Information Reduction in Credit Risk

Xin Guo
School of Operations Research and Industrial Engineering
Cornell University

http://www.orie.cornell.edu/∼xinguo
xg29@cornell.edu

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Joint with: R. Jarrow, Y. Zeng (Cornell U.)

Credit risk is to take the possibility of default into account for pricing and hedging derivatives.

Example: Bond with face value \$1

- Pricing of *default-free* bond under arbitrage-free setting

$$V(t,T) = E^Q\left[\exp\{-\int_t^T r_s ds\}|\mathcal{F}_t\right]$$

- Pricing of *defautable* bond under arbitrage-free setting

  Key: replace r with $r + \lambda a$, where a is the fractional loss, under proper conditions
  Duffie, Schroder, Skiadias (1996)
  Elliott, Jeanblanc, Yor (2000)

λ_t – default intensity (intuitively)

$$\lambda_t = \lim_{h \downarrow 0} \frac{P(t + h \geq \tau > t | \mathcal{F}_t)}{h}$$

- Measures the instantaneous likelihood of a default
- Key quantity in the *reduced form* model

Two main approaches in credit risk:

- Structural models
  e.g. Merton(1974), Black-Scholes (1973)
- Reduced form models
  e.g. Artzner and Delbaen (1995)
  Jarrow, Turnbull (1992, 1995)
  Madan and Unal (1995)
  Lando (1998)
  Duffie and Singleton (1999)
  Elliott, Jeanblanc, and Yor (2000)
  Jeanblanc and Rutkowski (2002)
  Bielecki and Rutkowski (2002)
  Belanger, Shreve, and Wong (2004)

Structural model

- Model default by modelling the dynamics of the assets of the firm
  E.g. default happens when the asset value of a firm ($X(t)$) falls to threshold level process L, i.e.

$$\tau = \inf\{t > 0, X(t) \leq L\}$$

  — first-passage time problem
- Default is usually *predictable* under general models if the asset value is continuous and observable

Main problem

- Asset value usually not observable
- Default estimation is computationally difficult

Reduced form model

- Default comes as a "surprise"
- There is an intensity of the arrival of default, λ_t,
- λ_t is an exogenous, random process
- Default can not be predicted: *totally inaccessible*

Advantage

Easy to implement for date fitting

Problem

Economic interpretation is not intuitive

Relationship between the two approaches

- Difference between the reduced form and structural models can be characterized by different filtrations: Default times become *predictable* or *totally inaccessible* with filtration changes

Totally inaccessible stopping times

E.g. jump times in a Poisson process under the natural filtration
— Its arrival is a "surprise"

Predictable stopping times

E.g. first passage times in a Brownian motion under the natural filtration
— Its arrival is "announced" by a sequence of stopping times

And ... How?

Let τ be a stopping time,
(e.g. a first passage time in a structure model).
Define

$$N_t = 1_{\{\tau \leq t\}}$$

Then

- N_t is a submartingale
- $\exists!$ predictable increasing process A_t, $A_0 = 0$, s.t. $N_t - A_t$ is a martingale. (Doob-Meyer)

Moreover, under *proper* assumptions,

- $\exists!$ λ_t s.t. $A_t = \int_0^t (1 - N_s)\lambda_s ds$, $t > 0$,

$$P(\tau > T|\mathcal{F}_t) = (1-N_t)E^P[\exp\{-\int_t^T \lambda_u du\}|\mathcal{F}_t]$$

Examples:

In a structural model, default time of first passage time type becomes totally inaccessible, with noisy/partial accounting information

Duffie, Lando (2004)

Cetin et. al. (2004)

Collin-Dufresne, Goldstein, and Helwege (2003)

Conviction: It is critical to understand mathematically how default and pricing change with different information structures, regardless of the model approach. In particular,

- Qualitatively
  — differentiating different default times
- Quantitatively
  — Analytical form for λ_t based on known information (i.e. simple filtration structure)

Q1: Given a filtration, which are appropriate sub-filtrations?

- Let $X(t)$ be the asset value process,

$$\tau = \inf\{t > 0 : X_t \leq x\}$$

- Two information structures:

  Complete vs. Partial and DELAYED

More precisely:

- Complete: Natural filtration $\mathcal{F}^X$
- Delayed filtration (deterministic): A sub-filtration $(\mathcal{F}_t^l)_{t\geq 0}$ of $(\mathcal{F}_t^X)_{t\geq 0}$ is *delayed at t*, if there is a positive number $\delta = \delta(t) > 0$ such that
$$\sigma(X_{t-\delta}) \subset \mathcal{F}_t^l \subset \mathcal{F}_{t-\delta}^X,$$

Now define

$$\mathcal{F}_t(\tau) = \mathcal{F}_t^l \vee \sigma(t \wedge \tau).$$

The filtration $(\mathcal{F}_t(\tau))_{t\geq 0}$ is *(deterministic) delayed filtration.*
This delayed filtration can be both discrete and continuous.

Theorem 1 Let X be a one-dimensional, time-homogeneous Markov process with a continuous sample path. Then
I) Under the natural filtration $(\mathcal{F}_t^X)_{t\geq 0}$, the default time τ is predictable.
II) Let $(\mathcal{F}_t(\tau))_{t\geq 0}$ be a deterministic delayed filtration. If τ has a continuous density function $f(x,t)$, and $0 < \inf_{t\leq T}\delta(t) \leq \sup_{t\leq T}\delta(t) < \infty$ for $\forall T > 0$, then τ has a default intensity

$$\frac{f(x - X_{t-\delta(t)}, \delta(t))}{P^{X_{t-\delta(t)}}(\tau > \delta(t))} 1_{\{\tau > t\}}.$$

Proof of II: Fix $t > 0$.

$$
\begin{aligned}
& \frac{1}{h} P(t < \tau \leq t + h | \mathcal{F}_t(\tau)) \\
= \; & \frac{1}{h} \frac{P(t < \tau \leq t + h | \mathcal{F}_t^l)}{P(\tau > t | \mathcal{F}_t^l)} 1_{\{\tau > t\}} \\
= \; & \frac{1}{h} \left(1 - \frac{P^{X_{t-\delta(t)}}(\tau > h + \delta(t))}{P^{X_{t-\delta(t)}}(\tau > \delta(t))} \right) 1_{\{\tau > t\}} \\
= \; & \frac{1}{h} \frac{P^{X_{t-\delta(t)}}(h + \delta(t) \geq \tau > \delta(t))}{P^{X_{t-\delta(t)}}(\tau > \delta(t))} 1_{\{\tau > t\}} \\
\to \; & \frac{f(x - X_{t-\delta(t)}, \delta(t))}{P^{X_{t-\delta(t)}}(\tau > \delta(t))} 1_{\{\tau > t\}}, \quad \text{as } h \downarrow 0.
\end{aligned}
$$

Furthermore, one can verify that that this limit indeed is the intensity process of interest by checking the Aven's lemma.

Remark 1:
The theorem above provides an explicit and analytic connection between structural and reduced-form models.
E.g. X satisfies $dX_t = \sigma(X_t)dW_t + b(X_t)dt$ where b and σ satisfy mild regularity conditions, then for τ_y (the first-passage time of X hitting the level y), there is a density $P^x(\tau_y \in dt) = \theta(t, x, y)dt$ (Pauwels (1987)). Hence, the existence of the corresponding default intensity is implied.

In particular,

Corollary Under a geometric Brownian motion model, assuming $\{\tau > t\}$, the default intensity of τ at time $t \in [t_k, t_{k+1})$ is

$$-\frac{\psi_t(\frac{\mu}{\sigma} - \frac{1}{2}\sigma, t - t_k, \frac{\log x - \log X_{t_k}}{\sigma})}{\psi(\frac{\mu}{\sigma} - \frac{1}{2}\sigma, t - t_k, \frac{\log x - \log X_{t_k}}{\sigma})}$$

(Collin-Dufresne, Goldstein, and Helwege (2003))

Remarks 2: About **H** - hypothesis.
Let τ be a r.v. on $(\Omega, \mathcal{G}, P)$, $F(t) = P(\tau \leq t)$ right continuous with $F(0) = 0$, $F(t) < 1$ for any $t > 0$, $N_t = 1_{\tau \leq t}$ and $\mathcal{H}_t = \sigma(N_u, u \leq t)$, then if $f = F'$ exists, $\mathcal{H}$ -intensity of τ is $\lambda(s) = f(s)/(1 - F(s))$.
(Elliott, Jeanblanc, Yor (2000))

Our framework does not satisfy this **H** -hypothesis.

Q2: Is this delayed filtration necessarily deterministic?

Generalization to

- One-dimensional strong Markov models with continuous sample path
- Regime switching models

under *stochastically* delayed filtration

Regime switching model

$X(t)$ be the asset value of a firm

$$dX(t) = X(t)(\mu(\epsilon(t))dt + \sigma(\epsilon(t))dW(t))$$

$\epsilon(t) \in \{a_1, \ldots, a_S\}$ be a finite-state continuous Markov chain, with jump times denoted as T_i

- Credit rating migration modelled by Markov chain (Jarrow, Lando, Turnbull (1997))

Corresponding filtration structures

- Complete case: $\mathcal{F}_t^{X,\epsilon} = \sigma(X_s, \epsilon(s) : s \leq t)$
- Partial case: $\mathcal{F}_t^{\mathcal{B}}$ is the filtration generated by three point processes

$$1_{\{\tau \leq t\}}$$

$$\sum_{i=1}^{\infty} X_{t_i} 1_{\{t_i \leq t\}}$$

$$\sum_{i=1}^{\infty} X_{T_i} 1_{\{T_i \leq t\}}$$

Theorem 2:
Under $\mathcal{F}^{\mathcal{B}}$, Assuming $t \in [t_k, t_{k+1})$, $\tau > t$, the default intensity for τ is

$$\lambda_t \;=\; -\frac{\psi_t(\theta_n, t', \frac{1}{\sigma_n}\log\frac{x}{X_{t_k \vee T_n}})}{\psi(\theta_n, t', \frac{1}{\sigma_n}\log\frac{x}{X_{t_k \vee T_n}})}.$$

where $\theta_n = \mu_n/\sigma_n - \sigma_n/2$, $n = \epsilon(t)$.

$$\begin{aligned}\psi(\theta, t, y) &= P(\inf_{0\le s\le t} W_s^{(\theta)} > y)\\ &= 1 - \int_0^t \frac{|y|}{\sqrt{2\pi s^3}} e^{-\frac{(y-\theta s)^2}{2s}} ds\end{aligned}$$

and $t' = t - t_k \vee T_n$ (time difference due to delayed information)

Q3: How about models with jumps?

Jump-diffusion model:

The asset value process X is then assumed to satisfy

$$X_t = X_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \prod_{0 < s \leq t, \Delta\epsilon(s) \neq 0} \xi_{\epsilon(s)},$$

- $\Delta\epsilon(s) := \epsilon(s) - \epsilon(s-)$
- For each state i of ϵ $(0 \leq i \leq S-1)$ assign a positive random variable ξ_i such that $(\xi_i)_{i \geq 1}$
- F_i the distribution of ξ_i $(0 \leq i \leq S-1)$
- $(\xi_i)_i$, ϵ and W are all independent with $P(\xi_i = 1) = 0$

In particular, if ϵ is a standard Poisson process, then X satisfies

$$\frac{dX_t}{X_{t-}} = \mu dt + \sigma dW_t + d(\sum_{i=1}^{\epsilon_t} (\xi_i - 1)).$$

Corresponding filtrations:

- Complete: the natural filtration $\mathcal{F}_t^{X,\epsilon}$
- Partial $\mathcal{F}_t^{\mathcal{B}}$: the augmented minimal filtration generated by ϵ and the point processes $1_{\{\tau \le t\}}$, $\sum_{n=0}^{\infty} X_{t_k} 1_{\{t_k \le t\}}$, $\sum_{n=1}^{\infty} \xi_{\epsilon(T_n)} 1_{\{T_n \le t\}}$ and $\sum_{n=0}^{\infty} X_{T_n} 1_{\{T_n \le t\}}$.

Theorem 3

1) Under $\mathcal{F}^{X,\epsilon}$, the total inaccessible part of τ has a default intensity $\sum_{i\neq\epsilon(t)} q_{\epsilon(t)i}\left[F_i(\frac{x}{X_t}-)+\frac{1}{2}\Delta F_i(\frac{x}{X_t})\right]$.

2) Under $\mathcal{F}^{\mathcal{B}}$, when $\tau > t$, $t_k \leq t < t_{k+1}$ and $T_n \leq t < T_{n+1}$, the default intensity is

$$d_t = -\frac{\psi_t(\theta, t', \frac{1}{\sigma}\log\frac{x}{X_{t_k\vee T_n}})}{\psi(\theta, t', \frac{1}{\sigma}\log\frac{x}{X_{t_k\vee T_n}})} + \sum_{j\neq\epsilon(t)} q_{\epsilon(t)j}\frac{\int_0^1 F_j(dz)\phi(\theta, t', \frac{1}{\sigma}\log\frac{x}{X_{t_k\vee T_n}}, \frac{1}{\sigma}\log\frac{x}{zX_{t_k\vee T_n}})}{\psi(\theta, t', \frac{1}{\sigma}\log\frac{x}{X_{t_k\vee T_n}})}$$

where $t' = t - t_k \vee T_n$, and for $y_1 \leq y_2$,

$$
\begin{aligned}
& \phi(\theta, t, y_1, y_2) \\
= \; & P(\inf_{s \leq t} W_s^{(\theta)} > y_1, W_t^{(\theta)} \leq y_2) \\
= \; & \Phi(\frac{y_2 - \theta t}{\sqrt{t}}) - \Phi(\frac{y_1 - \theta t}{\sqrt{t}}) \\
- \; & e^{2\theta y_1} \left[\Phi(\frac{y_2 - 2y_1 - \theta t}{\sqrt{t}}) - \Phi(\frac{-y_1 - \theta t}{\sqrt{t}}) \right],
\end{aligned}
$$

with $\Phi(x)$ being the distribution of a standard normal random variable.

Main Technical Ingredients for Proofs

- Meyer's previsibility theorem (Rogers and Williams (2000))

- Meyer's Laplacian Approximation: Let Z be an appropriate càdlàg positive supermartingale with $\lim_{t\to\infty} E\{Z_t\} = 0$. Let $Z = M - A$ be its Doob-Meyer decomposition. Define

$$A_t^h = \int_0^t \frac{Z_s - E\{Z_{s+h}|\mathcal{F}_s\}}{h} ds$$

  Then for any stopping time T

$$A_T = \lim_{h\to 0} A_T^h$$

  in the sense of the weak topology $\sigma(L^1, L^\infty)$

- Aven's lemma (1985) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space that satisfies usual hypothesis. Let $(N_t)_{t\geq 0}$ be a counting process. Assume that $E\{N_t\} < \infty$ for all t. Let $\{h_n\}_{n\geq 1} \downarrow 0$ and let $(Y_t^n)_{t\geq 0}$ be a measurable version of the process $(E\{N_{t+h_n} - N_t|\mathcal{F}_t\}/h_n)_{t\geq 0}$. Assume for measurable processes $(\lambda_t)_{t\geq 0} \geq 0$ and $(y_t)_{t\geq 0} \geq 0$:
  (i) for each t, $\lim_n Y_t^n = \lambda_t$ a.s.;
  (ii) for each t, there exists for almost all ω an $n_0 = n_0(t, \omega)$ such that

$$|Y_s^n(\omega) - \lambda_s(\omega)| \leq y_s(\omega), \qquad s \leq t, n \geq n_0$$

  (iii) $\int_0^t y_s ds < \infty$, a.s. $0 \leq t < \infty$.
  Then $(\int_0^t \lambda_s ds)_{t\geq 0}$ is the compensator of $(N_t)_{t\geq 0}$.

Morals of the story:

- Default intensity comes with the inaccuracy of the information: *NOISY* as in Duffie and Lando or *DELAYED*
- Default intensity is not (and should not be?) exogenous: it is *IMPLIED*
- There are (simple) filtrations that are mathematically tractable

Pricing (?) Defaultable Bonds with Incomplete Information:

- Zero-coupon bond paying \$1 at maturity time T if there is no default, and \$R at time T if the firm defaults before time T
- $R \in [0, 1]$, the recovery rate, is independent of the default process

General Markovian Setup:

- $Y = (Y_t)_{t \geq 0}$, the asset value process, be a (multi-dimensional) Markov process under a risk-neutral measure Q, with a general state space E
- U is a subset of E and $\tau_U := \inf\{t > 0 : Y_t \in U\}$.
- $V^{\mathcal{A}}(t,T)$ and $V^{\mathcal{B}}(t,T)$ of defaultable zero-coupon bond under the two different filtration structures, complete and partial, respectively.

Proposition Let $\{s_m\}_{m\geq 0}$ be a strictly increasing sequence of non-negative numbers ($s_0 = 0$), such that $s_m \uparrow \infty$. Under the filtration

$$\mathcal{F}_t^{\mathcal{B}} := \sigma(Y_{s_1}, \cdots, Y_{s_m}) \vee \sigma(t \wedge \tau_U)$$

for $s_m \leq t < s_{m+1}$,

$$V^{\mathcal{B}}(t,T) = 1_{\{\tau_U > t\}} \frac{V^{\mathcal{A}}(s_m, T)}{V^{\mathcal{A}}(s_m, t)}$$

Remarks:

- Both regime switching and jump diffusion models are incomplete, with non-unique equivalent martingale measures
- Less information does not lead a cheaper price for bond

Jeanblanc and Valchev (2004)
Chen and Kou (2005)

Summary:

- A simple and general framework for credit risk study
- Provide closed-form solutions and techniques to compute the default intensity λ_t, with known information

Computational Complexity

- Data needed for default estimation are in the public domain
- Parameter estimations via "implied" default intensity method

So ... What is now?

- Extension to general Markov process
- Bond pricing with incomplete information
- Optimal stopping time for companies releasing news
- Modelling recovery rate with incomplete information
- Data calibration: *implied* default intensity