An International Examination of Affine Term Structure Models and the Expectations Hypothesis

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Current Literature

- Well-documented stylized results of US yield curve
  - Rejection of expectations hypothesis
  - Predictive power of forward rates for bond returns
  - See, e.g., Campbell and Shiller (1991)
- Extensive study of affine term structure models
  - Extensive theoretical studies
  - Large literature of empirical examinations of US yield data
  - See, e.g., Dai and Singleton (2003) and Piazzesi (2003) for surveys of literature.
- But very few results on the behavior of foreign yield curve due to lack of foreign data

Objective of the Current Paper

- Examine the yield curve behavior in CA, GM, JP, UK and US
  - the expectations hypothesis (EH)
  - the predictive power of forward rates for bond returns
- Compare all affine term structure models in the five countries
  - Estimate all 3-factor models
  - Nested model comparison
  - Non-nested model comparison
- Check whether the best model can generate yield curve behavior in the data

Motivation - 1

- Why re-visit the expectations hypothesis in foreign data?
  - Provide good out-of-sample robustness check
  - Extend the few earlier studies with foreign data
    - earlier studies used only a subset (usually the short end) of the yield curve
    - earlier studies used data before 1992 in most cases
    - earlier studies generated mixed results: mostly opposite to the US findings
  - Serve as benchmark to examine dynamic affine term structure models

Motivation - 2

- Why extend the affine term structure model comparison to foreign data?
  - Fill a gap in the literature
  - Provide additional evidence on the performance of the model
  - An international comparison provides more information than that using a single yield curve:
    * The 5 countries have different monetary policies, different government bond markets, different experiences in interest rates and inflations
    * CA, UK, and US: high inflation in 1980s
    * JP: steady decline in asset values and the worry of deflation
    * GM: low interest rate and low inflation, unification, Euro

Summary of Results

- Three-factor models fit the data well in CA, JP, UK and US
- A model with regime shifts will likely improve the fit of GM yield curve
- The EH is rejected in all five countries
- Strongest predictive power of forward rates in US, followed by JP and GM, and then UK and CA
- A three-factor essential affine model performs the best in all five countries
- Simulation results are inconclusive due to large standard errors in parameter estimates

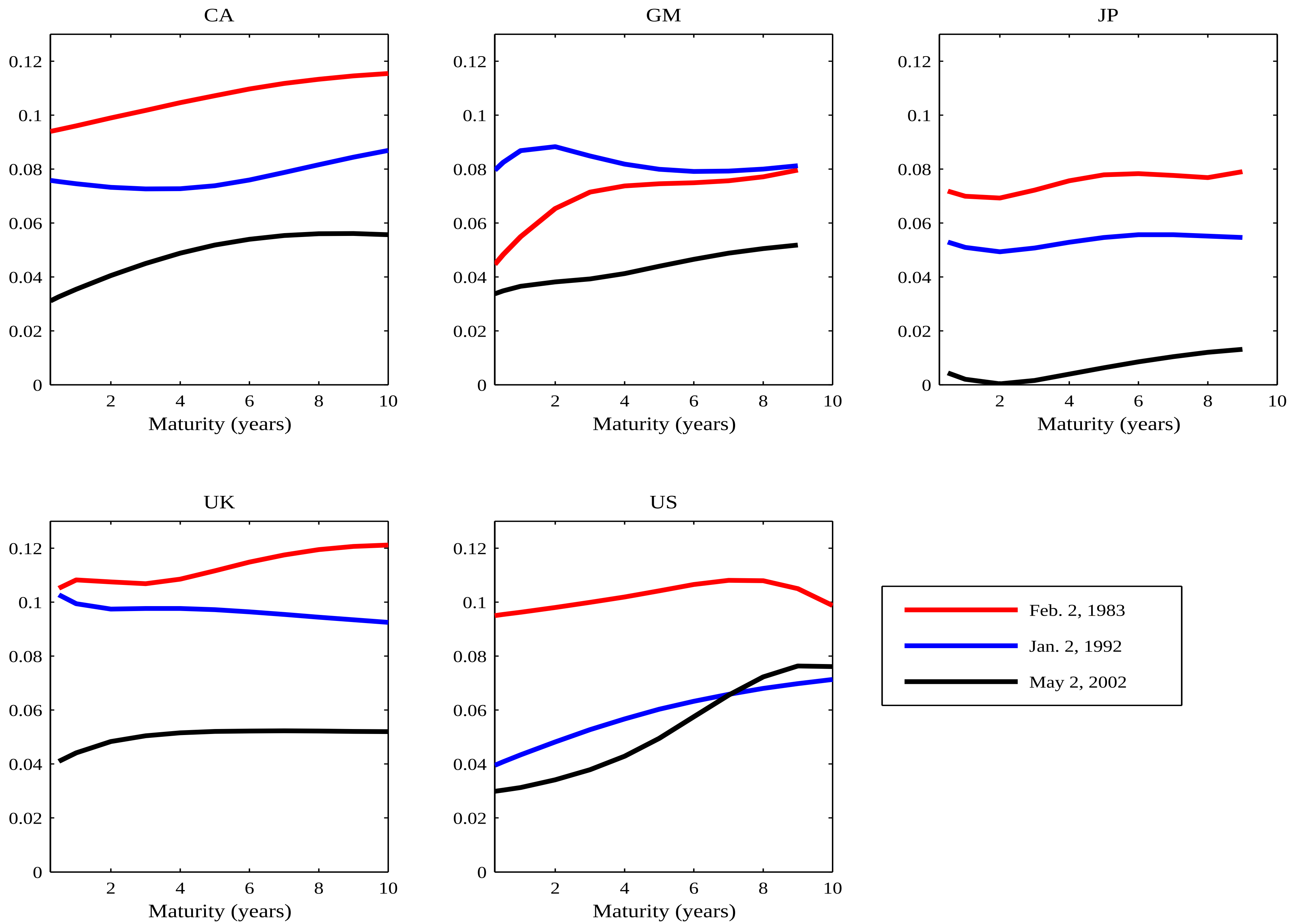
- **Thus, despite the differences in the 5 countries, the results are strikingly similar and consistent.**

Roadmap

- Yield Curve Data
- Yield Curve Behavior
- Affine Term Structure Models
- Model Estimation and Comparison
- Simulation results

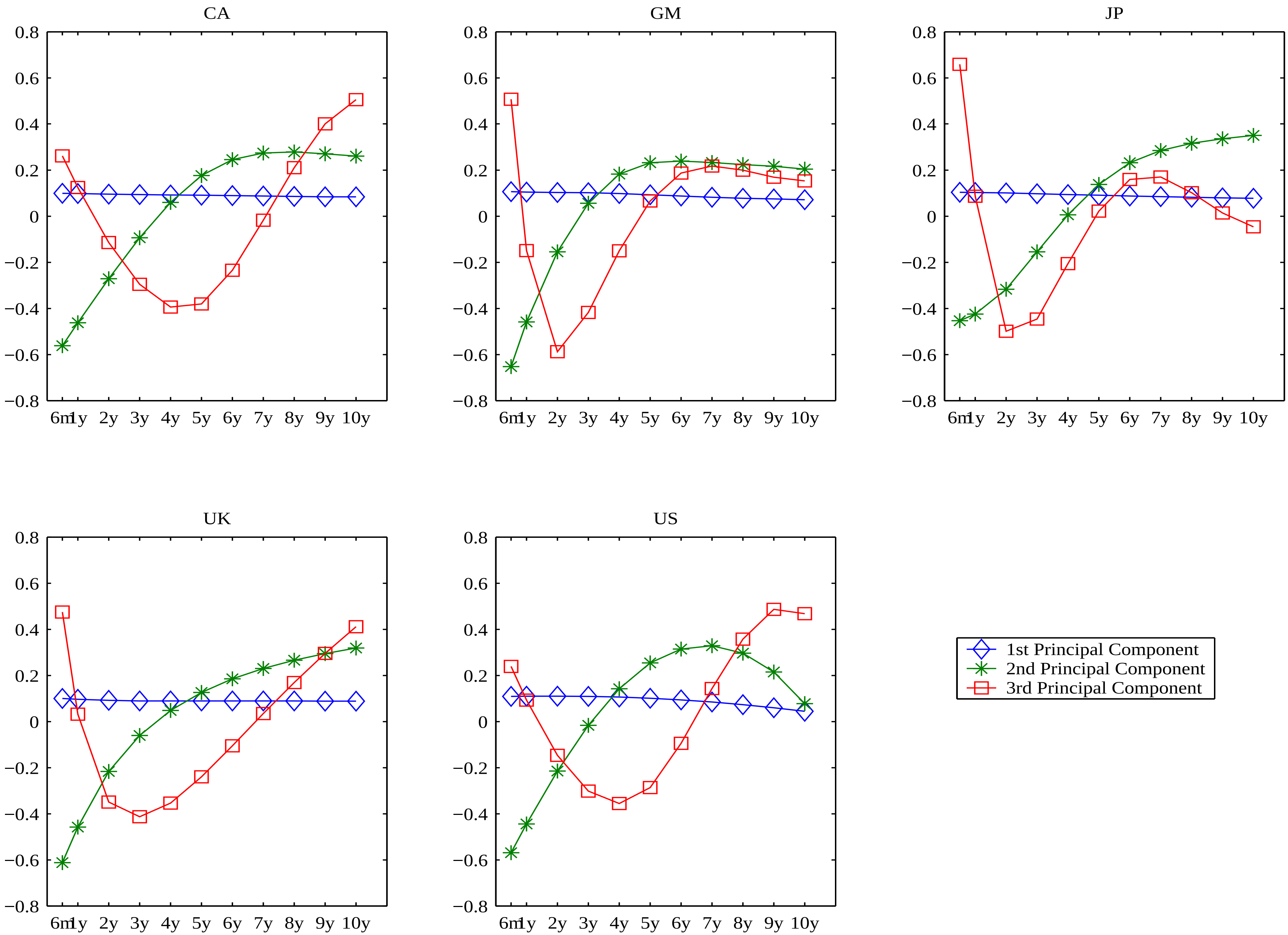
Yield Data

- Zero-coupon constant maturity nominal government bond yields
- Constructed from coupon bonds using a cubic spline at monthly frequency
- January 1983 to May 2002 (CA, JP and GM), January 2003 (US), and February 2003 (UK)
- Mean (standard deviation) yields tend to increase (decrease) with maturity in all countries
- All yields highly persistent
- High cross-correlation $\longrightarrow$ small number of factors driving the curve
  - highest in JP, and lowest in GM


CA
GM
JP
UK
US
0.12
0.1
0.08
0.06
0.04
0.02
0
2
4
6
8
10
Maturity (years)
Feb. 2, 1983
Jan. 2, 1992
May 2, 2002

Principal Component Analysis

Country	PC1	PC2	PC3	PC4	PC5	$\sum_{i=1}^{3} PC_i$
CA	95.3457	4.0966	0.5468	0.0098	0.0009	99.9892
GM	82.8400	15.6411	1.1409	0.2308	0.1108	99.6219
JP	98.2324	1.4424	0.2056	0.1097	0.0074	99.8804
UK	95.8408	3.8860	0.2449	0.0225	0.0054	99.9717
US	93.2608	3.9021	2.7467	0.0888	0.0013	99.9097


CA
GM
JP
UK
US
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
6m
1y
2y
3y
4y
5y
6y
7y
8y
9y
10y
1st Principal Component
2nd Principal Component
3rd Principal Component

Expectations Hypothesis

- EH: long yield equals the average of expected future short rate & a constant term premium

$$y_t^\tau = \alpha_\tau + \frac{1}{\tau}\sum_{k=0}^{\tau-1} \mathrm{E}_t\left[y_{t+12k}^1\right] \tag{1}$$

- Two empirical tests: equivalent under the null
  - Test 1: Slope of the yield curve forecasts the change in long term yield with $H_0 : a_1 = 1$

$$y_{t+12}^{\tau-1} - y_t^\tau = a_0 + a_1 \frac{y_t^\tau - y_t^1}{\tau - 1} + \epsilon, \quad \tau = 2, \cdots, 10. \tag{2}$$

  - Test 2: Slope of the yield curve forecasts the average change in the future short rates with $H_0 : b_1 = 1$

$$\frac{1}{\tau}\sum_{k=1}^{\tau-1}\left[y_{t+12k}^1 - y_t^1\right] = b_0 + b_1\left[y_t^\tau - y_t^1\right] + \epsilon, \quad \tau = 2, \cdots, 10. \tag{3}$$

Results from Test 1

Country	**Bond Maturity** τ								
	2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
CA	**0.669**	**0.384**	**0.180**	**0.081**	**0.081**	**-0.027**	**-0.316**	**-0.725**	**-1.140**
	(0.33)	**(0.32)**	**(0.31)**	**(0.33)**	**(0.36)**	**(0.40)**	**(0.45)**	**(0.51)**	**(0.56)**
	[0.70]	**[0.69]**	**[0.69]**	**[0.73]**	**[0.80]**	**[0.90]**	**[1.00]**	**[1.11]**	**[1.21]**
$\bar{R}^2$	**1.4**	**0.2**	**-0.3**	**-0.4**	**-0.4**	**-0.5**	**-0.2**	**0.5**	**1.4**
GM	**-0.041**	**-0.391**	**-0.481**	**-0.425**	**-0.435**	**-0.543**	**-0.679**	**-0.871**	
	(0.13)	**(0.16)**	**(0.17)**	**(0.19)**	**(0.22)**	**(0.25)**	**(0.29)**	**(0.33)**	
	[0.25]	**[0.28]**	**[0.33]**	**[0.36]**	**[0.40]**	**[0.47]**	**[0.56]**	**[0.65]**	
$\bar{R}^2$	**-0.4**	**2.4**	**3.0**	**1.7**	**1.4**	**1.7**	**2.0**	**2.6**	

Results from Test 1 (continued)

Country	Bond Maturity τ								
	2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
JP	**0.225**	**-0.662**	**-1.717**	**-2.268**	**-2.451**	**-2.362**	**-2.177**	**-1.982**	
	(0.33)	**(0.49)**	**(0.50)**	**(0.49)**	**(0.50)**	**(0.52)**	**(0.54)**	**(0.55)**	
	[0.64]	**[1.07]**	**[1.19]**	**[1.20]**	**[1.21]**	**[1.25]**	**[1.30]**	**[1.32]**	
$\bar{R}^2$	**-0.3**	**0.4**	**4.7**	**8.5**	**9.5**	**8.1**	**6.5**	**5.2**	
UK	**0.009**	**-0.033**	**-0.041**	**-0.053**	**-0.086**	**-0.150**	**-0.248**	**-0.380**	**-0.545**
	(0.22)	**(0.25)**	**(0.29)**	**(0.32)**	**(0.35)**	**(0.37)**	**(0.40)**	**(0.42)**	**(0.44)**
	[0.43]	**[0.49]**	**[0.56]**	**[0.64]**	**[0.72]**	**[0.80]**	**[0.87]**	**[0.94]**	**[1.01]**
$\bar{R}^2$	**-0.4**	**-0.4**	**-0.4**	**-0.4**	**-0.4**	**-0.4**	**-0.3**	**-0.1**	**0.2**
US	**-0.031**	**-0.260**	**-0.414**	**-0.406**	**-0.235**	**-0.222**	**-0.278**	**-0.238**	**-0.364**
	(0.28)	**(0.29)**	**(0.32)**	**(0.38)**	**(0.47)**	**(0.54)**	**(0.56)**	**(0.52)**	**(0.46)**
	[0.62]	**[0.64]**	**[0.70]**	**[0.82]**	**[0.96]**	**[1.02]**	**[1.05]**	**[1.07]**	**[0.97]**
$\bar{R}^2$	**-0.4**	**-0.1**	**0.3**	**0.1**	**-0.3**	**-0.4**	**-0.3**	**-0.4**	**-0.2**

Results from Test 2

Country	**Bond Maturity τ**								
	2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
CA	**0.834**	**0.832**	**0.896**	**0.976**	**1.000**	**1.043**	**1.087**	**1.128**	**1.144**
	(0.17)	**(0.12)**	**(0.09)**	**(0.08)**	**(0.07)**	**(0.07)**	**(0.07)**	**(0.07)**	**(0.08)**
	[0.35]	**[0.24]**	**[0.18]**	**[0.14]**	**[0.11]**	**[0.11]**	**[0.11]**	**[0.10]**	**[0.11]**
$\bar{R}^2$	**9.9**	**18.2**	**31.2**	**46.4**	**56.3**	**60.3**	**61.6**	**65.3**	**61.8**
GM	**0.479**	**0.526**	**0.677**	**1.298**	**1.401**	**1.453**	**1.504**	**1.530**	
	(0.07)	**(0.07)**	**(0.08)**	**(0.10)**	**(0.09)**	**(0.09)**	**(0.08)**	**(0.07)**	
	[0.13]	**[0.13]**	**[0.16]**	**[0.18]**	**[0.16]**	**[0.16]**	**[0.14]**	**[0.13]**	
$\bar{R}^2$	**19.3**	**21.6**	**28.1**	**46.0**	**57.2**	**62.6**	**69.6**	**78.3**	

Results from Test 2 (continued)

Country	Bond Maturity τ								
	2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
JP	**0.612**	**0.690**	**0.426**	**0.371**	**0.443**	**0.622**	**0.942**	**1.467**	
	(0.17)	**(0.20)**	**(0.18)**	**(0.16)**	**(0.15)**	**(0.15)**	**(0.16)**	**(0.15)**	
	[0.32]	**[0.41]**	**[0.39]**	**[0.33]**	**[0.30]**	**[0.31]**	**[0.38]**	**[0.30]**	
$\bar{R}^2$	**5.3**	**5.1**	**2.2**	**2.3**	**4.4**	**9.0**	**17.8**	**41.7**	
UK	**0.505**	**0.637**	**0.755**	**0.872**	**0.981**	**1.041**	**1.125**	**1.162**	**1.250**
	(0.11)	**(0.10)**	**(0.10)**	**(0.09)**	**(0.08)**	**(0.08)**	**(0.07)**	**(0.07)**	**(0.07)**
	[0.22]	**[0.16]**	**[0.17]**	**[0.16]**	**[0.16]**	**[0.15]**	**[0.14]**	**[0.13]**	**[0.13]**
$\bar{R}^2$	**8.1**	**15.2**	**22.9**	**32.2**	**42.6**	**51.2**	**60.2**	**67.1**	**73.3**
US	**0.485**	**0.715**	**0.786**	**1.143**	**1.414**	**1.387**	**1.369**	**1.299**	**1.114**
	(0.14)	**(0.12)**	**(0.14)**	**(0.13)**	**(0.11)**	**(0.09)**	**(0.07)**	**(0.06)**	**(0.04)**
	[0.31]	**[0.26]**	**[0.27]**	**[0.24]**	**[0.16]**	**[0.15]**	**[0.14]**	**[0.10]**	**[0.06]**
$\bar{R}^2$	**4.5**	**13.1**	**13.4**	**29.3**	**47.3**	**57.4**	**68.8**	**77.9**	**82.5**

Predictive Power of Forward Rates

- EH $\Longrightarrow$ excess bond return is not predictable
- Strong predictive power of forward rates found in US data
- Define excess bond return as

$$r_{t+12}^{\tau} \equiv \ln P_{t+12}(\tau - 1) - \ln P_t(\tau) - y_t^1, \quad \forall \tau \geq 2.$$

- Define the forward rates

$$f_t(0,1) = -\ln P_t(1) = y_t^1, \quad \text{and} \quad f_t\,(i-1,i) \equiv \ln P_t(i-1) - \ln P_t(i), \quad \forall i \geq 2.$$

- Test

$$r_{t+12}^{\tau} = c_0 + c_1 f_t(0,1) + c_2 f_t(2,3) + c_3 f_t(8,9) + e_t, \quad \tau = 2, \ldots, 10. \qquad (4)$$

Predictive Regression Results (CA)

	Bond Maturity τ								
Coefficient	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
c_0	**-0.007**	**-0.015**	**-0.024**	**-0.036**	**-0.049**	**-0.062**	**-0.076**	**-0.090**	**-0.102**
	[0.71]	**[0.90]**	**[1.12]**	**[1.37]**	**[1.60]**	**[1.76]**	**[1.89]**	**[1.98]**	**[2.01]**
c_1	**-0.112**	**-0.317**	**-0.560**	**-0.771**	**-0.898**	**-1.071**	**-1.343**	**-1.706**	**-2.136**
	[0.54]	**[0.82]**	**[1.03]**	**[1.14]**	**[1.13]**	**[1.18]**	**[1.32]**	**[1.52]**	**[1.73]**
c_2	**0.163**	**0.463**	**0.802**	**1.044**	**1.053**	**0.997**	**0.991**	**1.108**	**1.422**
	[0.89]	**[1.39]**	**[1.76]**	**[1.85]**	**[1.59]**	**[1.31]**	**[1.13]**	**[1.11]**	**[1.26]**
c_3	**0.097**	**0.154**	**0.216**	**0.354**	**0.639**	**1.022**	**1.458**	**1.866**	**2.135**
	[0.68]	**[0.59]**	**[0.60]**	**[0.78]**	**[1.18]**	**[1.62]**	**[2.02]**	**[2.29]**	**[2.36]**
$\bar{R}^2$	**2.6**	**4.6**	**6.8**	**8.1**	**8.2**	**8.7**	**10.1**	**11.7**	**13.0**

Predictive Regression Results (GM)

Coefficient	Bond Maturity τ								
	2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
c_0	**-0.013**	**-0.032**	**-0.037**	**-0.033**	**-0.033**	**-0.047**	**-0.079**	**-0.129**	
	[1.55]	**[1.81]**	**[1.41]**	**[1.01]**	**[0.88]**	**[1.05]**	**[1.52]**	**[2.21]**	
c_1	**-0.229**	**-0.455**	**-0.677**	**-0.836**	**-0.964**	**-1.084**	**-1.206**	**-1.369**	
	[2.32]	**[2.38]**	**[2.51]**	**[2.62]**	**[2.71]**	**[2.70]**	**[2.67]**	**[2.73]**	
c_2	**0.452**	**0.966**	**1.263**	**1.261**	**1.202**	**1.172**	**1.085**	**0.875**	
	[4.16]	**[4.31]**	**[3.89]**	**[3.19]**	**[2.61]**	**[2.18]**	**[1.77]**	**[1.30]**	
c_3	**0.038**	**0.086**	**0.131**	**0.280**	**0.523**	**0.897**	**1.544**	**2.576**	
	[0.21]	**[0.22]**	**[0.21]**	**[0.36]**	**[0.56]**	**[0.81]**	**[1.21]**	**[1.82]**	
$\bar{R}^2$	**27.5**	**33.4**	**32.0**	**26.7**	**23.2**	**21.7**	**22.4**	**25.9**	

Predictive Regression Results (JP)

	Bond Maturity τ								
Coefficient	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
c_0	**0.003**	**0.014**	**0.021**	**0.025**	**0.029**	**0.028**	**0.021**	**0.011**	
	[1.07]	**[2.30]**	**[2.27]**	**[2.05]**	**[1.88]**	**[1.50]**	**[0.99]**	**[0.48]**	
c_1	**-0.775**	**-1.903**	**-3.117**	**-4.150**	**-4.825**	**-5.220**	**-5.554**	**-5.807**	
	[3.60]	**[4.57]**	**[5.33]**	**[5.67]**	**[5.57]**	**[5.40]**	**[5.39]**	**[5.33]**	
c_2	**1.339**	**3.162**	**4.920**	**6.291**	**7.109**	**7.444**	**7.593**	**7.536**	
	[5.75]	**[6.82]**	**[7.49]**	**[7.70]**	**[7.57]**	**[7.40]**	**[7.32]**	**[6.88]**	
c_3	**-0.503**	**-1.192**	**-1.700**	**-2.023**	**-2.197**	**-2.106**	**-1.783**	**-1.303**	
	[4.11]	**[4.88]**	**[4.68]**	**[4.31]**	**[3.97]**	**[3.44]**	**[2.72]**	**[1.87]**	
$\bar{R}^2$	**28.6**	**34.8**	**36.3**	**34.6**	**30.5**	**26.1**	**23.5**	**22.5**	

Predictive Regression Results (UK)

Coefficient	Bond Maturity τ								
	2	3	4	5	6	7	8	9	10
c_0	0.000	-0.009	-0.020	-0.030	-0.039	-0.045	-0.047	-0.047	-0.043
	[0.02]	[0.84]	[1.23]	[1.44]	[1.53]	[1.52]	[1.43]	[1.28]	[1.08]
c_1	-0.439	-0.779	-1.058	-1.293	-1.485	-1.635	-1.747	-1.827	-1.883
	[2.77]	[2.82]	[2.82]	[2.76]	[2.65]	[2.50]	[2.34]	[2.18]	[2.04]
c_2	0.974	1.980	2.779	3.327	3.623	3.689	3.551	3.236	2.772
	[3.69]	[4.23]	[4.39]	[4.32]	[4.06]	[3.68]	[3.22]	[2.69]	[2.14]
c_3	-0.501	-1.022	-1.380	-1.534	-1.500	-1.309	-0.991	-0.572	-0.072
	[2.52]	[2.92]	[2.94]	[2.69]	[2.27]	[1.75]	[1.20]	[0.63]	[0.07]
$\bar{R}^2$	13.7	16.6	17.6	17.2	15.9	14.0	12.1	10.3	8.9

Predictive Regression Results (US)

Coefficient	Bond Maturity τ 2	3	4	5	6	7	8	9	10
c_0	**-0.087**	**-0.160**	**-0.228**	**-0.299**	**-0.377**	**-0.456**	**-0.524**	**-0.571**	**-0.583**
	[5.06]	**[5.19]**	**[5.24]**	**[5.41]**	**[5.85]**	**[6.28]**	**[6.44]**	**[6.31]**	**[5.85]**
c_1	**-0.632**	**-1.286**	**-1.922**	**-2.462**	**-2.855**	**-3.267**	**-3.775**	**-4.438**	**-5.293**
	[3.36]	**[3.98]**	**[4.43]**	**[4.60]**	**[4.56]**	**[4.58]**	**[4.73]**	**[5.24]**	**[6.40]**
c_2	**1.200**	**2.310**	**3.322**	**4.201**	**4.941**	**5.698**	**6.457**	**7.156**	**7.707**
	[5.50]	**[6.14]**	**[6.52]**	**[6.64]**	**[6.66]**	**[6.80]**	**[6.93]**	**[7.14]**	**[7.52]**
c_3	**0.605**	**1.139**	**1.669**	**2.263**	**2.946**	**3.617**	**4.213**	**4.708**	**5.033**
	[4.54]	**[5.09]**	**[5.47]**	**[5.86]**	**[6.44]**	**[6.92]**	**[7.10]**	**[7.16]**	**[7.18]**
$\bar{R}^2$	**31.0**	**32.4**	**31.9**	**30.6**	**30.6**	**32.1**	**33.3**	**33.6**	**33.7**

Expectations Hypothesis: Results Summary

- Test 1 produces no predictive results and rejects EH in all five countries.
- Test 2 generates strong predictive results, but rejects the null of $b_1 = 1$ in GM and US for long maturity bond yields.
- The tension between results from Test 1 and Test 2 is NOT caused by small sample bias.
- Simulation results suggest that the results are consistent with a model with time varying risk premium.
- Predictive power of the forward rates exists in all 5 countries, but the strength varies across the countries.
- **Can a dynamic affine term structure generate these rich features observed in the yield curve?**

Affine Term-Structure Models (1)

- $N-$vector of unobservable state variables $X(t) = (X_1(t), X_2(t), \cdots, X_N(t))$ follow an affine diffusion under the risk neutral measure Q:

$$dX(t) = \tilde{K}\left(\tilde{\theta} - X(t)\right) dt + \Sigma\sqrt{S(t)}dW^Q(t) \tag{5}$$

- $S(t)$'s i^{th} diagonal element is given by

$$S(t)_{ii} = \alpha_i + \beta_i' X(t) \tag{6}$$

- The instantaneous short rate $r(t)$ is an affine function of $X(t)$:

$$r(t) = \delta_0 + \delta_1' X(t) \tag{7}$$

- The price of a zero-coupon default-free bond with maturity τ is

$$P(t, \tau) = \exp\left(A(\tau) - B(\tau)' X(t)\right) \tag{8}$$

- $A(\tau)$ and $B(\tau)$ satisfy ODE.

Affine Term-Structure Models (2)

- Assume the pricing kernel is given by

$$\frac{dM}{M} = -rdt - \Lambda_t' dW$$

- An essential affine model requires that the price of risk Λ be given by:

$$\Lambda_t = \sqrt{S(t)} \left(\lambda_1 + S(t)^{-1} I \lambda_2 X(t) \right) \tag{9}$$

- If $\lambda_2 = 0$, then the model is completely affine.

  Only factors that affect the short rate volatility affect the price of risk.

- If $\lambda_2 \neq 0$, then the model is essentially affine.

  Factors driving the short rate volatility can be different from factors driving the price of risk.

- $A_m(N)E$ ($A_m(N)C$) denote an essentially (completely) affine N-factor model in which $m \leq N$ factors enter the variance matrix $S(t)$.

Estimation

- Limit $N \leq 3$.
- Parameter restriction and normalization to guarantee admissibility and identification.
- The bond yield y_t is a linear function of $X(t)$:

$$y_t(\tau_j) \equiv -\frac{\ln P_{j,t}}{\tau_j} = -A\left(\tau_j\right) + B\left(\tau_j\right)' X\left(t\right) + \epsilon_t(\tau_j), \quad j = 1, \cdots, 7, \tag{10}$$

- Data inputs are yields with maturities of 6 months, 1, 2, 3, 5, 7 and 10 years.
- Bond yields with maturities of 4, 6, 8, and 9 years are for out-of-sample tests.
- Use the approximate Kalman filter/QMLE.

Model Comparison

- Nested models: Wald tests.
- Non-nested models:
  - Schwarz Criterion (BIC)

$$BIC = \text{ log likelihood} - \frac{1}{2} N_\theta \ln(T)$$

  - In-sample absolute pricing error averaged across the 7 bonds used in the estimation:

$$PE_t = \frac{1}{J} \sum_{j=1}^{J} \left| \hat{y}_t\left(\tau_j\right) - y_t\left(\tau_j\right) \right|, \text{ and } PE = \frac{1}{T} \sum_{t=1}^{T} PE_t.$$

  - Out-of-sample absolute pricing error, PEO, is averaged across the 4 bonds not used in estimation.

- Three-factor models perform better than two- or one-factor models.
- $A_1(3)E$ is the best model in all five countries.

Comparison Results

		Affine Term Structure Models				
Country	Criterion	$A_0(3)E$	$A_1(3)E$	$A_2(3)E$	$A_2(3)C$	$A_3(3)C$
CA	*BIC*	10.024	**10.093**	9.662	9.565	9.690
CA	*PE*	33.9	31.5	40.2	32.1	**31.1**
CA	*PEO*	31.7	30.1	37.7	31.0	**29.7**
GM	BIC	**9.215**	9.197	8.957	9.108	9.084
GM	PE	23.6	**22.3**	24.7	23.4	24.6
GM	PEO	20.0	**18.7**	19.3	19.1	19.2
JP	*BIC*	9.331	**9.375**	9.336	9.333	9.272
JP	*PE*	22.2	**21.1**	22.1	22.1	22.1
JP	*PEO*	22.0	**21.3**	21.9	21.8	22.2
UK	*BIC*	10.164	**10.209**	10.096	10.064	10.151
UK	*PE*	30.7	**29.3**	31.0	31.5	30.3
UK	*PEO*	28.7	**27.9**	29.3	29.4	28.6
US	*BIC*	9.642	**9.658**	8.376	8.421	8.750
US	*PE*	32.4	**29.1**	34.9	34.8	35.4
US	*PEO*	32.8	**29.6**	35.4	34.5	34.2

Summary of Parameter Estimates in $A_1(3)E$

- Need to estimate 24 parameters
- Many parameters are estimated with large standard errors and are not statistically significant.
- In the short rate process $r(t) = \delta_0 + \delta_1' X(t)$, δ_0 and at least one element in δ_1 are significant.
- In the price of risk $\Lambda_t = \sqrt{S(t)}\,(\lambda_1 + S(t)^{-1} I \lambda_2 X(t))$:
  - At least one element in λ_1 is significant in CA, GM, UK and US.
  - At least one element in λ_2 is significant in CA and US.
- In the mean reversion matrix K:
  - K_{11} is small in all five countries $\Longrightarrow X_1$ is persistent.
  - K_{22} and K_{33} are larger than K_{11} and are estimated imprecisely $\Longrightarrow X_2$ and X_3 are more transient.

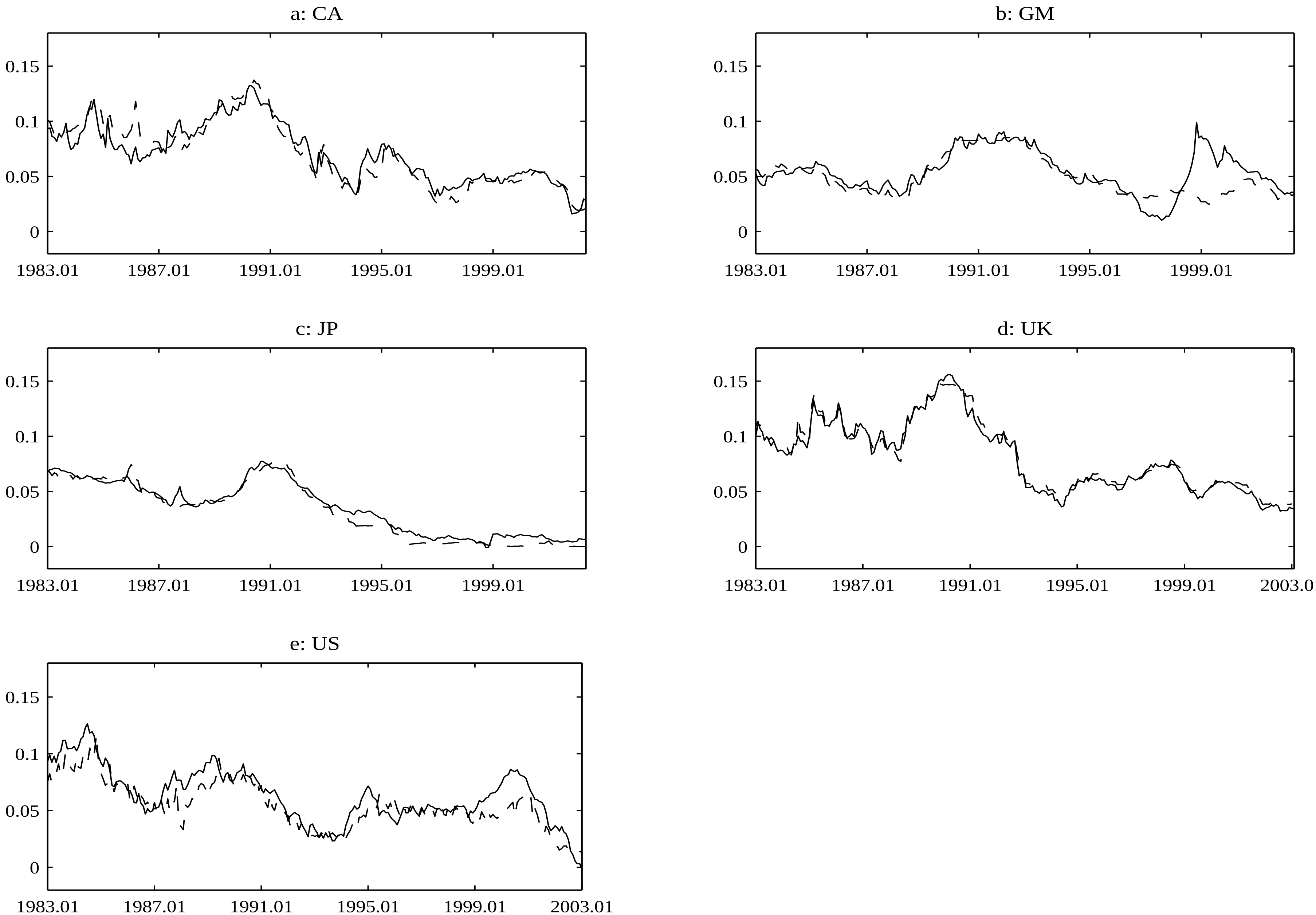
Model Fit

- The average (and standard deviation of) absolute prediction error is small $\Longrightarrow$ model fits the yield curve reasonally well.
- Average absolute prediction error declines with bond maturity $\Longrightarrow$ models fits the long end of the yield curve better.
- Occasional large mispricing of bond yields.
- Compare the one-month Treasury bill rate, TB_t, with the model-implied risk free rate $\hat{r}(t)$

$$\hat{r}(t) \equiv \hat{\delta}_0 + \hat{\delta}_1' \hat{X}(t), \tag{11}$$

Statistics	Variables	CA	GM	JP	UK	US
Mean	TB	7.10%	4.95%	3.21%	8.30%	5.46%
	$\hat{r}$	7.11%	5.38%	3.55%	8.13%	6.31%
Volatility	TB	2.98%	1.78%	2.63%	3.10%	2.04%
	$\hat{r}$	2.63%	1.90%	2.36%	3.19%	2.42%

TB (dashed line) vs. $\hat{r}$ (solid line)



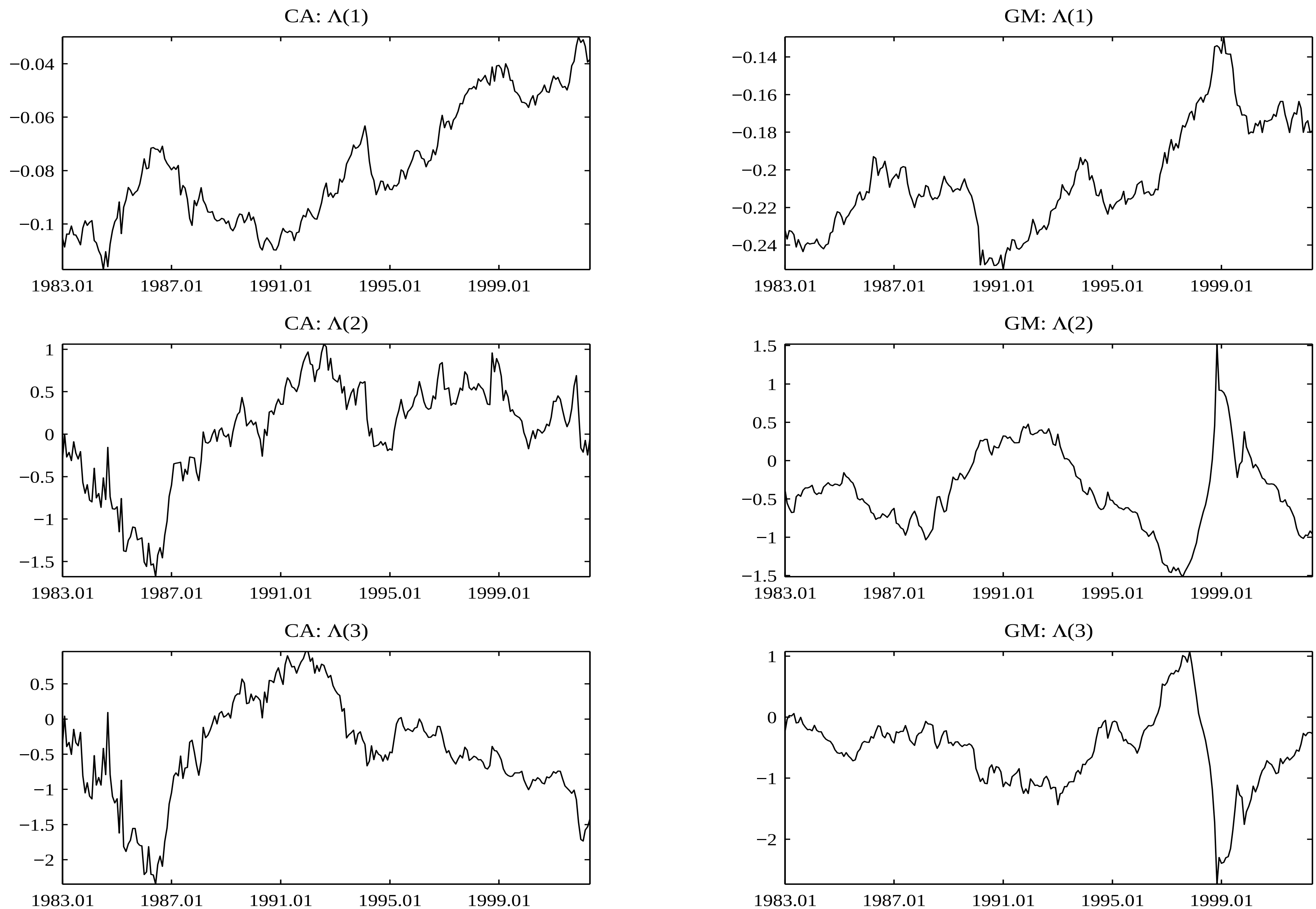
Model Implied Price of Risk - 1

- The estimated vector of price of risk

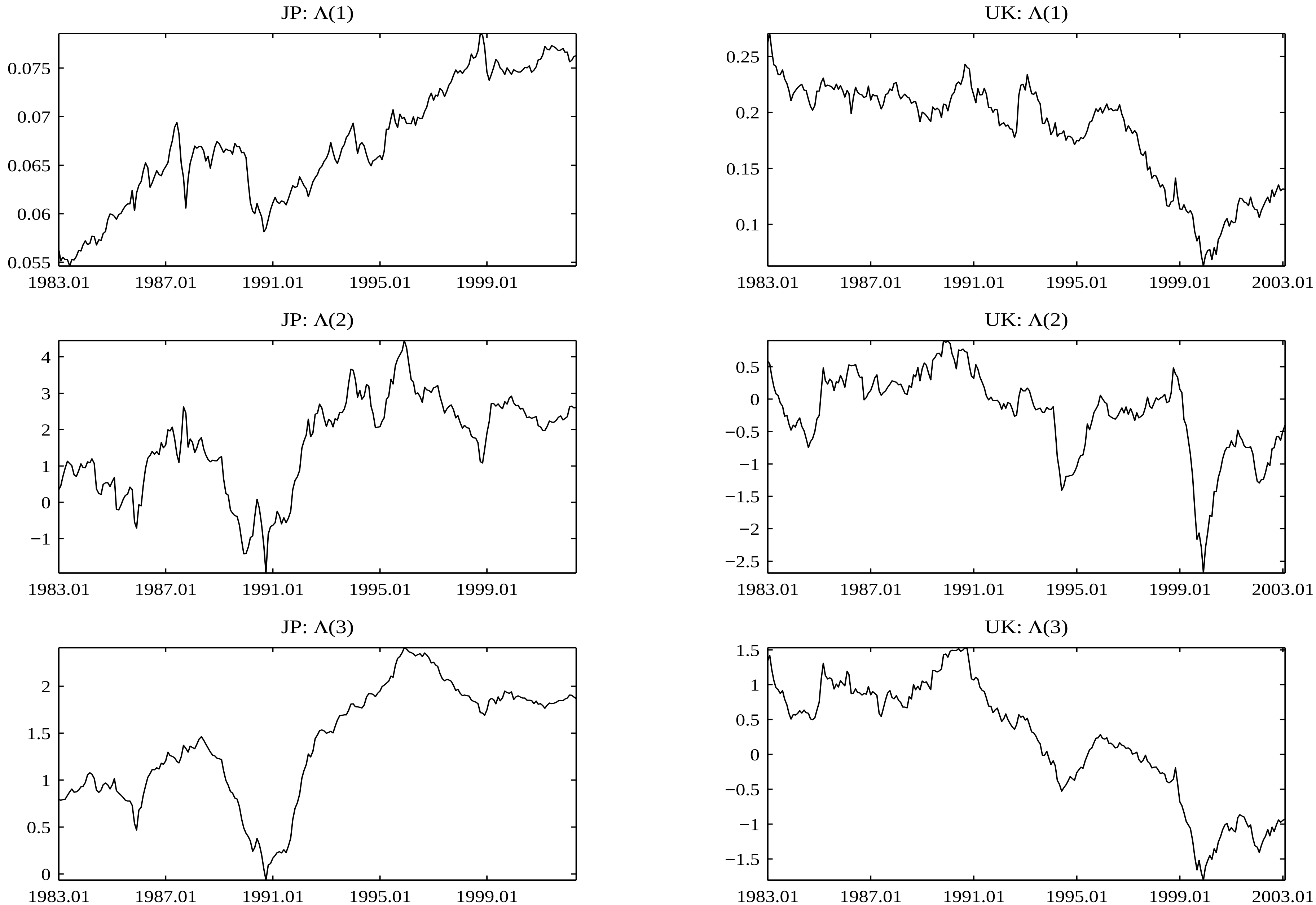
$$\hat{\Lambda}_t = \sqrt{\hat{S}\,(t)}\left(\hat{\lambda}_1 + \hat{S}(t)^{-1} I \hat{\lambda}_2 \hat{X}\,(t)\right), \qquad (12)$$

- The i^{th} element, $\hat{\Lambda}(i)$, is the price of risk associated with factor X_i.

$\hat{\Lambda}$

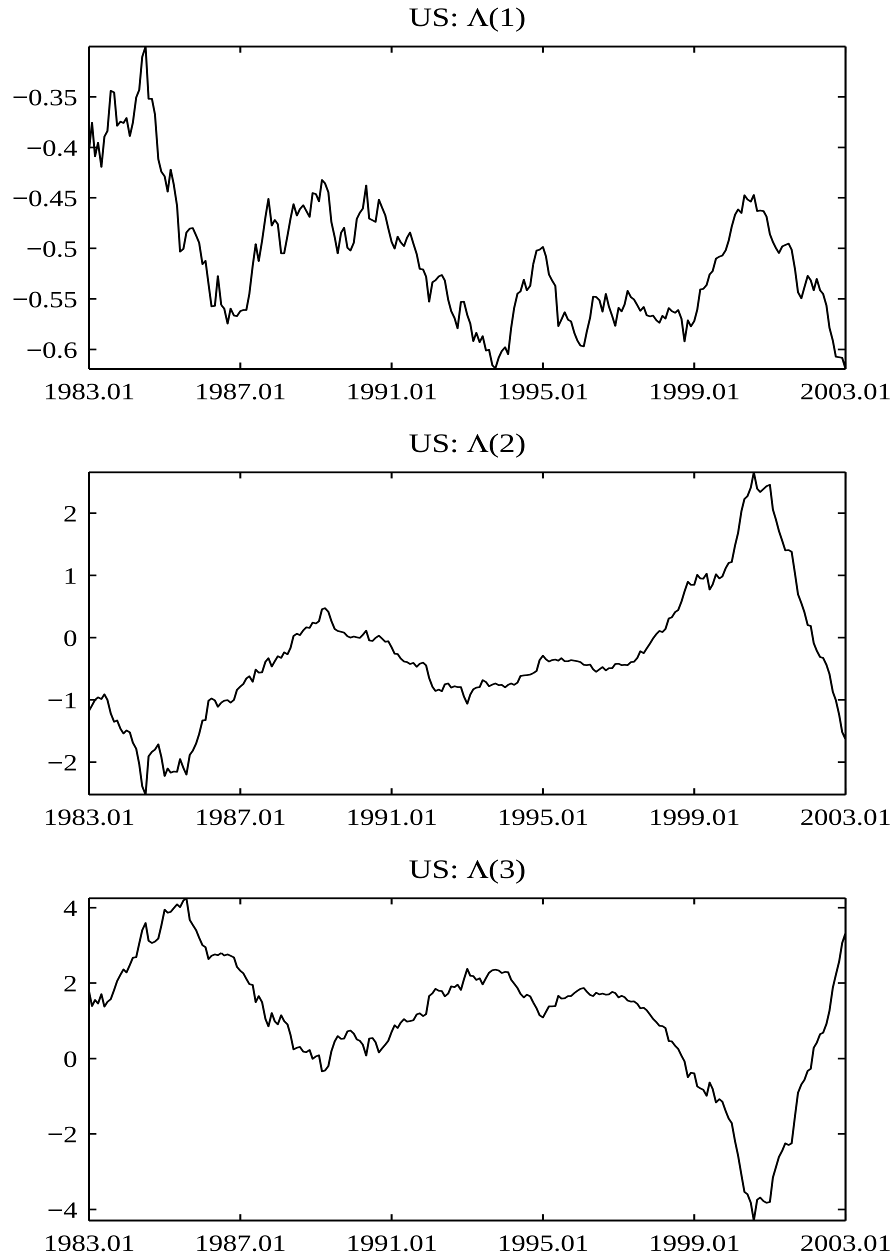

CA: Λ(1)
−0.04
−0.06
−0.08
−0.1
GM: Λ(1)
−0.14
−0.16
−0.18
−0.2
−0.22
−0.24
CA: Λ(2)
1
0.5
0
−0.5
−1
−1.5
GM: Λ(2)
1.5
1
0.5
0
−0.5
−1
−1.5
CA: Λ(3)
0.5
0
−0.5
−1
−1.5
−2
GM: Λ(3)
1
0
−1
−2
1983.01
1987.01
1991.01
1995.01
1999.01

$\hat{\Lambda}$



JP: Λ(1)
0.075
0.07
0.065
0.06
0.055
1983.01
1987.01
1991.01
1995.01
1999.01
UK: Λ(1)
0.25
0.2
0.15
0.1
1983.01
1987.01
1991.01
1995.01
1999.01
2003.01
JP: Λ(2)
4
3
2
1
0
−1
1983.01
1987.01
1991.01
1995.01
1999.01
UK: Λ(2)
0.5
0
−0.5
−1
−1.5
−2
−2.5
1983.01
1987.01
1991.01
1995.01
1999.01
2003.01
JP: Λ(3)
2
1.5
1
0.5
0
1983.01
1987.01
1991.01
1995.01
1999.01
UK: Λ(3)
1.5
1
0.5
0
−0.5
−1
−1.5
1983.01
1987.01
1991.01
1995.01
1999.01
2003.01

$\hat{\Lambda}$


US: Λ(1)
−0.35
−0.4
−0.45
−0.5
−0.55
−0.6
1983.01
1987.01
1991.01
1995.01
1999.01
2003.01
US: Λ(2)
2
1
0
−1
−2
1983.01
1987.01
1991.01
1995.01
1999.01
2003.01
US: Λ(3)
4
2
0
−2
−4
1983.01
1987.01
1991.01
1995.01
1999.01
2003.01

Model Implied Price of Risk - 2

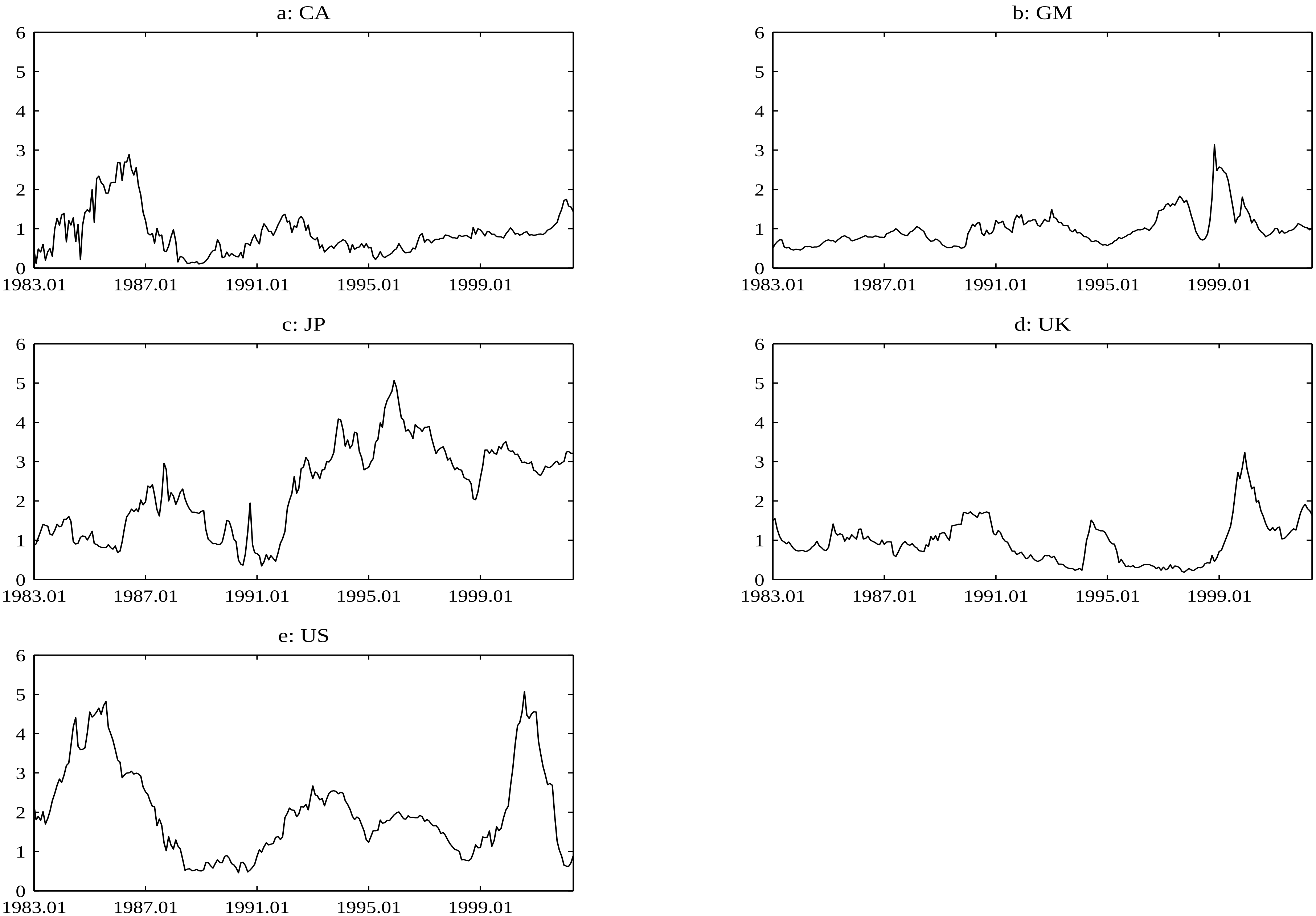
- The estimated maximum Sharpe ratio in the bond market has large sample mean but also large sample volatility:

$$\hat{\eta} \equiv \sqrt{\hat{\Lambda}\hat{\Lambda}'},$$

Statistics	Variables	CA	GM	JP	UK	US
Mean	$\hat{\eta}$	0.88	0.98	2.32	0.98	2.02
Volatility	$\hat{\eta}$	0.57	0.41	1.12	0.55	1.13

- **Estimates have large standard errors but the model implied series seem to move with important monetary and economic events.**

$\hat{\eta}$

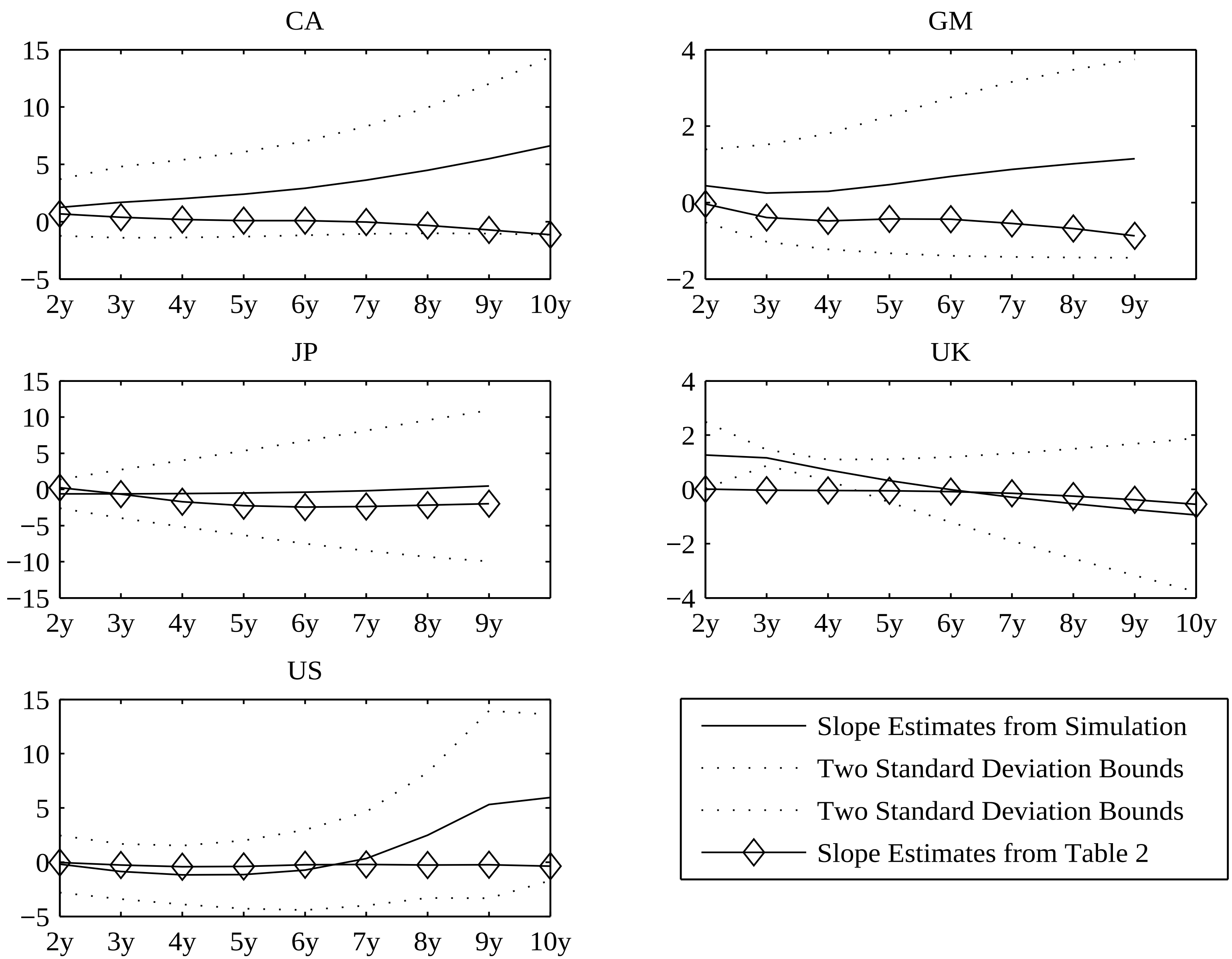


a: CA
b: GM
c: JP
d: UK
e: US
0
1
2
3
4
5
6
1983.01
1987.01
1991.01
1995.01
1999.01

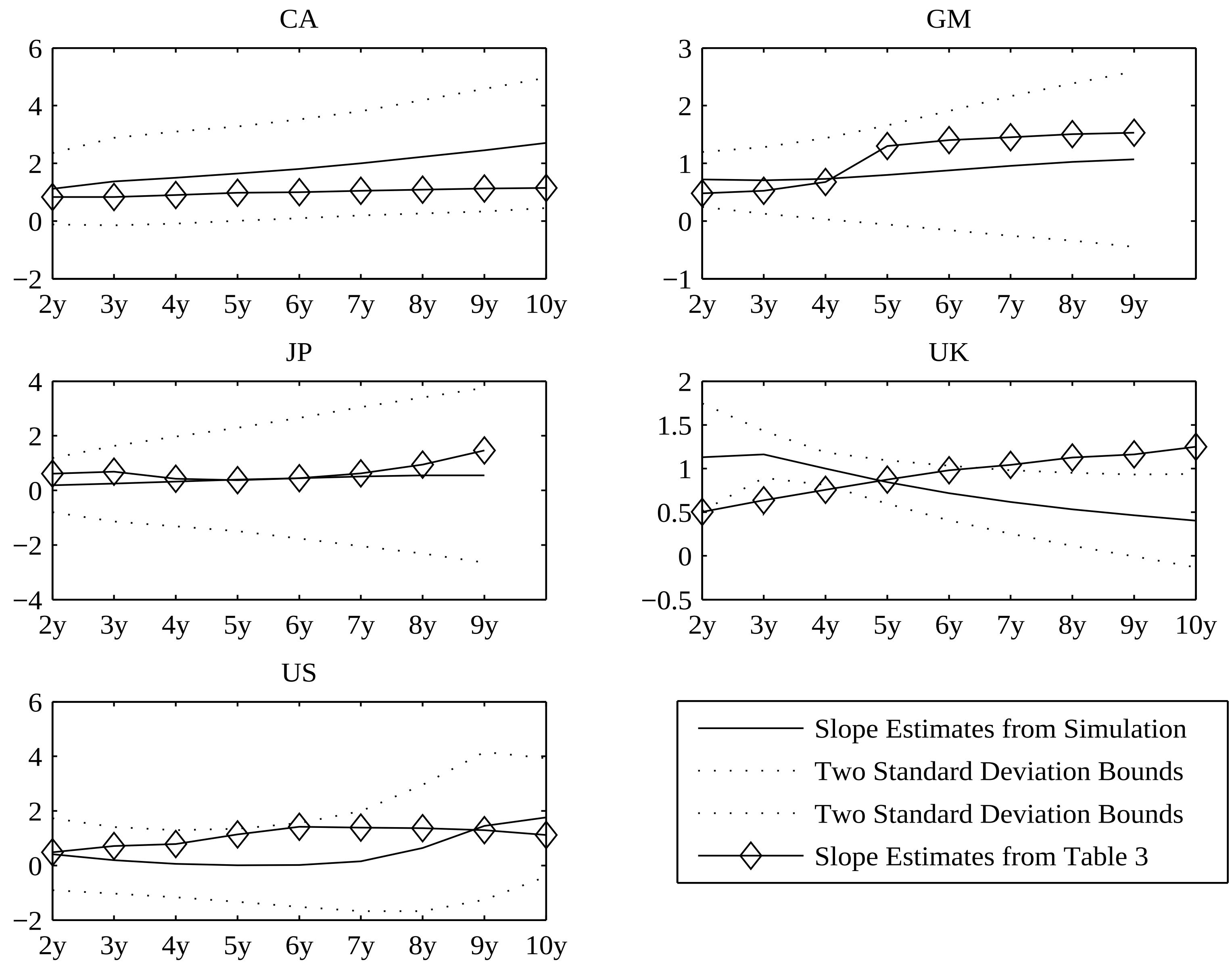
Model Implied Term Structure Behavior

- Most existing results are asymptotic and are only for the US data.
- Small sample behavior of the term structure behavior implied by the estimated $A_1(3)E$ model.
- Simulate 2000 time series of yields with maturities from one to ten years using estimated parameters.
- Each time series has 233-242 monthly observations.
- Carry out the two EH regressions and the predictive regression of the forward rates in the simulated yields.
- The mean and standard error of coefficient estimates across the 2000 simulated datasets are the simulated coefficients and their standard errors.

Test 1: $y_{t+12}^{\tau-1} - y_t^{\tau} = a_0 + a_1 \frac{y_t^{\tau} - y_t^{1}}{\tau-1} + \epsilon$

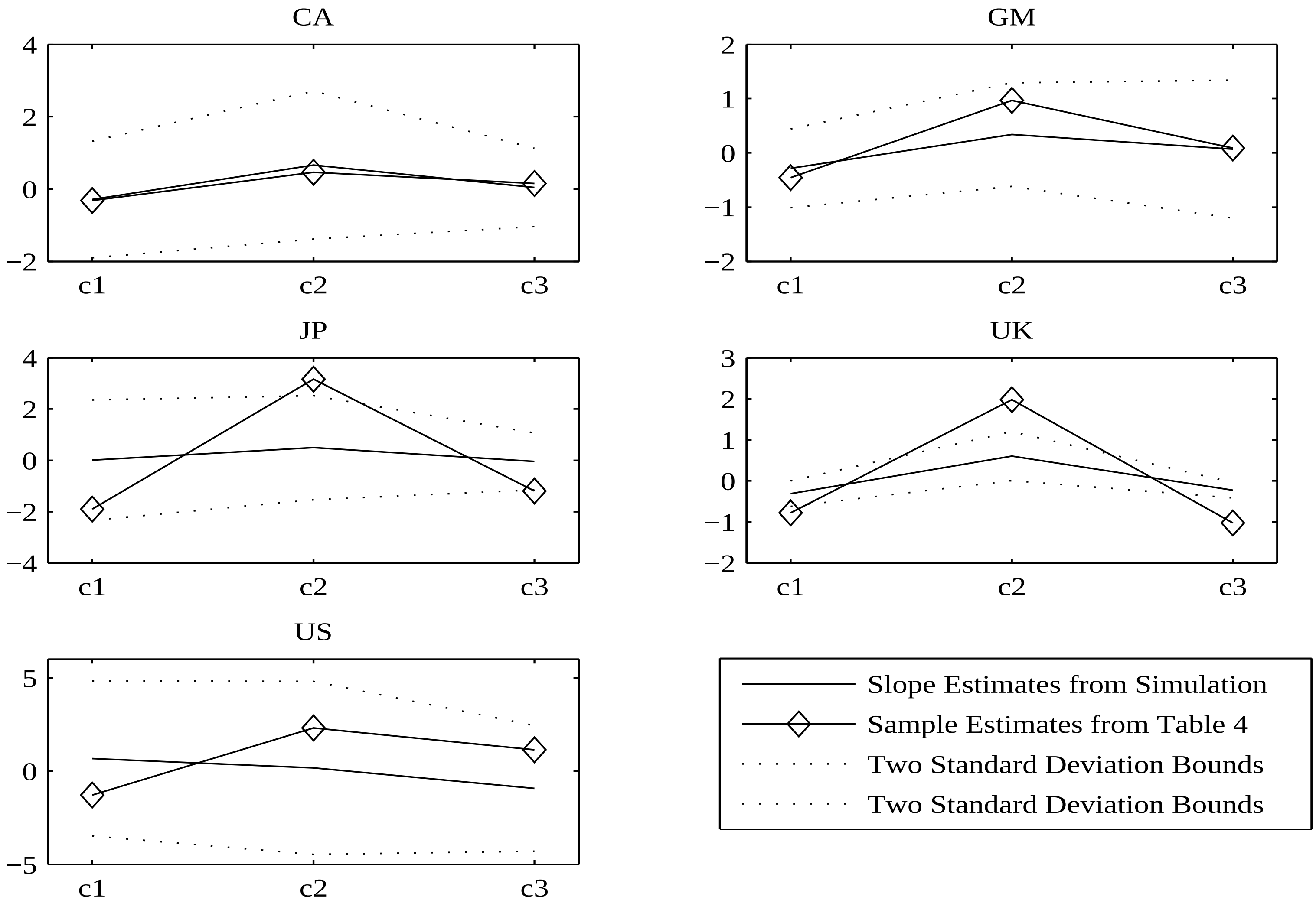


Test 2: $\frac{1}{\tau}\sum_{k=1}^{\tau-1}\left[y_{t+12k}^{1}-y_{t}^{1}\right]=b_0+b_1\left[y_t^{\tau}-y_t^{1}\right]+\epsilon$



Predictive Regression of Forward Rates

$$\mathbf{r^{\tau}_{t+12} = c_0 + c_1 f_t(0,1) + c_2 f_t(2,3) + c_3 f_t(8,9) + e_t}$$



Summary of the Simulation Results

- In general, the patterns observed in the data are NOT observed in the simulated estimates.
- But they are NOT statistically different.
- Simulated estimates have large standard errors, because parameters in the $A_1(3)E$ model are estimated with large errors.
- It is also possible that the small sample (Jan. 1983 to May 2002) leads to imprecise regression coefficient estimates (especially for Test 1).

- **Therefore, the small sample results are inconclusive.**

Conclusion

- Provide international evidence on yield curve behavior
- Estimate and compare the affine term structure models
- Examine the small sample behavior of the best affine term structure model

- **There are country specific empirical regularities**
- **But the qualitative results and their implications are consistent across all five countries**