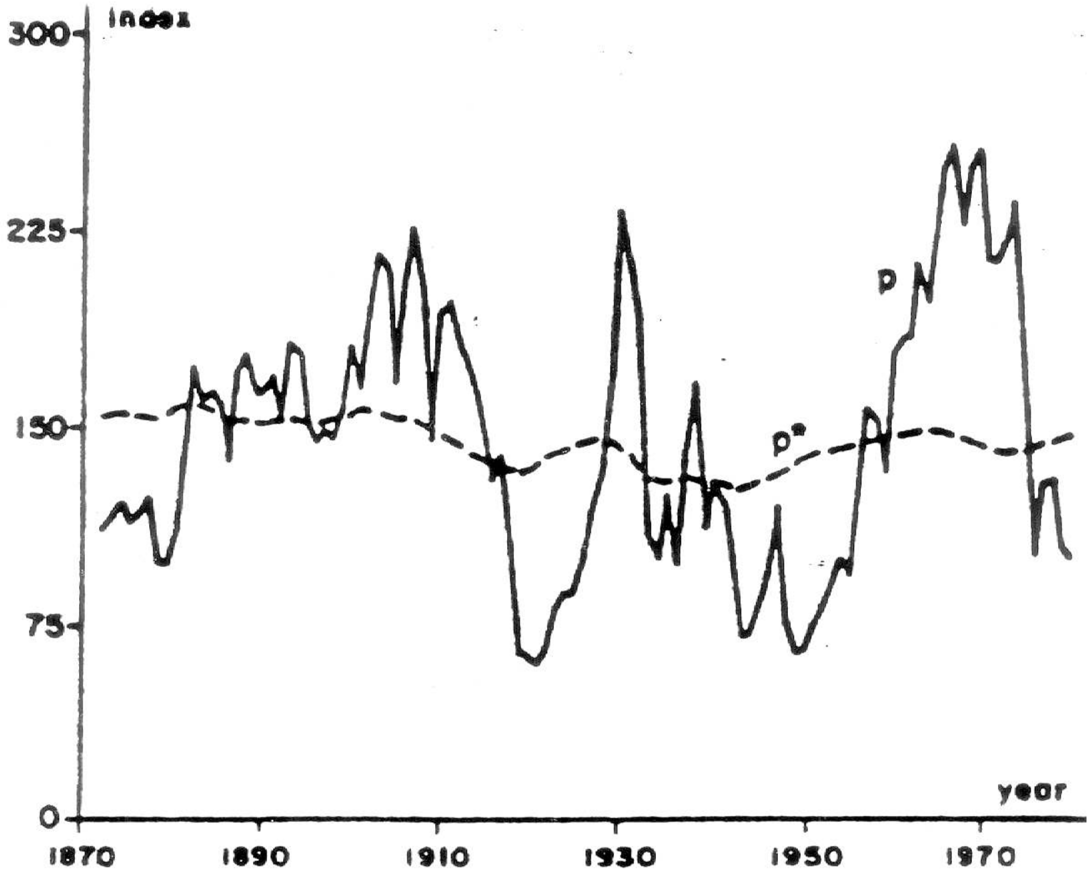
March 2005

Motivation: Excessive volatility of stock prices

Shiller (1981) and LeRoy-Porter (1981)

- **Compare**: Actual stock prices to present discounted value of dividends
  - Discount rate (i.e. expected return) assumed constant
- **Find**: Stock prices "too volatile" to be justified by subsequent dividends
- **Conclude**: Volatility is *excessive* relative to fundamentals



Solid line p: Real S&P Composite stock price index;

Dotted line p*: Ex post rational price; that is, PV of actual subsequent real detrended dividends

Our objective: Understand "Volatility Arbitrage"

- Excess volatility, if it exists, may be an indication that the financial market is not information efficient.

- If that is so, there must exist a trading strategy that allows a rational, intertemporally optimizing investor (a "risk arbitrageur") to take advantage of this inefficiency.

- **Our objective**:
  - Calculate and understand this strategy.
  - Understand implications of the strategy for excess volatility.

Contribution: Model

- **Construct** a framework where stock prices are excessively volatile, using the same modeling device as in Scheinkman and Xiong (2003)
- **Two classes of agents**, and one class (*overconfident*) believes that magnitude of correlation between innovations in signal and in unobserved growth rate is larger than it actually is.

  "In their enlightenment, economists will routinely incorporate as much "behavior" into their models as they observe in the real world." (Thaler, 1999, p. 17)
- **These agents overreact** to signals, which generates excessive volatility (measured relative to volatility under rational Bayesian learning).
- **In contrast** to Scheinkman and Xiong, our model is general equilibrium, we allow for short sales, and agents are risk averse.

Contribution: Optimal portfolio

- Obtain optimal dynamic trading strategy of rational investors.
- Show that portfolio of rational investors consists of four components:
  - **static** (i.e., Markowitz) portfolio based only on **current** expected stock returns and risk,
  - portfolios that **hedge** against **future** changes in state variables:
    - ⋆ **aggregate dividends**,
    - ⋆ **average belief** in market about expected dividend growth,
    - ⋆ **dispersion of beliefs** about expected dividend growth.

Contribution: Risk arbitrage

- Rational risk-arbitrageurs find it **beneficial** to trade on their belief that the market is being foolish
  - but when doing so, they must hedge future fluctuations in the market's foolishness.

- Risk arbitrage **cannot** be based just on a **current** price divergence;
  - the risk arbitrage must include a **protection** in case there is a deviation from that prediction, and
  - a prediction concerning **change in magnitude in disagreement**, and
  - it must also be based on a model of irrational behavior.

Contribution: Survival of irrational traders

- In contrast to Kogan, Ross, Wang, Westerfield (2004) and Yan (2004), we study survival of traders who are sometimes overoptimistic and sometimes overpessimistic, depending on sequence of signals.

- **Derive** the speed of impoverishment of the irrational traders

- **Find** that
  - presence of a few rational traders is not sufficient to eliminate effect of overconfident investors on excess volatility, and
  - overconfident investors may survive for a long time.

Other models of excess volatility

- **Bayesian learning** models: Brennan-Xia (2001); also Barsky and DeLong (1993) Timmermann (1993, 1996), Bullard-Duffy (1998)
- **Noise-trader** models: Irrational random demand for stocks
- **Changing discount rate** models: Menzly-Santos-Veronesi (2004), Bansal and Yaron (2004)
- **Incomplete markets**: Citanna-Schmedders ('02), Uppal-Bhamra ('05)
- **Bubbles**: A lot of (unjustified) volatility in terminal price, because of fluctuating risk of future bubble bursting

Model

Process for aggregate output

- Exogenous process for aggregate dividend (=earnings = consumption)
  - Output uncertainty: first source of risk (δ shock)

$$\frac{d\delta_t}{\delta_t} = f_t dt + \sigma_\delta dZ_t^\delta, \tag{1}$$

  - Expected value of rate of growth of dividend f is stochastic

$$df_t = -\zeta \left(f_t - \overline{f}\right) dt + \sigma_f dZ_t^f; \quad \zeta > 0, \tag{2}$$

Unobservable growth rate

- The expected growth rate is not unobserved by any investor; investors continuously form (filter) estimates of it, based on:
  - Realized dividend (δ shock again)
  - Realization of a signal "news" (s shock)

$$ds_t = f_t dt + \sigma_s dZ_t^s, \tag{3}$$

Agents: Two groups with identical preferences

- **Group A:** Irrational traders
  - They believe steadfastly that
    - ⋆ innovations in the signal have correlation $\phi \geq 0$ with innovations in the current expected rate of growth of dividends
    - ⋆ when, in fact, the true value of that correlation is **zero**

$$ds_t = f_t dt + \sigma_s \phi dZ_t^f + \sigma_s \sqrt{1 - \phi^2} dZ_t^s. \tag{4}$$

  - They overreact to signal and cause excess volatility in stock market
  - Otherwise, behave optimally
- **Group B:** Rational traders ("smart money" "hedge funds") with $\phi = 0$.

What should be their trading strategy?

Filtering

- The conditional expected values, $\widehat{f}^A$ and $\widehat{f}^B$ of f according to individuals of group A (deluded; $\phi \neq 0$) and group B (rational; $\phi = 0$) are:

$$d\widehat{f}_t^A = -\zeta\left(\widehat{f}_t^A - \overline{f}\right)dt + \frac{\gamma^A}{\sigma_\delta^2}\left(\frac{d\delta}{\delta} - \widehat{f}_t^A dt\right) + \frac{\phi\sigma_s\sigma_f + \gamma^A}{\sigma_s^2}\left(ds - \widehat{f}_t^A dt\right), \quad (5)$$

$$d\widehat{f}_t^B = -\zeta\left(\widehat{f}_t^B - \overline{f}\right)dt + \frac{\gamma^B}{\sigma_\delta^2}\left(\frac{d\delta}{\delta} - \widehat{f}_t^B dt\right) + \frac{\gamma^B}{\sigma_s^2}\left(ds - \widehat{f}_t^B dt\right), \quad (6)$$

where:

$$\gamma^A \triangleq \frac{\sqrt{\left(\zeta + \frac{\phi\sigma_f}{\sigma_s}\right)^2 + \left(1-\phi^2\right)\sigma_f^2\left(\frac{1}{\sigma_s^2} + \frac{1}{\sigma_\delta^2}\right)} - \left(\zeta + \frac{\phi\sigma_f}{\sigma_s}\right)}{\frac{1}{\sigma_s^2} + \frac{1}{\sigma_\delta^2}}, \quad (7)$$

$$\gamma^B \triangleq \frac{\sqrt{\zeta^2 + \sigma_f^2\left(\frac{1}{\sigma_s^2} + \frac{1}{\sigma_\delta^2}\right)} - \zeta}{\frac{1}{\sigma_s^2} + \frac{1}{\sigma_\delta^2}}. \quad (8)$$

Overconfidence

- The number γ^A (γ^B) is the steady-state variance of f as estimated by group A (B).
- These would normally be deterministic functions of time.
- But we assume, as did Scheinkman and Xiong, that
  - there has been a sufficiently long period of learning Group A to converge to that level of variance, irrespective of their prior, while,
  - at the same time, they have refused to use the same information to infer the magnitude of correlation.
- That is the exact degree to which they are being irrational.

Financial markets

- Financial market is complete
  - Securities available:
    - Equity = claim on total output
    - Bond = claim on one unit of output in ten years
    - Short-term bank deposit or T bill (instantaneously riskless)

Static formulation of the model

- To obtain a "static" formulation, we rewrite the SDEs in terms of processes that are Brownian motions under different probability measures.
- Consider a two-dimensional process $W^B = (W_\delta^B, W_s^B)$ that is Brownian under the probability measure that reflects expectations of Group B.
- By the definition of $\widehat{f}^B$, we can then write:

$$\frac{d\delta_t}{\delta_t} = \widehat{f}_t^B dt + \sigma_\delta dW_{\delta,t}^B, \tag{9}$$

$$ds_t = \widehat{f}_t^B dt + \sigma_s dW_{s,t}^B, \tag{10}$$

$$d\widehat{f}_t^A = \left[-\zeta\left(\widehat{f}^A - \overline{f}\right) + \left(\frac{\gamma^A}{\sigma_\delta^2} + \frac{\phi\sigma_s\sigma_f + \gamma^A}{\sigma_s^2}\right)\left(\widehat{f}_t^B - \widehat{f}^A\right)\right] dt + \frac{\gamma^A}{\sigma_\delta^2}\sigma_\delta dW_{\delta,t}^B + \frac{\phi\sigma_s\sigma_f + \gamma^A}{\sigma_s^2}\sigma_s dW_{s,t}^B \tag{11}$$

$$d\widehat{f}_t^B = -\zeta\left(\widehat{f}^B - \overline{f}\right) dt + \frac{\gamma^B}{\sigma_\delta} dW_{\delta,t}^B + \frac{\gamma^B}{\sigma_s} dW_{s,t}^B. \tag{12}$$

- The system made of (9), (10), (11) and (12) completely characterizes the evolution of the economy in the eyes of population B

Change from B's measure to A's measure

- The viewpoint of population A will be handled by means the change from B's measure to A's measure, which is given by:

$$\eta_t = \exp\left(-\frac{1}{2}\int_0^t \|\nu\|^2 \, dt - \int_0^t \nu_t^{\intercal} dW_t^B\right), \tag{13}$$

or

$$\frac{d\eta_t}{\eta_t} = -\nu_t^{\intercal} dW_t^B, \tag{14}$$

where

$$\nu_t = \left(\widehat{f}_t^B - \widehat{f}_t^A\right)\begin{bmatrix} \frac{1}{\sigma_\delta} \\ \frac{1}{\sigma_s} \end{bmatrix}. \tag{15}$$

Difference of opinion

- Process for the **difference of opinion**, $\widehat{g} \triangleq \widehat{f}^B - \widehat{f}^A$:

$$d\widehat{g}_t = -\left(\zeta + \frac{\gamma^A}{\sigma_\delta^2} + \frac{\phi\sigma_s\sigma_f + \gamma^A}{\sigma_s^2}\right)\widehat{g}_t dt + \frac{\gamma^B - \gamma^A}{\sigma_\delta} dW_{\delta,t}^B + \frac{\gamma^B - \left(\phi\sigma_s\sigma_f + \gamma^A\right)}{\sigma_s} dW_{s,t}^B. \tag{16}$$

- When $\widehat{g} > 0$, Group B investors comparatively **optimistic**.
- Also, $\widehat{g}$ (or its absolute value) is a measure of the **dispersion** of beliefs.
  - Because $\gamma^B - (\phi\sigma_s\sigma_f + \gamma^A) < 0$, a positive realization of the signal increment $dW_{s,t}^B$ causes Population B to become less optimistic relative to what it was before.

State variables

- Four state variables $\left\{\delta, \eta, \widehat{f}^B, \widehat{g} \triangleq \widehat{f}^B - \widehat{f}^A\right\}$.
- Driven by only two Brownians, W_δ^B and W_s^B because f is unobserved.
- The four variables are not independent of each other; since only two Brownians, the diffusion matrix of $\left\{\delta, \eta, \widehat{f}^B, \widehat{g}\right\}$ is a 4×2 matrix:

$$\begin{bmatrix} \delta\sigma_\delta > 0 & 0 \\ -\eta\frac{\widehat{g}}{\sigma_\delta} & -\eta\frac{\widehat{g}}{\sigma_s} \\ \frac{\gamma^B}{\sigma_\delta} > 0 & \frac{\gamma^B}{\sigma_s} > 0 \\ \frac{\gamma^B - \gamma^A}{\sigma_\delta} \geq 0 & \frac{\gamma^B - \left(\phi\sigma_s\sigma_f + \gamma^A\right)}{\sigma_s} < 0 \end{bmatrix}. \tag{17}$$

Two effects of imperfect learning

- Two distinct effects of imperfect learning.
  1. The first effect is instantaneous: $\widehat{g}$ has nonzero diffusion.
  2. The second effect is cumulative: since $\widehat{g}$ is stochastic and conditions the diffusion of η, it implies that η has a diffusion that can take large positive or negative values.

$$\begin{array}{c} \delta \\ \eta \\ \widehat{f}^B \\ \widehat{g} \end{array} \left[\begin{array}{cc} \delta\sigma_\delta > 0 & 0 \\ -\eta\frac{\widehat{g}}{\sigma_\delta} & -\eta\frac{\widehat{g}}{\sigma_s} \\ \frac{\gamma^B}{\sigma_\delta} > 0 & \frac{\gamma^B}{\sigma_s} > 0 \\ \frac{\gamma^B - \gamma^A}{\sigma_\delta} \geq 0 & \frac{\gamma^B - \left(\phi\sigma_s\sigma_f + \gamma^A\right)}{\sigma_s} < 0 \end{array} \right].$$

Individual optimization and equilibrium

- Power utility with **same** risk aversion $1 - \alpha$ & impatience rate ρ.
- Problem of population B:

$$\sup_{c} \mathbb{E}^B \int_0^\infty e^{-\rho t} \frac{1}{\alpha} \left(c_t^B\right)^\alpha dt, \tag{18}$$

subject to the static budget constraint:

$$\mathbb{E}^B \int_0^\infty \xi_t^B c_t^B dt = \overline{\theta}^B \mathbb{E}^B \int_0^\infty \xi_t^B \delta_t dt, \tag{19}$$

where ξ^B is change of measure from B's measure to risk-neutralized measure; $\overline{\theta}^B$ is initial endowment of shares.

- The FOC for consumption equates marginal utility $e^{-\rho t} \left(c_t^B\right)^{\alpha-1}$ to $\lambda^B \xi_t^B$, where λ^B is the Lagrange multiplier of the budget constraint.

Optimization problem of Population A

- Problem of A: maximize expected utility from lifetime consumption:

$$\sup_c \mathbb{E}^A \int_0^\infty e^{-\rho t} \frac{1}{\alpha} \left(c_t^A\right)^\alpha dt, \tag{20}$$

subject to the static budget constraint:

$$\mathbb{E}^A \int_0^\infty \xi_t^A c_t^A dt = \theta^A \mathbb{E}^A \int_0^\infty \xi_t^A \delta_t dt, \tag{21}$$

where ξ^A is change from A's measure to risk-neutralized measure and θ^A is A's initial share of equity.

- FOC for consumption equates marginal utility $e^{-\rho t} \left(c_t^A\right)^{\alpha-1}$ to $\lambda^A \xi_t^A$, where λ^A is the Lagrange multiplier of the budget constraint (21).

Optimization problem of Population A under B's measure

Using B's probability measure as the reference measure, the problem of A can be restated as:

$$\sup_c \mathbb{E}^B \int_0^\infty \eta_t \times e^{-\rho t} \frac{1}{\alpha} \left(c_t^A\right)^\alpha dt, \tag{22}$$

subject to the static budget constraint:

$$\mathbb{E}^B \int_0^\infty \xi_t^B c_t^A dt = \overline{\theta}^A \mathbb{E}^B \int_0^\infty \xi_t^B \delta_t dt. \tag{23}$$

The first-order condition for consumption in this case is

$$\eta_t \times e^{-\rho t} \left(c_t^A\right)^{\alpha - 1} = \lambda^A \xi_t^B, \tag{24}$$

where λ^A is the Lagrange multiplier of the budget constraint (23).

Complete-market equilibrium

- **Definition**: An equilibrium is a price system and a pair of consumption-portfolio processes such that
  1. investors choose their optimal consumption-portfolio strategies, given their perceived price processes;
  2. the perceived security price processes are consistent across investors; and,
  3. commodity and securities markets clear.

- The aggregate resource constraint is:

$$\left(\frac{\lambda^A \xi_t^B e^{\rho t}}{\eta_t}\right)^{\frac{1}{\alpha-1}} + \left(\lambda^B \xi_t^B e^{\rho t}\right)^{\frac{1}{\alpha-1}} = \delta_t. \tag{25}$$

Pricing measure and consumption-sharing rule

$$\xi_t^B e^{\rho t} = \left[\frac{\left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}{\delta_t} \right]^{1-\alpha}, \tag{26}$$

$$c_t^A = \delta_t \times \frac{\left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}, \quad c_t^B = \delta_t \times \frac{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}. \tag{27}$$

- The consumption-sharing rule is **linear** in δ because both groups have the same risk aversion.
- But the **slope** of the linear relation (the share of consumption allocated to each group) is stochastic because of the improper use of signal by individuals of Group A.

Solving for equilibrium

- Eqs. (26) & (27) allow us to solve for pricing measure and consumption as a function of δ_t, and current value of change of measure, η_t.
- Given the constant multipliers λ^A and λ^B, and given exogenous process for δ and η, we have now characterized the complete-market equilibrium.
- To relate the Lagrange multipliers λ^A and λ^B to initial endowments. requires the calculation of the wealth of each group.
- Four state variables in the Markovian recursive formulation: $\left\{\delta, \eta, \widehat{f}^B, \widehat{g} \triangleq \widehat{f}^B - \widehat{f}^A\right\}$

Results: Interest rate and market prices of risk

The risk-neutralized measure for Population i, ξ_t^i:

$$\xi_t^i = \delta_0^{\alpha-1} \exp\left(-\int_0^t r dt - \frac{1}{2}\int_0^t \left\|\kappa^i\right\|^2 dt - \int_0^t \left(\kappa^i\right)^\intercal dW^i\right). \tag{28}$$

Proposition 1 *In equilibrium, the instantaneous interest rate,* $r\left(\eta, \widehat{f}^B, \widehat{g}\right)$, *is:*

$$\begin{aligned} &= \rho + (1-\alpha)\widehat{f}^B - \frac{1}{2}(1-\alpha)(2-\alpha)\sigma_\delta^2 - (1-\alpha)\widehat{g} \times \frac{\left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}} \\ &-\frac{1}{2}\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{\sigma_\delta^2} + \frac{1}{\sigma_s^2}\right)\widehat{g}^2 \times \frac{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}\left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left[\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}\right]^2}, \end{aligned} \tag{29}$$

and the market prices of risk in the eyes of Population B and A are:

$$\kappa^{B}(\eta,\widehat{g}) = \begin{bmatrix} (1-\alpha)\,\sigma_\delta \\ 0 \end{bmatrix} + \widehat{g} \times \frac{\left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}} \begin{bmatrix} \frac{1}{\sigma_\delta} \\ \frac{1}{\sigma_s} \end{bmatrix}, \tag{30}$$

$$\kappa^{A}(\eta,\widehat{g}) = \kappa^{B}(\eta,\widehat{g}) - \widehat{g} \times \begin{bmatrix} \frac{1}{\sigma_\delta} \\ \frac{1}{\sigma_s} \end{bmatrix}. \tag{31}$$

- The interest rate is an increasing function of $\widehat{f}^B$.
- Effect of $\widehat{g}$ on interest rate is nonmonotonic and asymmetric, because $\widehat{g}$ affects both the mean of $\widehat{f}^A$ and $\widehat{f}^B$, and also the dispersion.
- Under agreement ($\widehat{g} = 0$), the prices of risk include a reward for output risk W_δ, but zero reward for signal risk W_s.
- With disagreement, both sets of investors realize that probability measure of other population will fluctuate randomly. Hence, they start charging a risk premium for vagaries of others.

Stock price, bond price, and wealth

- The equilibrium stock price, using the pricing measure of B is:

$$F\left(\delta,\eta,\widehat{f}^{B},\widehat{g},t\right) \triangleq \frac{1}{\xi_t^B}\mathbb{E}^B_{\delta,\eta,\widehat{f}^B,\widehat{g}}\int_t^\infty \xi_u^B \delta_u du. \tag{32}$$

$$= \delta^{1-\alpha}\mathbb{E}^B_{\delta,\eta,\widehat{f}^B,\widehat{g}}\int_t^\infty e^{-\rho(u-t)}\left[\frac{\left(\frac{\eta_u}{\lambda^A}\right)^{\frac{1}{1-\alpha}}+\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}}+\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}\right]^{1-\alpha}\delta_u^\alpha du.$$

- Wealth of Population B:

$$F^B\left(\delta,\eta,\widehat{f}^{B},\widehat{g},t\right) \triangleq \frac{1}{\xi_t^B}\mathbb{E}^B_{\delta,\eta,\widehat{f}^B,\widehat{g}}\int_t^\infty \xi_u^B c_u^B du \tag{33}$$

Computing the expected values to obtain prices and wealth

- To compute the expected values in (32) and (33), we need the joint conditional distribution of η_u and δ_u, given $\delta_t, \eta_t, \widehat{f}_t^A, \widehat{g}_t$ at t.

- The joint distribution is not easy to obtain but its **characteristic function** $\mathbb{E}^B_{\delta,\eta,\widehat{f}^B,\widehat{g}}\left[(\delta_u)^{\varepsilon}(\eta_u)^{\chi}\right]; \varepsilon, \chi \in \mathbb{C}$ can be obtained in closed form.

- We provide these closed-form expressions in the appendix.

Effect of irrationality on asset prices

- To illustrate effect of irrationality on securities prices, we specify numerical values for parameters of the model.

- The parameter values that we specify are based on estimation of models similar to ours in Brennan-Xia (2001) and Berrada (2004).

- All plots have two curves:
  - A red-dotted curve representing the case of $\phi = 0.00$
  - A blue-dashed curve representing the case of $\phi = 0.95$

Name	Symbol	Value
Parameters for aggregate endowment and the signal		
Long-term average growth rate of aggregate endowment	$\bar{f}$	0.015
Volatility of expected growth rate of endowment	σ_f	0.02
Volatility of aggregate endowment	σ_δ	0.13
Mean reversion parameter	ζ	0.10
Volatility of the signal (our choice)	σ_s	0.10
Parameters for the agents		
Agent A's correlation between signal and mean growth rate	ϕ	0.95
Agent B's correlation between signal and mean growth rate	—	0
Agent A's initial share of aggregate endowment	θ^A	0.50
Time-preference parameter for both agents	ρ	0.11
Relative risk aversion for both agents	$1-\alpha$	3

Average belief vs. dispersion of belief

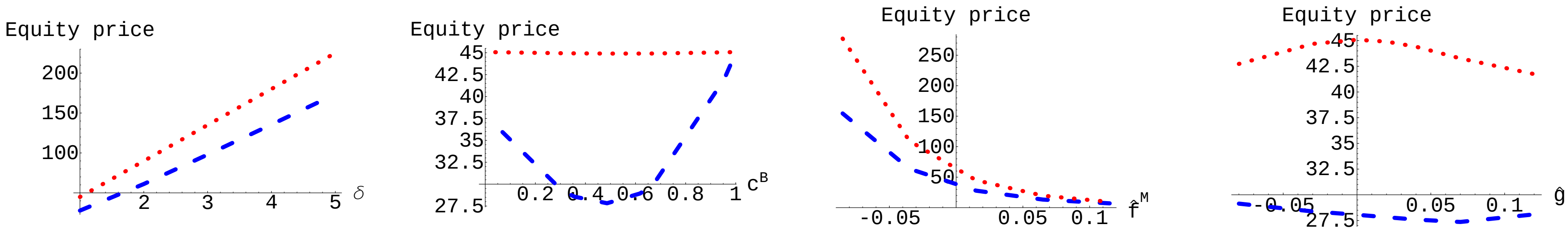
- The variable $\widehat{g}$ contributes both to the **average** of $\widehat{f}^A$ and $\widehat{f}^B$ and also to the**dispersion** between them.

- For purposes of interpretation and exposition, it is clearer to define $\widehat{f}^M$, the **population average belief** (where the weights are each population's share of consumption):

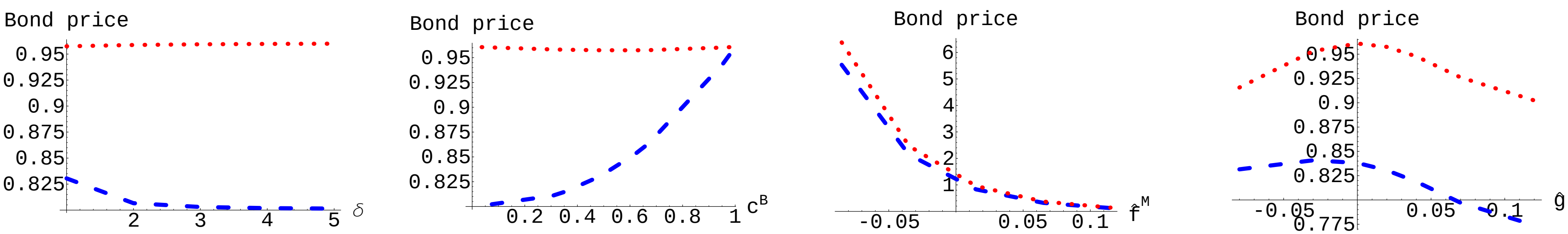
$$\widehat{f}^M \triangleq \frac{\widehat{f}^B \times \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \widehat{f}^A \times \left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}} \tag{34}$$

- So, we use as state variables: $\left\{\delta, \eta, \widehat{f}^M, \widehat{g}\right\}$

Stock Price

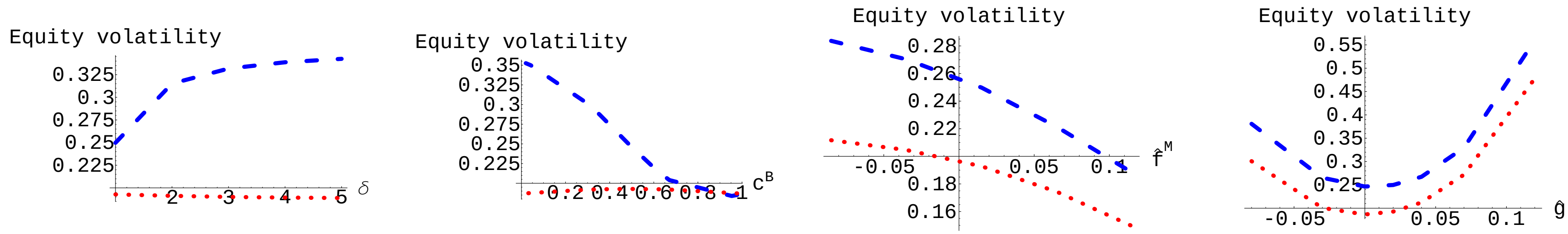


Bond Price

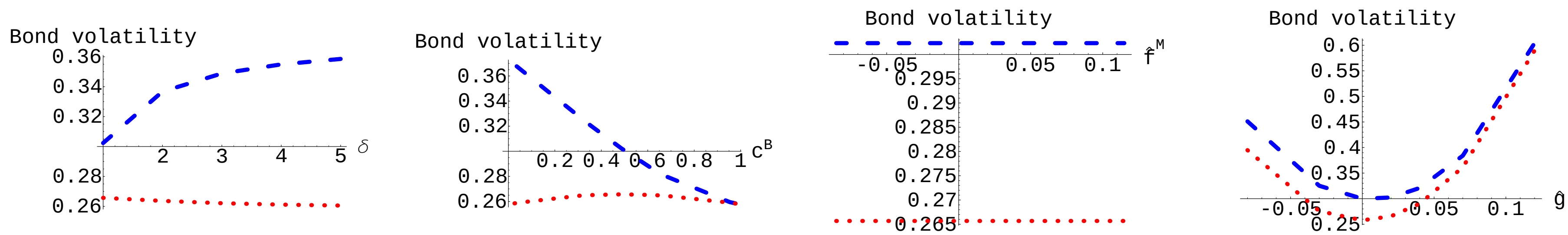


- Price levels are reduced by presence of irrational traders, because irrational traders add “noise” for which traders require a risk premium.
- Prices decrease as $\widehat{f}^M$ increases, because of decrease in marginal utility.

Stock Volatility

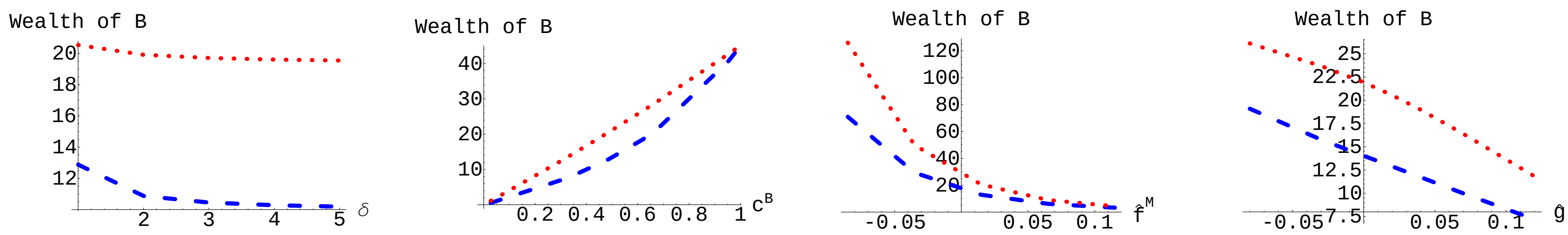


Bond Volatility



- Irrational agents create "noise" that raises stock & bond return volatility.
- The increase is moderate when $\widehat{g} = 0$, but without bound for $\widehat{g} \neq 0$
- As relative wealth of B decreases, volatility increases.

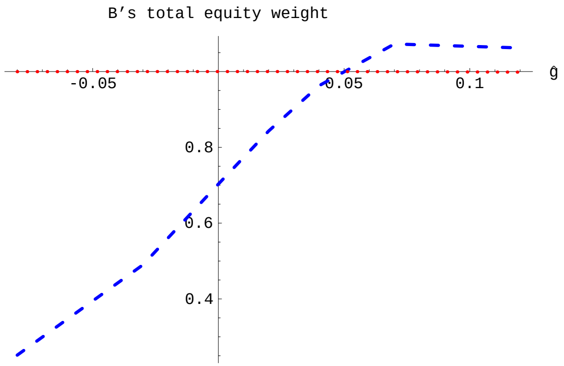
Wealth of Population B

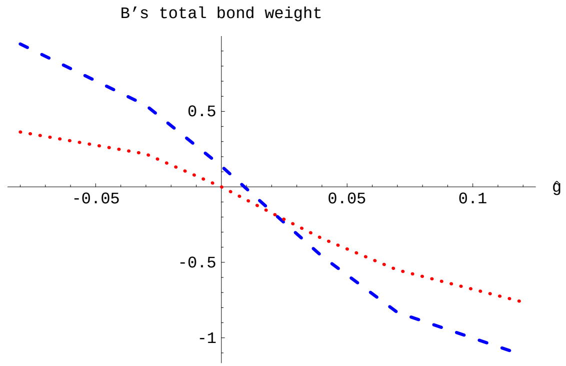


- As was the case for equity, a rise in $\widehat{f}^M$ decreases wealth (third plot).

- The second plot shows that the wealth of Population B rises monotonically with its consumption c^B.

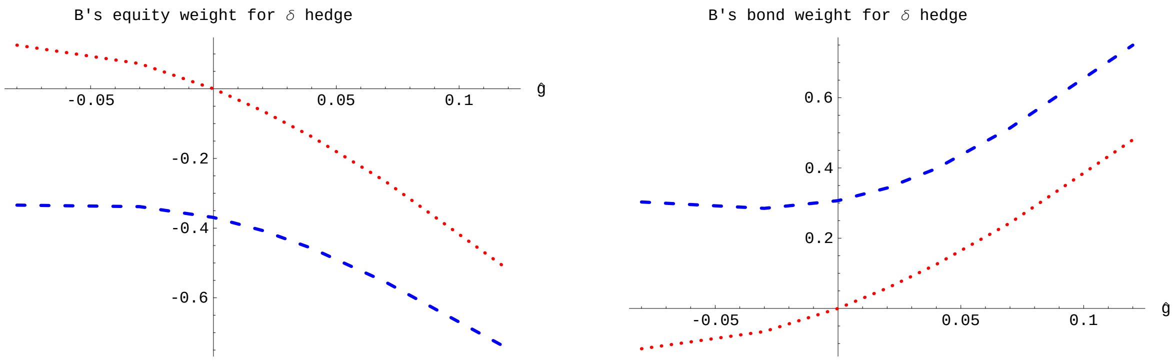
Total portfolio holdings of Population B





- Under rationality of A and agreement ($\widehat{g} = 0$), there A and B are the same: Hence, both are 100% in equity, with 0% in bonds.
- Under rationality (red dotted line) but with $\widehat{g} \neq 0$, B continues to be 100% in equity; but when it is optimistic ($\widehat{g} > 0$), it shorts the bond (and invests proceeds in short-term deposit). This is because future expected growth rate is deterministic.
- Under irrationality (blue dashed line), B holds less than it would be in a rational market, unless he/she is extremely optimistic.
- Because stocks and bonds are positively correlated, the investor compensates the smaller weight in equity with a larger weight on bonds.

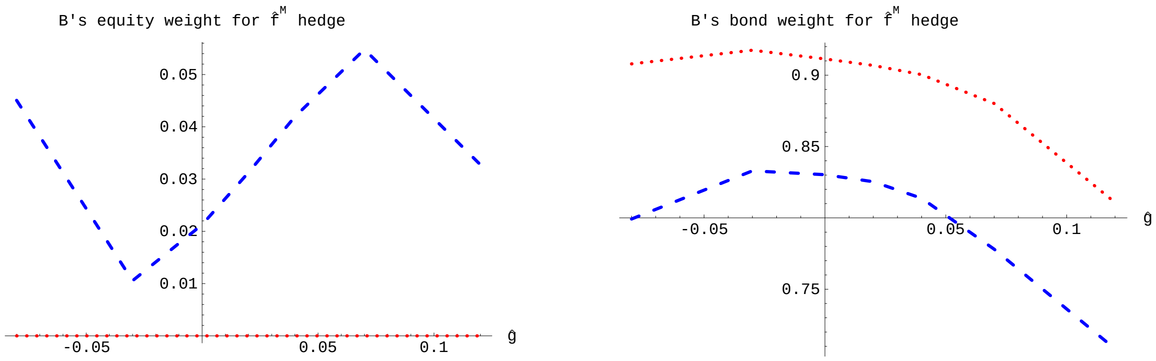
Population B's portfolio to hedge changes in δ



- Under rationality, the hedge against δ risk is zero when $\widehat{g} = 0$ and changes sign with $\widehat{g}$.

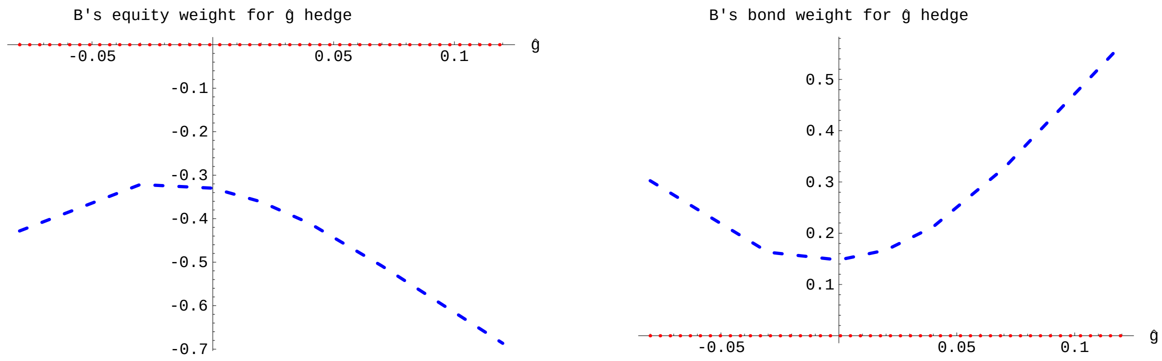
- An increase in δ reduces the value of B's wealth and raises consumption and so is "good news". The hedge is, therefore, to hold equity negatively and to hold the bond positively.

Population B's portfolio to hedge changes in $\widehat{f}^M$



- Under rationality, the hedge against $\widehat{f}^M$ contains no equity: It is entirely made up of the bond because the bond is exposed to $\widehat{f}^M$ risk exclusively

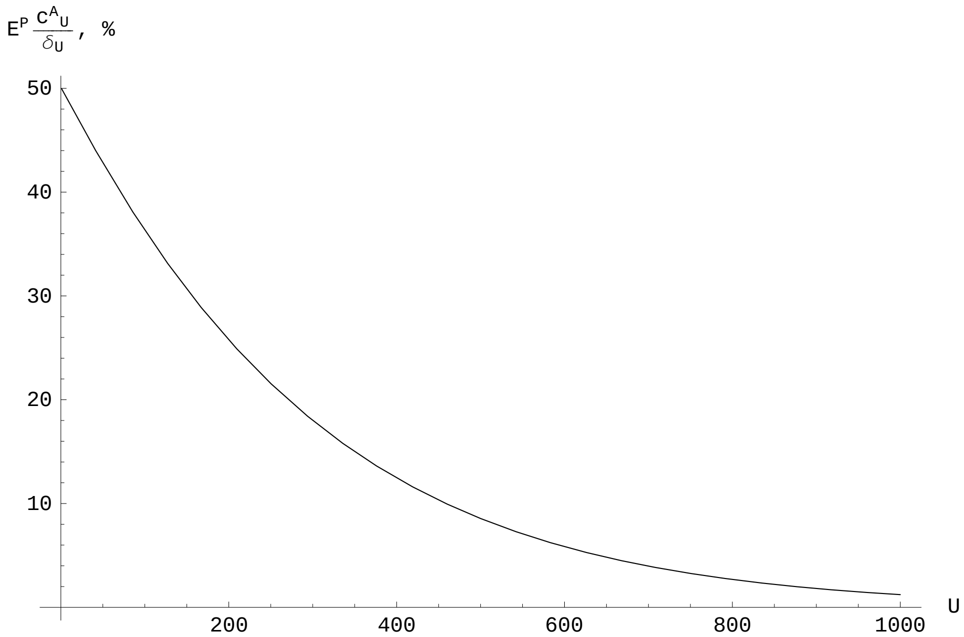
Population B's portfolio to hedge changes in $\widehat{g}$



- Under rationality, the hedge against changes in $\widehat{g}$ is always zero.

- Even when investors happen to agree today about the expected rate of growth of dividends, they know that they will almost certainly disagree tomorrow, and so will wish to hedge this.

Survival of irrational (overconfident) agents



- This figure shows expected value of Population A's consumption share as a function of time measured in years.
- Even after 200 years the overconfident agents consume 25% of the aggregate dividends.

Conclusions

- We have demonstrated that a rational risk-arbitrageur finds it beneficial to trade on his/her belief that the market is being foolish.
  - When doing so, however, must hedge future fluctuations in market's foolishness.
  - Illustrates that risk arbitrage cannot be based just on current price divergence; must also include a protection in case there is a deviation from that prediction.
  - So strategy must be based on a model of irrational behavior and a prediction concerning the speed of convergence.
- The optimal strategy for the rational risk-arbitrageur is:
  - holding less than the entire equity market;
  - shorting bonds—an important part of the portfolio;
  - putting the residual in cash.
- Irrational investors are not eliminated quickly; excess volatility persists.