A Futures Trading Model with Transaction Costs

Karel Janeček
Johann Radon Institute for Computational
and Applied Mathematics
Linz, Austria

Steven E. Shreve
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, USA

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The day of colonial independence.

Background on proportional transaction costs λ

- Magill & Constantinides, *J. Economic Theory*, 1976.
  Proportional transaction costs into Merton's optimal consumption model.
- Hodges & Neuberger, *Rev. Futures Markets*, 1989.
  Option pricing in the presence of proportional transaction costs.
- Constantinides, *J. Political Economy*, 1986.
  Numerical computation of liquidity premium; unable to compute for λ below 50 basis points.
- Fleming, Grossman, Vila & Zariphopoulou, 1990.
  In model with consumption at final time only, found liquidity premium $O(\lambda^{2/3})$.
- Davis & Norman, *Math. Operations Research*, 1990.
  Rigorous treatment of Magill/Constantinides model.

- Shreve & Soner, *Ann. Applied Probability*, 1994.

  Viscosity solution analysis of Magill/Constantinides model. Found liquidity premium $O(\lambda^{2/3})$.

- Whalley & Wilmott, *Math. Finance*, 1997.

  Formal asymptotic expansion in powers of $\lambda^{\frac{1}{3}}$ for option pricing problem.

- Janeček & Shreve, *Finance and Stochastics*, 2004.

  Viscosity solution derivation of first two terms in asymptotic expansion of value function.

- Rogers, *Mathematics of Finance*, 2004.

  Simple heuristic probabilistic explanation of $O(\lambda^{\frac{2}{3}})$ effect.

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  (ii) Attempt to escape the Markov bonds.

  (iii) Probability is more fun.

Futures Trading Model

(Mike Harrison: "Work on the simplest problem you don't understand.")

- Futures price: $F(t) = F(0) + \alpha t + \sigma W(t)$.
- Number of futures contracts held: $Y(t) = L(t) - M(t)$.
- Cash in money market:

$$dX(t) = Y(t)\,dF(t) - \lambda\big(dL(t) + dM(t)\big) + rX(t)\,dt - c(t)X(t)\,dt.$$

Introduce trading proportional to wealth:

$$dL(t) = X(t-)\,d\ell(t), \quad dM(t) = X(t-)\,dm(t).$$

Then

$$\begin{aligned} dY(t) &= X(t-)\big(d\ell(t) - dm(t)\big), \\ dX(t) &= Y(t)\big(\alpha\,dt + \sigma dW(t)\big) - \lambda X(t-)\big(d\ell(t) + dm(t)\big) \\ &\quad + X(t)\big(r - c(t)\big)\,dt. \end{aligned}$$

Parameter assumptions:

$$\alpha > 0, \quad \sigma > 0, \quad r > 0, \quad \lambda > 0.$$

- Solvency region: $\mathcal{S} \triangleq \{(x,y) : x - \lambda y > 0, x + \lambda y > 0\}$.
- Utility function:

$$U_p(C) \triangleq \begin{cases} \frac{1}{1-p} C^{1-p} & \text{if } p > 0, p \neq 1, \\ \log C & \text{if } p = 1. \end{cases}$$

- Value function:

$$v(x,y) \triangleq \sup_{\ell,m,c} \mathbb{E} \int_0^\infty e^{-\beta t} U_p\big(c(t)X(t)\big)\, dt \quad \forall (x,y) \in \mathcal{S}.$$

- Homotheticity: For $\gamma > 0$,

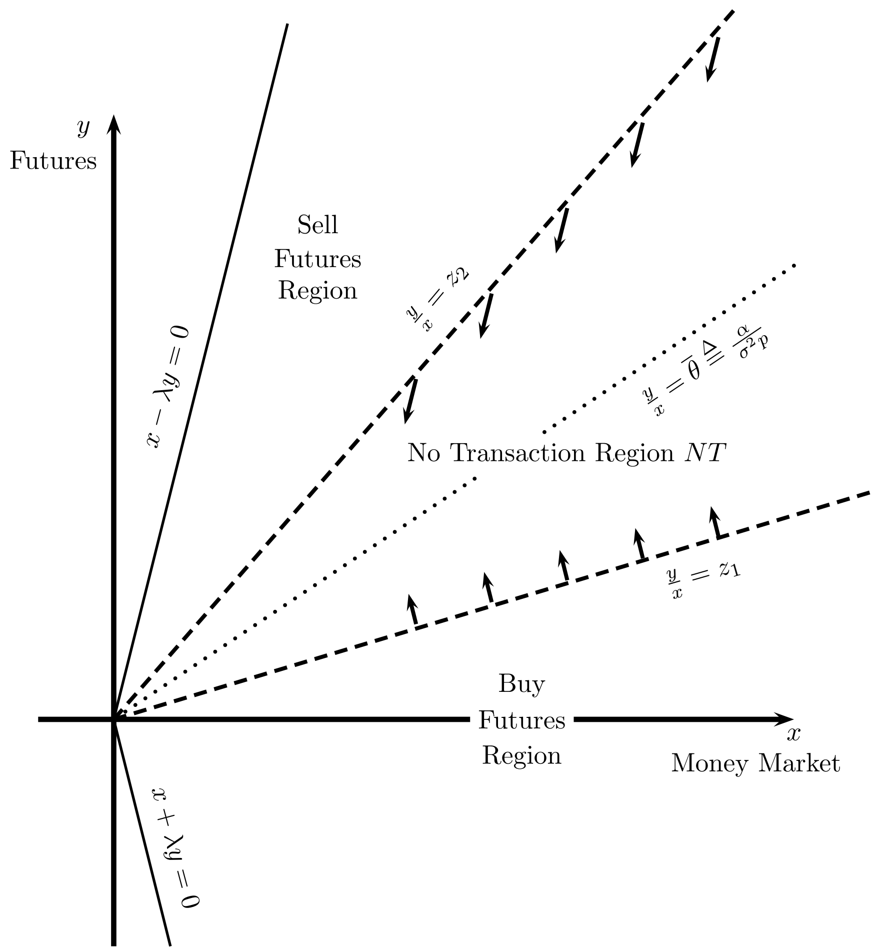
$$v(\gamma x, \gamma y) = \begin{cases} \gamma^{1-p} v(x,y) & \text{if } p > 0, p \neq 1, \\ v(x,y) + \frac{1}{\beta} \log \gamma & \text{if } p = 1. \end{cases}$$

- Hamilton-Jacobi-Bellman equation:

$$\min \left\{ \beta v - (rx + \alpha y) v_x - \frac{1}{2} \sigma^2 y^2 v_{xx} - \widetilde{U}_p(v_x), \lambda v_x - v_y, \lambda v_x + v_y \right\} = 0,$$

where

$$\widetilde{U}_p(\widetilde{C}) \triangleq \max_{C>0} \left\{ U_p(C) - C\widetilde{C} \right\}.$$



y
Futures
Sell
Futures
Region
$x - \lambda y = 0$
$\frac{y}{x} = z_2$
$\frac{y}{x} = \overline{\theta} \stackrel{\Delta}{=} \frac{\alpha}{\sigma^2 p}$
No Transaction Region NT
$\frac{y}{x} = z_1$
Buy
Futures
Region
x
Money Market
$x + \lambda y = 0$

Simplification for this presentation: $0 < p < 1$.

Zero transaction cost:

If λ were zero, the value function would be

$$v_0(x) = \frac{1}{1-p} A^{-p} x^{1-p} \quad \forall x \geq 0,$$

where

$$A \triangleq \frac{\beta - r(1-p)}{p} - \frac{\alpha^2(1-p)}{2\sigma^2 p^2}$$

must be assumed to be positive. The optimal investment proportion is

$$\frac{Y(t)}{X(t)} = \overline{\theta} \triangleq \frac{\alpha}{\sigma^2 p}.$$

The optimal consumption proportion is $c(t) \equiv A$.

Investment proportion

$$\theta(t) \triangleq \frac{Y(t)}{X(t)}$$

HJB equation shows that optimal consumption is a function $c\big(\theta(t)\big)$ of $\theta(t)$. This function is a constant plus $O(\lambda)$.

Evolution of $\theta(t)$:

$$\begin{aligned} d\theta(t) \;=\; & \theta(t)\big(-r + c\big(\theta(t)\big) + \alpha\theta(t) + \sigma^2\theta^2(t)\big)\,dt - \sigma\theta^2(t)\,dW(t) \\ & + \big(1+\lambda\theta(t)\big)\,d\ell(t) - \big(1-\lambda\theta(t)\big)\,dm(t). \end{aligned} \tag{1}$$

Optimal $\theta(t)$ is a doubly-reflected diffusion in the interval $[z_1, z_2]$. The drift and diffiusion are bounded uniformly in λ.

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Method of analysis

- Fix a Lipschitz function $c(\cdot)$, and "small" positive numbers w_1 and w_2.
- Let $\theta(t)$ be given by (1) with reflection occuring at the boundaries of $[\overline{\theta}(1 - w_1), \overline{\theta}(1 + w_2)]$.
- We estimate the cost associated with this consumption and investment strategy.

Decomposing the loss

$c(\cdot)$, $w_1 > 0$ and $w_2 > 0$ given. Let $\theta(\cdot)$ be given by (1). Choose also a constant $c > 0$. Define three processes, all with initial condition 1.

Process with investment ratio $\overline{\theta}$ and paying no transaction cost:

$$dX_0(t) = X_0(t)\left[(r - c + \alpha\overline{\theta})\,dt + \sigma\overline{\theta}\,dW(t)\right].$$

Process with investment ratio $\theta(\cdot)$ and paying no transaction cost:

$$dX_1(t) = X_1(t)\left[\big(r - c + \alpha\theta(t)\big)\,dt + \sigma\theta(t)\,dW(t)\right].$$

Process with investment ratio $\theta(\cdot)$ and paying transaction cost:

$$dX_2(t) = X_2(t)\left[\big(r - c + \alpha\theta(t)\big)\,dt + \sigma\theta(t)\,dW(t) - \lambda\big(d\ell(t) + dm(t)\big)\right].$$

For $i = 1, 2, 3$, define $u_i \triangleq \mathbb{E}\int_0^\infty e^{-\beta t} U_p\big(cX_i(t)\big)\,dt$.

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For $i = 1, 2, 3$, define $u_i \triangleq \mathbb{E}\int_0^\infty e^{-\beta t} U_p\big(cX_i(t)\big)\,dt$.

- Loss due to displacement $u_0 - u_1$ increases with w_1 and w_2.
- Loss due to transaction $u_1 - u_2$ decreases with w_1 and w_2.
- We choose w_1 and w_2 to balance these marginal losses.
- At the $O(\lambda)$ level of accuracy, $c(\cdot)$ is irrelevant.
- After optimizing over w_1 and w_2, we optimize over c.

Theorem 1 *The displacement loss is*

$$u_0 - u_1 = \frac{u_0 p(1-p)\sigma^2 \overline{\theta}^2}{6\big(pA + (1-p)c\big)}(w_1^2 - w_1 w_2 + w_2^2) + O\big((w_1 + w_2)^3\big).$$

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IDEA OF PROOF:

$$X_0(t) = \exp\left\{\left(r - c + \alpha\overline{\theta} - \frac{1}{2}\sigma^2\overline{\theta}^2\right)t + \sigma\overline{\theta}W(t)\right\},$$

$$X_1(t) = \exp\left\{\int_0^t \left(r - c + \alpha\theta(u) - \frac{1}{2}\sigma^2\theta^2(u)\right)du + \int_0^t \sigma\theta(u)\,dW(u)\right\}.$$

A trick learned from Chris Rogers shows

$$\mathbb{E}X_1^{1-p}(t) = \mathbb{E}X_0^{1-p}(t)\cdot\overline{\mathbb{E}}\exp\left\{(1-p)\sigma\int_0^t \big(\theta(u)-\overline{\theta}\big)\,d\overline{W}(u) - \frac{1}{2}(1-p)\sigma^2\int_0^t \big(\theta(u)-\overline{\theta}\big)^2\,du\right\},$$

where

$$\overline{W}(t) \triangleq W(t) - (1-p)\sigma\overline{\theta}t$$

is a Brownian motion under $\overline{\mathbb{E}}$.

Set

$$I(t) \triangleq (1-p)\sigma \int_0^t \big(\theta(u) - \overline{\theta}\big)\, d\overline{W}(u),$$

$$R(t) \triangleq \frac{1}{2}(1-p)\sigma^2 \int_0^t \big(\theta(u) - \overline{\theta}\big)^2\, du.$$

Then

$$\overline{\mathbb{E}} \exp\big(I(t) + R(t)\big) = \overline{\mathbb{E}}\left[1 + I(t) + R(t) + \frac{1}{2}I^2(t) + I(t)R(t) + \frac{1}{2}R^2(t)\right] + \overline{\mathbb{E}} \sum_{n=3}^{\infty} \frac{1}{n!}\big(I(t) + R(t)\big)^n.$$

The interesting terms are

$$\overline{\mathbb{E}}R(t) = \frac{1}{2}(1-p)\sigma^2 \int_0^t \overline{\mathbb{E}}\big(\theta(u) - \overline{\theta}\big)^2\, du,$$

$$\overline{\mathbb{E}}\left[\frac{1}{2}I^2(t)\right] = (1-p)^2\sigma^2 \int_0^t \overline{\mathbb{E}}\big(\theta(u) - \overline{\theta}\big)^2\, du.$$

If $\theta(u)$ were uniform on $[\overline{\theta}(1-w_1), \overline{\theta}(1+w_2)]$, then

$$\overline{\mathbb{E}}\big(\theta(u) - \overline{\theta}\big)^2 = \frac{1}{\overline{\theta}(w_1 + w_2)} \int_{\overline{\theta}(1-w_1)}^{\overline{\theta}(1+w_2)} (x - \overline{\theta})^2\, dx = \overline{\theta}^2(w_1^2 - w_1 w_2 + w_2^2).$$

In fact, $\theta(u)$ is almost uniform and

$$\overline{\mathbb{E}}\big(\theta(u) - \overline{\theta}\big)^2 = \overline{\theta}^2(w_1^2 - w_1 w_2 + w_2^2) + O\big((w_1 + w_2)^3\big).$$

One can use the maximal martingale inequality and Hölder's inequality to show the other terms are also $O\big((w_1 + w_2)^3\big)$.

Theorem 2 *The transaction loss is*

$$u_1 - u_2 = \frac{u_0(1-p)\sigma^2\overline{\theta}^3}{pA + (1-p)c} \cdot \frac{\lambda}{w_1 + w_2} + O(\lambda) + O\left(\frac{\lambda^2}{(w_1 + w_2)^2}\right).$$

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Idea of Proof:

$$X_1(t) = \exp\left\{\int_0^t \left(r - c + \alpha\theta(u) - \frac{1}{2}\sigma^2\theta^2(u)\right) du + \int_0^t \sigma\theta(u) dW(u)\right\},$$

$$X_2(t) = X_1(t) \exp\left\{-\lambda\big(\ell(t) + m(t)\big)\right\}.$$

Therefore,

$$X_1^{1-p}(t) - X_2^{1-p}(t) = Z(t)\Delta(t)Y(t),$$

where

$$Z(t) = \exp\left\{(1-p)\sigma\int_0^t \theta(u)\, dW(u) - \frac{1}{2}(1-p)^2\sigma^2\int_0^t \theta^2(u)\, du\right\},$$

$$\Delta(t) = \exp\left\{(1-p)\int_0^t \left(r - c + \alpha\theta(u) - \frac{1}{2}p\sigma^2\theta^2(u)\right) du\right\},$$

$$Y(t) = 1 - \exp\left\{-\lambda(1-p)\big(\ell(t) + m(t)\big)\right\}.$$

$\Delta(t)$ is nearly deterministic. Task is to estimate $Y(t)$ under the measure induced by Z.

Estimation of $Y(t)$

$$\begin{aligned} Y(t) &= 1 - \exp\left\{-\lambda(1-p)\big(\ell(t)+m(t)\big)\right\} \\ &= \lambda(1-p)\big(\ell(t)+m(t)\big) - \frac{1}{2}\lambda^2(1-p)^2\big(\ell(t)+m(t)\big)^2 e^{-\xi(t)}. \end{aligned}$$

For a doubly reflected Brownian motion on $[\overline{\theta}(1-w_1), \overline{\theta}(1+w_2)]$, the sum of the local times at the boundaries has expectation

$$\frac{t}{\overline{\theta}(w_1+w_2)}.$$

Here we have

$$\lambda(1-p)\mathbb{E}\big(\ell(t)+m(t)\big) = \frac{(1-p)t}{\overline{\theta}} \cdot \frac{\lambda}{w_1+w_2} + O(\lambda).$$

The second-order term has expectation

$$O\left(\frac{\lambda^2}{(w_1+w_2)^2}\right).$$

(Easy when $0 < p < 1$; requires estimates obtained from solving a control problem when $p > 1$.)

Putting the pieces together.

The loss from displacement and transaction is

$$u_0 - u_2 = \frac{u_0(1-p)\sigma^2\overline{\theta}^2}{pA+(1-p)c}\left(\frac{\lambda\overline{\theta}}{w_1+w_2}+\frac{p}{6}(w_1^2-w_1w_2+w_2^2)\right)$$
$$+O(\lambda)+O\big((w_1+w_2)^3\big)+O\left(\frac{\lambda^2}{(w_1+w_2)^2}\right).$$

Minimization of

$$\frac{\lambda\overline{\theta}}{w_1+w_2}+\frac{p}{6}(w_1^2-w_1w_2+w_2^2)$$

over w_1 and w_2 results in

$$w_1(\lambda)=w_2(\lambda)=\left(\frac{3\lambda\overline{\theta}}{2p}\right)^{1/3},$$

and a minimal value of

$$\left(\frac{9\overline{\theta}^2 p}{32}\right)^{\frac{1}{3}}\lambda^{2/3}.$$

Theorem 3 (Maximization over investment strategies) *For every $\eta \in (0, \frac{1}{3})$, we have*

$$\sup_{0<w_1,w_2\le\lambda^\eta} u_2 = u_0 \left[1 - \frac{(1-p)\sigma^2\overline{\theta}^{8/3}}{pA+(1-p)c}\left(\frac{9p}{32}\right)^{1/3}\lambda^{2/3}\right] + O(\lambda^{5/6}). \qquad (2)$$

Theorem 4 (Maximization over constant consumption) *The right hand side of (2) depends on c in the term u_0 and also in the term $pA+(1-p)c$. The maximum over c is attained by $c = A + O(\lambda^{1/2})$ and this maximum value is*

$$v_0(1) - \frac{\sigma^2\overline{\theta}^{8/3}}{A^{1+p}}\left(\frac{9p}{32}\right)^{1/3}\lambda^{2/3} + O(\lambda^{5/6}).$$

The leading term in the loss due to transaction and displacement is

$$\frac{\sigma^2\overline{\theta}^{8/3}}{A^{1+p}}\left(\frac{9p}{32}\right)^{1/3}\lambda^{2/3}.$$

This is in fact the loss for all $p > 0$, not just for $0 < p < 1$.

Bold Conjecture (stochastic volatility)

$$d\sigma^2(t) = \kappa\big(\sigma^2(t)\big)\,dt + \nu\big(\sigma^2(t)\big)\,dB(t),$$

where B is independent of W.

For $\lambda = 0$, optimal portfolio is $\frac{Y(t)}{X(t)} = \overline{\theta}(t) \triangleq \dfrac{\alpha}{p\sigma(t)}$.

Expect for $\lambda > 0$, we should keep $\theta(t) \triangleq \frac{Y(t)}{X(t)}$ in $\big[\overline{\theta}(t)\big(1 - w_1(t)\big), \overline{\theta}(t)\big(1 + w_2(t)\big)\big]$.

Diffusion for $\theta(t) - \overline{\theta}(t)$ is now $D(t) \triangleq \sigma^2\overline{\theta}^4(t) + \dfrac{\overline{\theta}^2(t)\nu^2\big(\sigma(t)\big)}{\sigma^4(t)}$.

Rate of utility loss from transaction should be $(1-p)D(t) \cdot \dfrac{\lambda}{\overline{\theta}(t)(w_1(t) + w_2(t))}$.

Expect $\theta(t)$ to be approximately uniform in the interval, and the rate of loss of utility from displacement as before:

$$\frac{1}{6}p(1-p)\sigma^2(t)\overline{\theta}^2(t)\big(w_1^2(t) - w_1(t)w_2(t) + w_2^2(t)\big).$$

Balance marginal losses to get $w_1(t) = w_2(t) = \left(\dfrac{3}{2p\overline{\theta}(t)}\right)^{\frac{1}{3}} \left(\overline{\theta}^2(t) + \dfrac{\nu^2(\sigma(t))}{\sigma^4(t)}\right)^{\frac{1}{3}} \lambda^{\frac{1}{3}}$.