A discretionary stopping problem with applications to the optimal timing of investment decisions

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The Problem

- Consider the following discretionary stopping problem:

$$\text{maximise } \mathbb{E}_x\left[e^{-r\tau}g(X_\tau)\ \right] \text{ over all stopping times } \tau,$$

$$\text{where } dX_t = b(X_t)\,dt + \sigma(X_t)dW_t, \text{ with } \ X_0 = x.$$

Applications

- When should you sell an asset that you have been endowed with?
  - X_t models the asset's price at time t;
  - $g(x) = x$.
- Exercise of perpetual American options:
  - X_t models the underlying asset's price at time t;
  - $g(x) = (x - K)^+$, where $K > 0$ is the strike.
- Real options:
  - X_t models the expected discounted payoff obtained from a project, realised at time t (the book value);
  - $g(x) = x - K$, where $K > 0$ is the cost of initialising the project.

- Mean-reverting Itô diffusions provide better models for the price processes of assets that exist in equilibrium than a geometric Brownian motion.

- General payoff functions, rather than affine ones provide significant additional modelling flexibility.

  In particular, they allow for *utility* based decision making.

- Introducing a state-dependent discounting rate allows for modelling the likelihood of an investment's default.

Problem Formulation

- State process

$$dX_t = b(X_t)\,dt \;+\; \sigma(X_t)\,dW_t, \quad X_0 = x > 0,$$

where W is a standard, one-dimensional Brownian motion, and b, σ are given deterministic functions.

- We assume that this SDE has a unique, in the sense of probability law, non-explosive solution such that $X_t > 0$, for all $t > 0$:

The functions $b, \sigma : \,]0,\infty[\to \mathbb{R}$ satisfy the following conditions:

$$\sigma^2(x) > 0, \text{ for all } x \in \,]0,\infty[,$$

$$\forall x \in]0,\infty[, \; \exists \varepsilon > 0 : \quad \int_{x-\varepsilon}^{x+\varepsilon} \frac{1 + |b(s)|}{\sigma^2(s)} ds < \infty.$$

Also, if we define

$$u(x) := \int_{x_0}^{x} \Big[p_{x_0}(x) - p_{x_0}(y) \Big]\, m(dy),$$

then $\lim_{x\downarrow 0} u(x) = \lim_{x\to\infty} u(x) = \infty$.

- A stopping strategy is a collection

$$\mathbb{S}_x = \left((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, X, W), \tau \right).$$

- Given an initial condition $x > 0$, the objective is to maximise

$$J(\mathbb{S}_x) := \mathbb{E}_x \left[e^{-\Lambda_\tau} g(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right],$$

over all stopping strategies $\mathbb{S}_x$. Here,

$$\Lambda_t := \int_0^t r(X_s)\, ds.$$

- The value function is defined by

$$v(x) := \sup_{\mathbb{S}_x} J(\mathbb{S}_x), \quad \text{for } x > 0.$$

Geometric Brownian Motion

- Suppose that

$$dX_t = bX_t\,dt + \sigma X_t\,dW_t, \quad X_0 = x > 0,$$

  $g(x) = x - K$ and $r =$ constant.

- In this case

$$e^{-rt}X_t = xe^{(b-r)t}e^{-\frac{1}{2}\sigma^2 t + \sigma W_t},$$

  and it follows that

  – if $b > r$, then $v(x) = \infty$ and there is no optimal strategy;

  – if $b < r$ and $K = 0$, then $v(x) = x$ and $\tau^* \equiv 0$ is optimal;

  – if $b < r$ and $K > 0$, then the solution is not totally trivial.

- Note that, when $b \geq r$,

$$\lim_{t\to\infty} \mathbb{E}_x \left[e^{-\Lambda_t} g(X_t)\right] = \lim_{t\to\infty} \mathbb{E}_x \left[e^{-rt} X_t\right] > 0.$$

Assumptions on g

- Observe that

$$\mathbb{E}_x\left[e^{-\Lambda_t} g(X_t)\right] = g(x) + \mathbb{E}_x\left[\int_0^t e^{-\Lambda_s} f(X_s)\, ds\right],$$

  where

$$f(x) := \frac{1}{2}\sigma^2(x) g''(x) + b(x) g'(x) - r(x) g(x).$$

  - If $f(x) < 0$, for all x, then $v(x) = g(x)$ and $\tau^* \equiv 0$ is optimal.
  - If $f(x) > 0$, for all x, then $v \equiv \infty$ and there is no optimal strategy.

- We therefore assume that there exists $x_1 > 0$ such that

$$f(x) \equiv \frac{1}{2}\sigma^2(x) g''(x) + b(x) g'(x) - r(x) g(x) \begin{cases} > 0, & \text{for } x < x_1, \\ < 0, & \text{for } x > x_1. \end{cases}$$

- We impose the *Lisbon condition*:

  There exists an $l :]0,\infty[\to [1,\infty[$ such that $\lim_{x\to\infty} l(x) = \infty$,

$$\lim_{t\to\infty} \mathbb{E}_x \left[e^{-\Lambda_t} l(X_t) \big| g(X_t) \big| \right] = 0,$$

  and

$$\frac{1}{2}\sigma^2(x)(lg)''(x) + b(x)(lg)'(x) - r(x)(lg)(x) \leq 0, \text{ for } x > x_2,$$

  for some $x_2 > x_1$.

- Note that the Lisbon condition implies the validity of the *transversality condition*

$$\lim_{t\to\infty} \mathbb{E}_x \left[e^{-\Lambda_t} \big| g(X_t) \big| \right] = 0,$$

- There exist constants $C, \varepsilon > 0$ such that

$$\begin{aligned} g(x) &\geq -C, \quad \text{for all } x > 0, \\ g(x) &> 0, \quad \text{for all } x > \frac{1}{\varepsilon}, \\ g(x) < C \text{ and } g'(x) &> -C, \quad \text{for all } x < \varepsilon. \end{aligned}$$

- There exist constants $C, x_3 > 0$ and $j \geq 1$ such that

$$\sigma^2(x) \leq C(1 + x^j), \quad \text{for all } x > 0,$$
$$[\sigma(x)g'(x)]^2 \leq C(1 + x^j), \quad \text{for all } x > x_3,$$

and

$$\int_0^t \mathbb{E}\left[X_s^j\right] ds < \infty, \quad \text{for all } t \geq 0.$$

- There exists a constant $r_0 > 0$ such that

$$r(x) \geq r_0, \quad \text{for all } x > 0.$$

Examples of g

- The following choices satisfy our assumptions for a wide range of diffusions X:

$$g(x) = \xi x^\eta - K, \qquad g(x) = (\xi x^\eta - K)^+,$$
$$g(x) = \xi \ln(x + \gamma) - K, \qquad g(x) = -\xi e^{-\gamma x} - K,$$

where $\xi, \eta, \gamma > 0$ and $K \in \mathbb{R}$.

The Hamilton-Jacobi-Bellman (HJB) Equation

- At any given time, say at time 0, there exist two possible actions: *wait* or *stop*.
- *Wait* for Δt and then continue optimally is associated with
$$v(x) \geq \mathbb{E}_x \left[e^{-\Lambda_{\Delta T}} v(X_{\Delta T}) \right].$$
Using Itô's formula, dividing by Δt and taking the limit $\Delta t \downarrow 0$ yields
$$\frac{1}{2}\sigma^2(x)v''(x) + b(x)v'(x) - r(x)v(x) \leq 0.$$
- *Stop* is associated with
$$v(x) \geq g(x).$$
- We therefore expect that the value function v should identify with a solution to the HJB equation
$$\max\left\{\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x),\ g(x) - w(x)\right\} = 0.$$

The Solution to the Discretionary Stopping Problem

- We look for a critical point x^*, such that it is optimal to
  - wait while $X_t < x^*$,
  - stop as soon as X hits $[x^*, \infty[$.
- We therefore look for a solution to the HJB equation such that

$$\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) = 0, \quad \text{for } x \in]0, x^*[,$$

and

$$w(x) = g(x), \quad \text{for } x \in [x^*, \infty[.$$

- Every solution to the ODE

$$\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) = 0$$

is given by

$$w(x) = A\psi(x) + B\phi(x),$$

for some constants $A, B \in \mathbb{R}$, where the functions ψ and ϕ are defined by

$$\psi(x) := \begin{cases} \mathbb{E}_x[e^{-\Lambda_{\tau_z}}], & \text{for } x < z, \\ 1/\mathbb{E}_z[e^{-\Lambda_{\tau_x}}], & \text{for } x \geq z, \end{cases} \qquad \text{(increasing)}$$

and

$$\phi(x) := \begin{cases} 1/\mathbb{E}_z[e^{-\Lambda_{\tau_x}}], & \text{for } x < z, \\ \mathbb{E}_x[e^{-\Lambda_{\tau_z}}], & \text{for } x \geq z, \end{cases} \qquad \text{(decreasing)}$$

for a given choice of $z > 0$.

- When X is a geometric Brownian motion, i.e., when

$$dX_t = bX_t\,dt + \sigma X_t\,dW_t,$$

for some constants b, σ,

$$\psi(x) = x^n \quad \text{and} \quad \phi(x) = x^m,$$

where $m < 0 < n$ are appropriate constants.

- When X is a square-root mean reverting process, i.e., when

$$dX_t = \kappa(\theta - X_t)\,dt + \sigma\sqrt{X_t}\,dW_t,$$

for some constants $\kappa, \theta, \sigma > 0$ such that $\kappa\theta - \frac{1}{2}\sigma^2 > 0$,

$$\psi(x) = {}_1F_1\left(\frac{r}{\kappa}, \frac{2\kappa\theta}{\sigma^2}; \frac{2\kappa}{\sigma^2}x\right) \quad \text{and} \quad \phi(x) = U\left(\frac{r}{\kappa}, \frac{2\kappa\theta}{\sigma^2}; \frac{2\kappa}{\sigma^2}x\right).$$

- When X is an exponential Ornstein-Uhlenbeck process, i.e., when

$$dX_t = \left(\kappa(\theta - \ln X_t) + \frac{1}{2}\sigma^2\right) X_t\, dt + \sigma X_t\, dW_t,$$

for some constants $\kappa, \theta, \sigma > 0$,

$$\psi(x) = \begin{cases} \frac{\Gamma(\frac{r+\kappa}{2\kappa})}{\pi} U\left(\frac{r}{2\kappa}, \frac{1}{2}; \frac{\kappa}{\sigma^2}(\theta - \ln x)^2\right), & \text{for } x \le e^\theta, \\ {}_1F_1\left(\frac{r}{2\kappa}, \frac{1}{2}; \frac{\kappa}{\sigma^2}(\theta - \ln x)^2\right), & \text{for } x > e^\theta, \end{cases}$$

and

$$\phi(x) = \begin{cases} {}_1F_1\left(\frac{r}{2\kappa}, \frac{1}{2}; \frac{\kappa}{\sigma^2}(\theta - \ln x)^2\right), & \text{for } x \le e^\theta, \\ \frac{\Gamma(\frac{r+\kappa}{2\kappa})}{\pi} U\left(\frac{r}{2\kappa}, \frac{1}{2}; \frac{\kappa}{\sigma^2}(\theta - \ln x)^2\right), & \text{for } x > e^\theta. \end{cases}$$

- When X is a geometric Ornstein-Uhlenbeck process, i.e., when

$$dX_t = \kappa(\theta - X_t)X_t\,dt + \sigma X_t\,dW_t,$$

for some constants $\kappa, \theta, \sigma > 0$,

$$\psi(x) = x^n\,{}_1F_1\left(n, 2n + \frac{2\kappa\theta}{\sigma^2}; \frac{2\kappa x}{\sigma^2}\right),$$
$$\phi(x) = x^n U\left(n, 2n + \frac{2\kappa\theta}{\sigma^2}; \frac{2\kappa x}{\sigma^2}\right).$$

- We consider solutions to the HJB equation given by

$$w(x) = \begin{cases} A\psi(x) + B\phi(x), & \text{if } x < x^*, \\ g(x), & \text{if } x \geq x^*, \end{cases}$$

- Since the payoff function is increasing and bounded for small x, we expect that the same is true for the value function, which implies $B = 0$.

- To specify A and x^*, we appeal to the "smooth-pasting" condition of optimal stopping that requires the value function to be C^1 at the free boundary point x^*. This implies

$$A\psi(x^*) = g(x^*) \quad \text{and} \quad A\psi'(x^*) = g'(x^*).$$

  This system implies that x^* must satisfy $q(x^*) = 0$, where

$$q(x) := g(x)\psi'(x) - g'(x)\psi(x), \quad \text{for } x > 0.$$

- Under the assumptions that we have imposed, there exists a unique x^* satisfying $q(x^*) = 0$.