Hedging Basket Credit Derivative Claims: A Local Risk-Minimization Approach

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Motivation

- Basket credit derivatives have seen an exponential growth over the last few years especially with the emergence of standard indices such as iTraxx and CDX.

- The main credit derivative structures are: Credit Default Swaps, First-To-Default Swaps, Collateralized Debt Obligations.

- FTDs and CDOs are correlation products, their value depends on the likelihood of multiple defaults. They are usually hedged with single-name CDS.

- Correlation risk introduces a market incompleteness. To address this problem, we use a local-risk minimization approach.

- Bi-modal nature of credit distributions means that we need to hedge spread risk and default risk.

Credit Derivatives

- **Credit Default Swap:** is a bilateral agreement whereby the protection buyer makes a series of premium payments until the maturity of the trade or default, depending on which occurs first. In return, he receives a protection against the default of the reference entity. In the event of default, the protection seller makes a payment equal to the difference between the face value of the obligation and its recovery value after default.

- **First-To-Default Swap:** is a default swap where the protection seller takes on exposure to the first entity suffering a credit event within a basket. Similarly, we can have second-to-default swaps, third-to-default swaps,and so forth.

- **CDO (or first loss swap):** is a default swap where the protection seller commits to cover all the losses incurred on a portfolio within a pre-defined range. In return, the protection buyer pays a periodic premium on a notional that amortizes with losses on the portfolio.

Correlation and Copulas

Definition 1 *A copula function is the multivariate distribution function of n random variables uniformly distributed on* $[0,1]$.

Theorem 2 *(Sklar (1959)) For n random variables $(X_1, ..., X_n)$ with marginal distribution functions $(F_1, ..., F_n)$ and joint distribution function F, there exists an n-dimensional copula $\mathbf{C}$ such that for all $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$,*

$$F(x_1, x_2, ..., x_n) = \mathbf{C}(F_1(x_1), F_2(x_2), ..., F_n(x_n))$$

if $F_1, F_2, ..., F_n$ are continuous, then $\mathbf{C}$ is unique.

- Examples:
  - Gaussian Copula / T-Copula.
  - Marshall-Olkin Copula.

Marshall-Olkin Copula

- The Marshall-Olkin copula was traditionally used in reliability theory to model the failure rate in multi-component systems. The failure of each component is assumed to be contingent on some independent Poisson shocks.

- This is also known as a multivariate Poisson model or a Poisson shock model.

- It was first used in the context of basket credit derivatives pricing in Duffie (1998), then in Duffie and Garleanu (2001).

- It has a number of useful analytical results for the aggregate portfolio distribution (see Lindskog and McNeil (2003)).

Quadratic Hedging Approaches

- **Complete Market**: perfect replication by self-financing strategies.

- **Incomplete Market**:

  - **Local Risk Minimization**: allow mean self-financing strategies and minimize the "Expected Additional Cost"

$$r_t(\varphi) = \mathbb{E}\left[\left(C_{t+1}(\varphi) - C_t(\varphi)\right)^2 |\mathcal{G}_t\right],$$

    The value process of a local risk-minimizing strategy is given by

$$V_t^H = \widehat{\mathbb{E}}\left[H\,|\mathcal{G}_t\right],$$

    where $\widehat{\mathbb{E}}\left[.\,|\mathcal{G}_t\right]$ is the conditional expectation operator under the **Minimal Martingale Measure** $\widehat{P}$.

  - **Mean Variance Hedging**: keep the self-financing property and give-up the perfect replication

    find $V_0 \in \mathbb{R}$ and $\alpha \in \Theta$ such that $\mathbb{E}\left[(V_0 + G_T(\alpha) - H)^2\right]$ is minimal.

The Model (1)

We work in an economy represented by a probability space $(\Omega, \mathcal{G}, P)$ and a time horizon $T^* \in (0, \infty)$, on which is given a d-dimensional Brownian motion W, and a set of n non-negative random variables $(\tau_1, ..., \tau_n)$ representing the default times of the obligors in the economy.

We introduce an $\mathbb{R}^d$-valued Itô process X_t, describing the evolution of the state-variables in the economy, which solves the following SDE

$$dX_t = \alpha\left(X_t\right) dt + \beta\left(X_t\right) dW_t,$$

for some continuous functions $\alpha_k : \mathbb{R}^d \to \mathbb{R}$ and $\beta_{kj} : \mathbb{R}^d \to \mathbb{R}^d$, $1 \leq k \leq d$, $1 \leq j \leq d$.

We denote by $\{\mathcal{F}_t\}_{0 \leq t \leq T^*}$ the filtration generated by X and augmented with the P-null sets of $\mathcal{G}$:

$$\mathcal{F}_t = \sigma\left(X_s : 0 \leq s \leq t\right) \vee \mathcal{N}.$$

The Model (2)

We introduce, for each obligor i, the right-continuous process $D_t^i \triangleq \mathbf{1}_{\{\tau_i \leq t\}}$ indicating whether the firm has defaulted or not. We denote by $\left\{\mathcal{H}_t^i\right\}$ the filtration generated by this process

$$\mathcal{H}_t^i \triangleq \mathcal{F}_t^{D^i} = \sigma\left(D_s^i : 0 \leq s \leq t\right).$$

The agents' filtration is the one generated by the economic state variables and the default processes

$$\mathcal{G}_t \triangleq \mathcal{F}_t \vee \left[\bigvee_{i=1}^{n} \mathcal{H}_t^i\right].$$

Assumption. We assume that the default times are correlated and we allow for multiple instataneous joint defaults. The multivariate dependence is defined by a Marshall-Olkin copula.

The Model (3)

There exists a set of m independent Cox processes $N_t^{c_j}$ with continuous bounded intensities $\lambda^{c_j}(X_t)$, which can trigger simultaneous defaults.

Conditional on a trigger event of type c_j, at time $\theta_r^{c_j}$, we draw an independent Bernoulli variable $A_{\theta_r^{c_j}}^{i,j}$ with a given probability $p^{i,j} \in [0, 1]$, indicating if obligor i has defaulted or not.

The process N_t^i defined as

$$N_t^i \triangleq \sum_{j=1}^{m} \sum_{\theta_r^{c_j} \leq t} A_{\theta_r^{c_j}}^{i,j},$$

is also a Cox process with intensity

$$\lambda^i (X_t) = \sum_{j=1}^{m} p^{i,j} \lambda^{c_j} (X_t) .$$

The Model (4)

The default time τ_i is defined as the first jump time of the Cox process N_t^i:

$$\tau_i = \inf \left\{ t : N_t^i > 0 \right\}.$$

This can be formally described by the following SDE

$$dD_t^i = \left(1 - D_{t^-}^i\right) \sum_{j=1}^{m} A_t^{i,j} dN_t^{c_j}.$$

The Marshall-Olkin filtration is much larger that the one accessible to the agents. It contains the evolution of the common trigger events and the “conditional” Bernoulli events:

$$\widetilde{\mathcal{G}}_t = \mathcal{F}_t \vee \left[\bigvee_{j=1}^{m} \mathcal{F}_t^{N^{c_j}} \right] \vee \left[\bigvee_{j=1}^{m} \bigvee_{i=1}^{n} \mathcal{F}_t^{A^{i,j}} \right].$$

The Problem (1)

In our economy, we assume that we have $(n+1)$ primary assets available for hedging with price processes $S^i = \left(S^i_t\right)_{0 \leq t \leq T^*}$.

The first asset S^0 is the money-market account, i.e., $S^0_t = \exp\left(\int_0^t r_s ds\right)$. it will be used as numeraire, and all quantities will be expressed in units of S^0.

We shall consider zero-coupon credit derivatives or contingent claims of the European type.

The hedging assets. S^i will represent the zero-coupon defaultable bond maturing at T linked to obligor i; i.e., it pays 1 if obligor i survives until time T, or 0 otherwise. The payoff at maturity is defined as:

$$S^i_T \triangleq 1 - D^i_T.$$

The Problem (2)

In practice, zero-coupon defaultable bonds are not traded in the market. They can, however, be extracted from the prices of liquid default swap instruments with different maturities.

Definition. A contingent claim is a $\mathcal{G}_T$-measurable random variable H_T describing the payoff at maturity T of a financial instrument.

Example 1. The payoff of a k^{th}-to-default (zero-coupon note) maturing at T is defined as:

$$H_T^{(k)} \triangleq \mathbf{1}_{\{\sum_{i=1}^{n} D_T^i < k\}},$$

it will pay 1 if there are less than k defaults in the basket or 0 otherwise. The most common structure in this category is a first-to-default, $H_T^{(1)}$, which pays 1 if no obligor in the basket defaults before T.

The Problem (3)

Example 2. Assuming that the recovery rate for obligor i is a constant proportion $R^i \in [0,1)$,the payoff of a CDO (zero-coupon note) covering the portfolio losses, which fall in some range $[K_1, K_2]$, where $0 \leq K_1 < K_2 \leq 1$, is

$$H_T^{(K_1,K_2)} = \frac{1}{K_2 - K_1} \max\left(\min\left(\sum_{i=1}^{n} \left(1 - R^i\right) D_T^i - K_1, 0\right), K_2 - K_1\right).$$

We shall consider the problem of pricing and hedging zero-coupon contingent claims by dynamically trading the hedging assets S. The contingent claims, in this context, include credit derivatives of the basket type.

The Problem (4)

As shown in Föllmer and Schweizer (1991), for non-attainable claims, a locally risk-minimizing strategy is chracterized by: (a) its cost process must be a martingale, (b) the cost process is orthogonal to M^S the martingale part of the price process S.

This is equivalent to having the following decomposition

$$H_T = H_0 + \int_{]0,t]} \left(\alpha_t^{H_T}\right)^{tr} dS_t + L_t^{H_T},$$

where L^{H_T} is a martingale orthogonal to M^S. This is known as the Föllmer-Schweizer decomposition.

Our goal is to find an analytical result for $\left(\alpha_t^{H_T}\right)$.

The Equivalent Fatal Shock Model (1)

We use the "Equivalent Fatal Shock Model" (see Lindskog and McNeil (2003)) as a tool to equivalently describe the Marshall-Olkin model. This provides an explicit representation of the marked point process, which will be used throughout.

Let $\mathbf{\Pi_n}$ be the set of all subsets of $\{1, ..., n\}$. For each $\pi \in \mathbf{\Pi_n}$, we introduce the point process N_t^π, which counts the number of shocks in $(0, t]$ resulting in joint defaults of the obligors in π **only**:

$$N_t^\pi = \sum_{j=1}^{m} \sum_{r=1}^{N_t^{c_j}} A_{\theta_r^{c_j}}^{\pi,j},$$

where

$$A_t^{\pi,j} = \prod_{i \in \pi} A_t^{i,j} \prod_{i \notin \pi} \left(1 - A_t^{i,j}\right).$$

We have the key result of the fatal shock representation (see Proposition 4 in Lindskog and McNeil (2003)).

The Equivalent Fatal Shock Model (2)

Proposition 3 *(Fatal shock representation).* $(N^\pi)_{\pi \in \mathbf{\Pi_n}}$ *are independent Cox processes, with intensities*

$$\lambda^\pi(X_t) = \sum_{j=1}^{m} p^{\pi,j} \lambda^{c_j}(X_t),$$
$$\text{where } p^{\pi,j} = \prod_{i \in \pi} p^{i,j} \prod_{i \notin \pi} \left(1 - p^{i,j}\right).$$

Furthermore, we define the random time τ_π as the first jump time of the Cox process N^π:

$$\tau_\pi = \inf\{t \geq 0 : N_t^\pi > 0\},$$

the default process $D_t^\pi = \mathbf{1}_{\{\tau_\pi \leq t\}}$, and the filtration

$$\mathcal{H}_t^\pi = \mathcal{F}_t^{D^\pi} = \sigma\left(D_t^\pi : 0 \leq s \leq t\right).$$

For $\pi \in \mathbf{\Pi_n}$, the stopped process $M_t^\pi \triangleq D_t^\pi - \int_0^{t \wedge \tau_\pi} \lambda^\pi(X_s)\, ds$ is a $\left(P, \left\{\widetilde{\mathcal{G}}_t\right\}\right)$-martingale.

The Equivalent Fatal Shock Model (3)

Lemma 4 *(Relationship between filtrations). We have*

$$\bigvee_{i=1}^{n} \mathcal{H}_t^i \subset \bigvee_{\pi \in \mathbf{\Pi_n}} \mathcal{H}_t^\pi .$$

Lemma 5 *(Obligor description using the fatal shock representation).*

1. *The Cox process N^i is given by*

$$N_t^i = \sum_{\pi \in \mathbf{\Pi_n}} \mathbf{1}_{\{i \in \pi\}} N_t^\pi ,$$

and its intensity is

$$\lambda^i (X_t) = \sum_{\pi \in \mathbf{\Pi_n}} \mathbf{1}_{\{i \in \pi\}} \lambda^\pi (X_t) .$$

2. *The default time τ_i is given by*

$$\tau_i = \min \{ \tau_\pi : \pi \in \mathbf{\Pi_n}, i \in \pi \} .$$

The Marked Point Process Representation (1)

The Marshall-Olkin model is defined on the filtration $\left\{\widetilde{\mathcal{G}}_t\right\}$, which is larger that the one available to investors, namely $\{\mathcal{G}_t\}$.

We shall use the generic tools of the MO model, however the local characteristics of the MPP representation are derived for the $\{\mathcal{G}_t\}$ filtration.

We define the sequence of ordered default times $(T_0, T_1, ..., T_n) : T_0 = 0 \leq T_1 \leq ... \leq T_n$, and identities of the defaulted obligors as:

$$\begin{aligned} T_0 &= 0,\ Z_0 = \emptyset; \\ T_k &= \min\left\{\tau_i : 1 \leq i \leq n,\ \tau_i > T_{k-1}\right\}; \\ Z_k &= \pi \text{ if } T_k = \tau_i \text{ for all } i \in \pi, \text{ and } \pi \in \mathbf{\Pi_n}; \end{aligned}$$

The mark space of this point process is $E \triangleq \mathbf{\Pi_n}$, the set of all subsets of $\{1, ..., n\}$.

The Marked Point Process Representation (2)

The sequence $(T_k, Z_k)_{k\geq 1}$ defines a MPP with couting measure

$$\begin{aligned} \mu(\omega, dt \times dz) &: (\Omega, \mathcal{G}) \to ((0,\infty) \times E, (0,\infty) \otimes \mathcal{E}), \\ \int_0^t \int_E H(\omega, t, z)\, \mu(\omega, dt \times dz) &= \sum_{k\geq 1} H(\omega, T_k(\omega), Z_k(\omega))\, \mathbf{1}_{\{T_k(\omega)\leq t\}}, \end{aligned}$$

and $(P, \{\mathcal{G}_t\})$-intensity kernel

$$\lambda_t(\omega, dz)\, dt = \lambda_t(\omega)\, \Phi_t(\omega, dz)\, dt,$$

where λ_t is the non-negative $\{\mathcal{G}_t\}$-predictable process

$$\lambda_t = \sum_{\pi \in \mathbf{\Pi_n}} \mathbb{E}\left[1 - D_t^{\pi} \,|\mathcal{G}_t\right] \lambda^{\pi}(X_t),$$

and $\Phi_t(\omega, dz)$ is the probability transition kernel from $\left(\Omega \times [0,\infty), \mathcal{G} \otimes \mathcal{B}_+\right)$ into $(E, \mathcal{E})$

$$\Phi_t(\omega, \pi) = \frac{\mathbb{E}\left[1 - D_t^{\pi} \,|\mathcal{G}_t\right] \lambda^{\pi}(X_t)}{\lambda_t}, \text{ for } \pi \in \mathbf{\Pi_n},$$

with $\Phi_t(.) = 0$ if $\lambda_t = 0$. $(\lambda_t, \Phi_t(dz))$ are the $(P, \{\mathcal{G}_t\})$-local characteristics of $\mu(dt \times dz)$.

The Marked Point Process Representation (3)

For $1 \leq i \leq n$, the compensated point process M_t^i is given by

$$M_t^i = \int_0^t \int_E \mathbf{1}_{\{i \in z\}} \left(\mu \left(dt \times dz \right) - \lambda_t \left(dz \right) dt \right),$$

which can be written as

$$M_t^i = \sum_{\pi \in \mathbf{\Pi_n}} \mathbf{1}_{\{i \in z\}} M_t^{\pi | \{\mathcal{G}_t\}}$$

where $M^{\pi | \{\mathcal{G}_t\}}$ is the $\{\mathcal{G}_t\}$-adapted version of the compensated point process M^π:

$$M_t^{\pi | \{\mathcal{G}_t\}} \triangleq \mathbb{E} \left[D_t^\pi \, | \mathcal{G}_t \right] - \int_0^t \mathbb{E} \left[1 - D_s^\pi \, | \mathcal{G}_s \right] \lambda^\pi \left(X_s \right) ds.$$

This MPP representation makes formal the idea that the mark space of the default times $(\tau_1, .., \tau_n)$ is $\mathbf{\Pi_n}$ since joint defaults are allowed. Here, we have fixed the mark space, but as default events occur, we put zero probability mass for the states of $\mathbf{\Pi_n}$, which cannot occur anymore.

Equivalent Local Martingale Measures (1)

Assumption. We assume that the dynamics of the zero-coupon defaultable bonds, under P, are given by

$$dS_t^i = S_{t^-}^i \left(\mu_t^i dt + \left(\sigma_t^i \right)^{tr} dW_t - dM_t^i \right),$$

where μ_t^i and σ_t^i are $\{\mathcal{G}_t\}$-predictable processes, uniformly bounded in t and ω, and regular enough to ensure that the prices S_t^i are bounded for almost all $\omega \in \Omega$.

Assumption. We assume that the single name assets are not redundant.

No arbitrage. There are no arbitrage opportunities if and only if there exists a probability measure Q equivalent to P under which the (discounted) security prices S are local martingales.

To classify the equivalent probability measures, we use the following Girsanov transformation (see Jacod and Shiryaev (1987)).

Equivalent Local Martingale Measures (2)

Theorem 6 *Let θ be a d-dimensional $\{\mathcal{G}_t\}$-predictable process and let $\phi(t,z)$ be a $\{\mathcal{G}_t\}$-predictable E-indxed nonnegative process such that:*

$$\int_0^t \|\theta_s\|^2 \, ds < \infty, \ \int_0^t \int_E |\phi(s,z)| \, \lambda_s(dz) \, ds < \infty,$$

for finite t. Define the process L by

$$dL_t = L_{t^-} \left(\sum_{k=1}^{d} \theta_t^k dW_t^k + \int_E (\phi(s,z) - 1) (\mu(dt \times dz) - \lambda_t(dz) \, dt) \right),$$

and $L_0 = 1$. Suppose that $\mathbb{E}^P[L_t] = 1$, for all finite t. Then, there exists a probability measure Q equivalent to P such that

1. *we have $dW_t = \theta_t dt + d\widetilde{W}_t$, where $\widetilde{W}$ is a d-dimensional Brownian motion under Q;*

2. *the counting measure $\mu(dt \times dz)$ has a $(Q, \{\mathcal{G}_t\})$-intensity kernel given by*

$$\widetilde{\lambda}_t(dz) \, dt = \phi(t,z) \, \lambda_t(dz) \, dt.$$

Equivalent Local Martingale Measures (3)

Every probability measure Q locally equivalent to P has the same structure as described above.

We restrict our set of ELMMs to the ones constructed with:

1. θ is a d-dimensional$\{\mathcal{F}_t\}$-predictable process;

2. the E-indexed $\{\mathcal{G}_t\}$-predictable process $\phi(t, z)$ takes the form

$$\phi(t, \pi) = \phi^{\pi}(X_t), \text{ for } \pi \in \mathbf{\Pi_n},$$

   where $(\phi^{\pi})_{\pi \in \mathbf{\Pi_n}}$ is a set of strictly positive continuous bounded functions $\phi^{\pi} : \mathbb{R}^d \to \mathbb{R}_+$.

This allows us to preserve the Cox-process assumptions under the change of measure Q; in other words, the intensities are still driven by the d-dimensional Itô process X.

Equivalent Local Martingale Measures (4)

The dynamics of the zero-coupon defaultable bonds under the equivalent probability measure Q are given by

$$dS_t^i = S_{t-}^i \left(\widetilde{\mu}_t^i dt + \left(\sigma_t^i\right)^{tr} d\widetilde{W}_t - \int_E \mathbf{1}_{\{i \in z\}} \left(\mu\left(dt \times dz\right) - \widetilde{\lambda}_t\left(dz\right) dt \right) \right),$$

where

$$\begin{aligned} \widetilde{\mu}_t^i &= \mu_t^i + \left(\sigma_t^i\right)^{tr} \theta_t - \int_E \mathbf{1}_{\{i \in z\}} \left(\phi\left(t, z\right) - 1\right) \lambda_t\left(dz\right) \\ &= \mu_t^i + \left(\sigma_t^i\right)^{tr} \theta_t - \sum_{\pi \in \mathbf{\Pi_n}} \mathbf{1}_{\{i \in \pi\}} \left(\phi^\pi\left(X_t\right) - 1\right) \mathbb{E}\left[1 - D_t^\pi \left|\mathcal{G}_t\right.\right] \lambda^\pi\left(X_t\right). \end{aligned}$$

To ensure absence of arbitrage, the drift $\widetilde{\mu}_t^i$ under Q is equal to zero:

$$0 = \mu_t^i + \left(\sigma_t^i\right)^{tr} \theta_t - \sum_{\pi \in \mathbf{\Pi_n}} \mathbf{1}_{\{i \in \pi\}} \left(\phi^\pi\left(X_t\right) - 1\right) \mathbb{E}\left[1 - D_t^\pi \left|\mathcal{G}_t\right.\right] \lambda^\pi\left(X_t\right).$$

This system of linear equations classifies the set of ELMM that we consider. We have n equations (one for each security) and $2^n - 1 + d$ unknowns (corresponding to each source of risk). The market is incomplete, hence the ELMM is not unique.

The Minimal Martingale Measure (1)

Assumption. We assume that the matrix

$$\left[\left(\sigma_t^i\right)^{tr}\sigma_t^j + \sum_{\pi\in\mathbf{\Pi_n}} \mathbf{1}_{\{i\in\pi\}}\mathbf{1}_{\{j\in\pi\}}\mathbb{E}\left[1 - D_t^\pi\,|\mathcal{G}_t\right]\lambda^\pi\left(X_t\right)\right]_{\substack{1\le i\le n\\ 1\le j\le n}}$$

is invertible for all $t \in [0, T^*]$.

Proposition 7 *(Minimal martingale measure). Define the n-dimensional local martingale M^S*

$$\left(M_t^S\right)^i = \int_0^t S_{s^-}^i\left(\left(\sigma_s^i\right)^{tr} dW_s - \int_E \mathbf{1}_{\{i\in z\}}\left(\mu\left(ds\times dz\right) - \lambda_s\left(dz\right)ds\right)\right),$$

and the n-dimensional predictable process $\widehat{\lambda}$ as the solution of the linear system

$$\mu_t^i = \sum_{j=1}^n \widehat{\lambda}_t^j S_{t^-}^j\left[\left(\sigma_t^i\right)^{tr}\sigma_t^j + \sum_{\pi\in\mathbf{\Pi_n}} \mathbf{1}_{\{i\in\pi\}}\mathbf{1}_{\{j\in\pi\}}\mathbb{E}\left[1 - D_t^\pi\,|\mathcal{G}_t\right]\lambda^\pi\left(X_t\right)\right].$$

Then, the minimal martingale measure is given by the Doléans-Dade exponential $\mathcal{E}\left(-\int \widehat{\lambda}^{tr} dM^S\right)_t$.

Proof. The Doob-Meyer decomposition of the price process S is given by

$$\begin{aligned} S_t &= S_0 + M_t^S + A_t^S, \\ \left(A_t^S\right)^i &= \int_0^t S_{s-}^i \mu_s^i ds, \\ \left(M_t^S\right)^i &= \int_0^t S_{s-}^i \left(\left(\sigma_s^i\right)^{tr} dW_s - \int_E \mathbf{1}_{\{i\in z\}} \left(\mu\left(ds \times dz\right) - \lambda_s\left(dz\right) ds\right)\right). \end{aligned}$$

The predictable covariance process of M is

$$\left\langle M^{S^i}, M^{S^j} \right\rangle = \int_0^t S_{s-}^i S_{s-}^j \left(\left(\sigma_s^i\right)^{tr} \sigma_s^j + \int_E \mathbf{1}_{\{i\in z\}} \mathbf{1}_{\{j\in z\}} \lambda_s\left(dz\right)\right) ds$$

The finite variation process A_t^S can be expressed as

$$\left(A_t^S\right)^i = \left(\int_0^t d\left\langle M^S \right\rangle_s \widehat{\lambda}_s\right)^i = \sum_{j=1}^n \int_0^t \widehat{\lambda}_s^j d\left\langle M^{S^i}, M^{S^j} \right\rangle_s,$$

where the predictable process $\widehat{\lambda}$ is given by inverting the following linear system

$$\mu_t^i = \sum_{j=1}^n \widehat{\lambda}_t^j S_{t^-}^j \left[\left(\sigma_t^i\right)^{tr} \sigma_t^j + \sum_{\pi\in\mathbf{\Pi_n}} \mathbf{1}_{\{i\in\pi\}} \mathbf{1}_{\{j\in\pi\}} \mathbb{E}\left[1 - D_t^\pi \,|\mathcal{G}_t\right] \lambda^\pi\left(X_t\right)\right].$$

The uniform boundedness in t and ω of the mean-variance trade-off process

$$\widehat{K}_t = \int_0^t \left(\widehat{\lambda}_s\right)^{tr} d\left\langle M^S\right\rangle_s \widehat{\lambda}_s = \sum_{i,j=1}^{n} \int_0^t \widehat{\lambda}_s^i \widehat{\lambda}_s^j d\left\langle M^{S^i}, M^{S^j}\right\rangle_s$$

ensures that the minimal martingale measure is given by the Doléans-Dade exponential as shown in Föllmer and Schweizer (1991). ■

The Minimal Martingale Measure (2)

Signed measure. In general, the minimal martingale measure for discontinuous processes is only a signed measure since $\mathcal{E}\left(-\int \widehat{\lambda}^{tr} dM^S\right)_{T^*}$ can reach negative values. To ensure that $\widehat{P}$ is a true probability measure, we need additional assumptions. We can write the process $\left(\widehat{Z}_t\right)_{t\geq 0}$ as

$$\widehat{Z}_t = \exp\left(-\int_0^t \left(\widehat{\lambda}_s\right)^{tr} d\overline{M}_s^S - \frac{1}{2}\int_0^t \left(\widehat{\lambda}_s\right)^{tr} d\left\langle \overline{M}^S\right\rangle_s \widehat{\lambda}_s\right) \prod_{s\leq t}\left(1 - \left(\widehat{\lambda}_s\right)^{tr} \Delta M_s^S\right),$$

where $\overline{M}_t^S \triangleq M_t^S - \sum_{s\leq t} \Delta M_s^S$ is the continuous part of the martingale M^S.

The jump part is given by

$$\sum_{s\leq t}\left(\Delta M_s^S\right)^i = -\int_0^t \int_E S_{s-}^i \mathbf{1}_{\{i\in z\}} \mu\left(ds\times dz\right) = -\sum_{T_k\leq t} S_{T_k^-}^i \mathbf{1}_{\{i\in Z_k\}},$$

$$\prod_{s\leq t}\left(1 - \left(\widehat{\lambda}_s\right)^{tr} \Delta M_s^S\right) = \prod_{T_k\leq t}\left(1 + \sum_{i=1}^n \widehat{\lambda}_{T_k}^i S_{T_k^-}^i \mathbf{1}_{\{i\in Z_k\}}\right).$$

The Minimal Martingale Measure (3)

$\widehat{Z}_{T^*}$ is strictly positive if all the factors $\left(1+\sum_{i=1}^{n}\widehat{\lambda}^{i}_{T_k}S^{i}_{T_k^-}\mathbf{1}_{\{i\in Z_k\}}\right)$ are positive.

A useful property of the minimal martingale measure is that if $(\alpha_t)_{t\geq 0}$ is a locally risk-minimizing strategy for the T-claim H_T, then the value process is given by

$$V_t(\alpha) = \widehat{\mathbb{E}}\left[H_T \,|\mathcal{G}_t\right].$$

Next, we shall work under the minimal martingale measure and all expectations will be taken under this measure. Moreover, the FS decomposition will done under $\widehat{P}$.

This is similar to the approach taken in Föllmer and Sondermann (1986) where a “good” martingale measure is chosen, then the local risk minimization is done with respect to this measure.

Martingale Representation (1)

The agents' filtration $\{\mathcal{G}_t\}$ is generated by the Brownian motion $\widetilde{W}$ and the MPP $\mu(dt \times dz)$ with $\left(\widehat{P}, \{\mathcal{G}_t\}\right)$-intensity kernel $\widetilde{\lambda}_t(dz)$.

The martingale generator is $\left(\widetilde{W}, \left(\mu(dt \times \{z\}) - \widetilde{\lambda}_t(\{z\})\right)_{z \in \mathbf{\Pi_n}}\right)$(see Jacod and Shiryaev (1987) Chap III Corollary 4.31).

Proposition 8 *(Martingale representation of H_t). The $\{\mathcal{G}_t\}$-martingale $H_t = \widehat{\mathbb{E}}[H_T | \mathcal{G}_t]$, $t \in [0, T^*]$, where H_T is a $\mathcal{G}_T$-measurable random variable, integrable with respect to $\widehat{P}$, admits the following integral representation*

$$H_t = H_0 + \int_0^t (\xi_s)^{tr} \, d\widetilde{W}_s - \int_0^t \int_E \zeta(s,z) \left(\mu(dt \times dz) - \widetilde{\lambda}_s(dz)\, ds\right),$$

where ξ is a d-dimensional $\{\mathcal{G}_t\}$-predictable process and $\zeta(s,z)$ is an E-indexed $\{\mathcal{G}_t\}$-predictable process $\zeta(s,z)$ such that

$$\int_0^t \|\xi_s\|^2 \, ds < \infty, \ \int_0^t \int_E \zeta(s,z)\, \widetilde{\lambda}_s(dz)\, ds < \infty,$$

almost surely.

Martingale Representation (2)

This can be written as

$$H_t = H_0 + \int_0^t (\xi_s)^{tr} \, d\widetilde{W}_s - \sum_{\pi \in \mathbf{\Pi_n}} \int_{]0,t]} \zeta(s,\pi) \, d\widetilde{M}_s^{\pi|\{\mathcal{G}_t\}},$$

where $\widetilde{M}^{\pi|\{\mathcal{G}_t\}}$ is the $\{\mathcal{G}_t\}$-adapted version of the compensated point process $\widetilde{M}^{\pi}$:

$$\widetilde{M}_t^{\pi|\{\mathcal{G}_t\}} \triangleq \mathbb{E}\left[D_t^{\pi} \,|\mathcal{G}_t\right] - \int_0^t \mathbb{E}\left[1 - D_s^{\pi} \,|\mathcal{G}_s\right] \widetilde{\lambda}^{\pi}(X_s) \, ds.$$

In order to replicate the claim H_T, one needs to match the diffusion terms ξ_s^i, $1 \le i \le d$, and the jump-to-default terms $[-\zeta(s,\pi)]$ for each possible default state.

Computing the Hedging Strategy: Main Result (1)

We use the martingale representation in Proposition 8 to derive the local risk-minimization hedging strategy. As shown in Föllmer and Schweizer (1991), this is equivalent to finding the FS-decomposition

$$H_T = H_0 + \int_{]0,t]} (\alpha_t)^{tr} \, dS_t + L_t.$$

Our goal is to establish an analytical result, which derives single-name hedges $\left(\alpha^i\right)_{1 \le i \le n}$ in terms of the martingale representation processes ξ and $\zeta(., \pi)$, $\pi \in \mathbf{\Pi_n}$.

Computing the Hedging Strategy: Main Result (2)

The strategy $(\alpha_t)_{t \geq 0}$ can be computed as

$$\alpha_t = d\left\langle M^S \right\rangle_t^{-1} d\left\langle M^S, V(\alpha) \right\rangle_t,$$

where the value process is given by

$$V_t(\alpha) = \widehat{\mathbb{E}}\left[H_T \,|\mathcal{G}_t\right], \text{ for } t \in [0, T].$$

This follows from the FS-decomposition of H under $\widehat{P}$ and the projection of $V_t(\alpha)$ on the $\widehat{P}$-martingale $\int_{]0,t]} (\alpha_s)^{tr} dS_s$.

Theorem 9 *(Local risk-minimization hedging strategy). The local risk-minimization hedging strategy of a general (basket) contingent claim with single name instruments is given by the solution of the following linear system,*

for $1 \leq k \leq n$,

$$\begin{aligned} & \sum_{i=1}^{n} \alpha_t^i \left[S_{t-}^i S_{t-}^k \left[\left(\sigma_t^i\right)^{tr} \sigma_t^k + \int_E \mathbf{1}_{\{i \in z\}} \mathbf{1}_{\{k \in z\}} \widetilde{\lambda}_t(dz) \right] \right] \\ = \; & S_{t-}^k \left(\sigma_t^k\right)^{tr} \xi_t + \int_E \zeta(t, z) \, S_{t-}^k \mathbf{1}_{\{k \in z\}} \widetilde{\lambda}_t(dz). \end{aligned}$$

Proof. M^S is defined as

$$\left(M_t^S\right)^i = \int_0^t S_{s^-}^i \left(\left(\sigma_s^i\right)^{tr} d\widetilde{W}_s - \int_E \mathbf{1}_{\{i\in z\}} \left(\mu\left(ds \times dz\right) - \widetilde{\lambda}_s\left(dz\right) ds\right)\right),$$

and the predictable covariance is

$$d\left\langle M^S\right\rangle_t^{i,j} = d\left\langle M^{S^i}, M^{S^j}\right\rangle_t = S_{t-}^i S_{t-}^j \left(\left(\sigma_t^i\right)^{tr} \sigma_t^j + \int_E \mathbf{1}_{\{i\in z\}} \mathbf{1}_{\{j\in z\}} \widetilde{\lambda}_t\left(dz\right)\right) dt.$$

The value process $V_t\left(\alpha\right) = \widehat{\mathbb{E}}\left[H_T \left|\mathcal{G}_t\right.\right]$ is given by the martingale representation

$$V_t\left(\alpha\right) = \widehat{\mathbb{E}}\left[H_T \left|\mathcal{G}_t\right.\right] = H_0 + \int_0^t \left(\xi_s\right)^{tr} d\widetilde{W}_s - \int_0^t \int_E \zeta\left(s, z\right) \left(\mu\left(ds \times dz\right) - \widetilde{\lambda}_s\left(dz\right) ds\right).$$

Hence, we have

$$d\left\langle M^S, V\left(\alpha\right)\right\rangle_t^i = S_{t-}^i \left(\left(\sigma_t^i\right)^{tr} \xi_t + \int_E \mathbf{1}_{\{i\in z\}} \zeta\left(t, z\right) \widetilde{\lambda}_t\left(dz\right)\right) dt.$$

■

Computing the Hedging Strategy: Main Result (3)

This can be written explicitly in terms of the model parameters.

Applying Itô's lemma and using the Markovian property of X, we find an explicit expression of the dynamics of S^i under the martingale measure $\widehat{P}$.

Lemma 10 *(Single-name price process representation). We have*

$$S_t^i = S_0^i - \int_0^t \int_E \widetilde{s}^i(s, X_s)\, \mathbf{1}_{\{i \in z\}} \Big(\mu(ds \times dz) - \widetilde{\lambda}_s(dz)\, ds\Big) + \int_0^t \Big(1 - D_s^i\Big) \sum_{j=1}^d \sum_{k=1}^d \frac{\partial \widetilde{s}^i(s, X_s)}{\partial x_j} \beta_{jk}(X_s)\, d\widetilde{W}_s^k,$$

where $\widetilde{s}^i(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *is defined as*

$$\widetilde{s}^i(t,x) \triangleq \widehat{\mathbb{E}}_{(t,x)}\left[\exp\left(-\int_t^T \widetilde{\lambda}^i(X_s)\, ds\right)\right].$$

Computing the Hedging Strategy: Main Result (4)

Lemma 10 establishes the martingale representation for the single-name securities whose payoff is $H_T = 1 - D_T^i$:

$$
\begin{aligned}
\zeta^{1-D_T^i}(t,z) &= \mathbf{1}_{\{i \in z\}} \widetilde{s}^i(t, X_t), \text{ for } z \in \mathbf{\Pi_n}, \\
\left(\xi_t^{1-D_T^i}\right)^k &= \left(1 - D_t^i\right) \sum_{j=1}^{d} \frac{\partial \widetilde{s}^i(t, X_t)}{\partial x_j} \beta_{jk}(X_t).
\end{aligned}
$$

The hedging strategy is solution of

for $1 \leq k \leq n$

$$
\begin{aligned}
& \sum_{i=1}^{n} \alpha_t^i \left[\int_E \zeta^{1-D_T^k}(t,z)\, \zeta^{1-D_T^i}(t,z)\, \widetilde{\lambda}_t(dz) + \left(\xi_t^{1-D_T^k}\right)^{tr} \xi_t^{1-D_T^i} \right] \\
= \; & \int_E \zeta(t,z)\, \zeta^{1-D_T^k}(t,z)\, \widetilde{\lambda}_t(dz) + \left(\xi_t^{1-D_T^k}\right)^{tr} \xi_t.
\end{aligned}
$$

Note that this problem combines both default risk and spread risk.

Applications (1)

We consider a first-to-default (basket) contingent claim whose payoff is

$$H_T^{(1)} = \prod_{i=1}^{n} \left(1 - D_T^i\right).$$

The price of this claim, at time t, is

$$H_t^{(1)} = \widehat{\mathbb{E}}\left[\prod_{i=1}^{n} \left(1 - D_T^i\right) |\mathcal{G}_t\right].$$

We can show that it can be expressed as

$$H_t^{(1)} = \left[\prod_{i=1}^{n} \left(1 - D_t^i\right)\right] \widetilde{h}^{(1)}(t, X_t),$$

where the function $\widetilde{h}^{(1)}(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is defined as

$$\begin{aligned}
\widetilde{h}^{(1)}(t, X_t) &= \widehat{\mathbb{E}}_{(t,x)}\left[\exp\left(-\int_t^T \widetilde{\lambda}^{(1)}(s, X_s)\, ds\right)\right], \\
\widetilde{\lambda}^{(1)}(t, x) &= \sum_{j=1}^{m} \left[1 - \prod_{i=1}^{n} \left(1 - \widetilde{p}^{i,j}\right)\right] \widetilde{\lambda}^{c_j}(X_t).
\end{aligned}$$

Applications (2)

Using Itô's lemma and some algebra, we find

$$\begin{aligned} dH_t^{(1)} &= -\int_E \widetilde{h}^{(1)}(t, X_t) \left(\mu(dt \times dz) - \widetilde{\lambda}_t(dz)\, dt \right) \\ &\quad + \left[\prod_{i=1}^n \left(1 - D_t^i \right) \right] \sum_{j=1}^d \sum_{k=1}^d \frac{\partial \widetilde{h}^{(1)}(t, X_t)}{\partial x_j} \beta_{jk}(X_t)\, d\widetilde{W}_t^k . \end{aligned}$$

This gives the processes of the martingale representation

$$\begin{aligned} \zeta^{H_T^{(1)}}(t, z) &= \widetilde{h}^{(1)}(t, X_t), \text{ for all } z \in \mathbf{\Pi_n}, \\ \left(\xi_t^{H_T^{(1)}} \right)^k &= \left[\prod_{i=1}^n \left(1 - D_t^i \right) \right] \sum_{j=1}^d \frac{\partial \widetilde{h}^{(1)}(t, X_t)}{\partial x_j} \beta_{jk}(X_t), \end{aligned}$$

which can be plugged into the linear system of Theorem 9. Inverting the latter gives the single-name hedge ratios of the first-to-default basket claim.

Conclusion

- We have addressed the problem of hedging basket credit derivatives with single-name instruments in a Marshall-Olkin Model.

- We have used the Equivalent Fatal Shock model to derive the Marked Point Process representation.

- This is then used to classify the set of ELMMs and to derive the Minimal Martingale Measure $\widehat{P}$.

- Working under $\widehat{P}$, we have applied a Martingale Representation Theorem to the value process, and established the local risk-minimization hedging strategy.

- We have worked out the First-To-Default example to illustrate our result.

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