Different Approaches to the Volatility Surface: from Lévy Processes to Local Lévy

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The use of stochastic time changes in option pricing and hedging

Two examples : Geman-Yor (1992 , 1993)

a) In order to find the "exact" price of an Asian option in the Black-Scholes formula

- use the property that a geometric Brownian motion is a *time-changed* squared-Bessel process

$$S(t) = BESQ(X(t))$$

  where the time change X(t) is completely specified.

- search for the Laplace transform of the option price with respect to maturity.

b) Propose to use a stochastic time change to address the issue of stochastic volatility

For example, an option is sold by a financial institution at time 0 on the basis of a volatility b.
In reality, $\sigma(s)$ is a stochastic process and one is interested (see Leland-Rubinstein ; Bick) in the first time τ_b such that $\int_0^{\tau_b} [\sigma(s)]^2 \, ds = b^2 \, T$

Taking the Hull & White framework

$$\frac{d S_t}{S_t} = \mu \, dt + \sigma d \, W_t^1$$

$$\frac{d y_t}{y_t} = \theta \, dt + \zeta \, d \, W_t^2$$

Geman-Yor provide the Laplace transform of the density f_b of τ_b, hence the L.T of the mean through Fubini. Hence the average time to replication can be computed by inversion of the L.T.

→ Eydeland-Geman (RISK, 1995) invert the Laplace transform of these quantities.
For instance, in the case of the Asian call option, the Laplace transform

$$\phi(h) = \int_0^{+\infty} e^{-h\tau} C(\tau) d\tau$$

is first "normalized" into

$$\varphi_1(h) = \varphi(h) e^{\alpha h}$$

where α is chosen to make φ regular in the domain of complex numbers with positive real part

→ Because $\varphi 1$ is holomorphic in the $h \geq 0$, its inverse Laplace transform maybe expressed in terms of its inverse Fourier transform

→ For computational efficiency, E-G use the Fast Fourier Transform and derive the inversion of the function $\varphi 1$ (or equivalently, φ)

→ Fu-Madan (1996) use the Abate-Whitt procedure to perform these inversions and obtain similar results

The Case against the Geometric Brownian Motion Model for Stock Prices

→ Fat tails

→ Distributions more peaked than the normal distribution

→ Jumps in trajectories

→ Deviations of market option prices from Black-Scholes prices : Smiles, short-dated options…

→ Having the same volatility under the statistical measure and the risk-adjusted probability measure is a very hard constraint.

The Possible Approaches to the Volatility Surface

→ If Black-Scholes' assumptions were verified in the markets, the volatility surface would be reduced to a point
In fact, it is indeed a surface for all underlyings : equities, interest rates, commodities

→ The possible answers
- Use "local volatitlity" arguments as in Dupire (1994), Derman and Kani (1994)
- Introduce stochastic volatility as a second state-variable while keeping a continuous process for (S(t))
- Use Lévy processes to represent the dynamics of the stock price (S(t))
- Add stochastic volatility in the Lévy processes
- Go to a "local Lévy" representation and benefit from two sources of "skew": *leverage and Lévy*

Transaction Clock and Normality of Returns

Geman-Ané (1996) and Ané-Geman (2000) legitimate and extend Clark's remarkable results through the two key steps :

→ By the first fundamental theorem of asset pricing, No arbitrage implies that asset prices are semimartingales under the physical measure P.

→ Monroe's theorem (1978) establishes that "any semimartingale is a time-changed Brownian motion".

Hence the (log) price may be written

$$S(t) = W(T(t))$$

where T(t) is an almost increasing process (and may not be a subordinator).

Transaction Time as the Natural Path to the Mixture of Distributions Hypothesis

Viewing the return process as a time-changed Brownian motion

$$Y(t) = X(T(t))$$

we can write

$$P(Y_t \in dy) = \sum_u P\lfloor X(T(t)) \in dy / T(t) = u \rfloor \bullet f_T(u)$$

Assuming the independence of the processes X and T, we obtain

$$P(Y(t) \in dy) = \sum_u P(X(u) \in dy) f_T(u)$$

and the unconditional distribution of Y appears as an (a priori infinite) mixture of normal distributions where the mixing factor is the time change (hence, the market activity).

The Case for Pure Jump Processes
(Geman-Madan-Yor : 2000, 2001)

→ Index and stock prices are moving by jumps

→ Pure jump processes are of finite variation, as are real price processes

→ Writing S(t) = W(T(t)) it is clear that the continuity of the process (S(t)) is equivalent to the continuity of the process T(t)

f T(t) is continuous, then

$$T(t) = \int_0^t a(u)du + \int_0^t b(u)dZ(U)$$

Because T(t) is increasing, b(u) ≡ 0 and the time change is locally deterministic. This is an undesirable property since that we believe T(t) is related to locally random market activity like the arrival of orders or information

The CGMY Process
(Journal of Business, 2002)

→ It allows for more flexibility in the representation of positive and negative moves: the Lévy density of the CGMY process is defined by

$$k_{CGMY}(x)\begin{cases} \dfrac{Ce^{-Mx}}{x^{1+Y}} & x > 0 \\ \dfrac{Ce^{-G|x|}}{|x|^{1+Y}} & x < 0 \end{cases}$$

→ The parameter Y captures the "fine" structure of the process in the following way

$Y < -1$	FA and not CM
$-1 \leq Y < 0$	FA and CM (finite activity and complete monotonicity)
$0 \leq Y < 1$	IA and FV(infinite activity and finite variation)
$1 \leq Y < 2$	IV

→ The CGMY characteristic function is obtained by integration as

$$\log[\phi_{CGMY}(u)] = Ct\Gamma(-y)\{(M-iu)^{Y} - M^{Y} + (G+iu)^{Y} - G^{Y}\}$$

→ The $CGMY_e$ is the process obtained by addition of a diffusion

$$X_{CGMY_e}(t) = X_{CGMY}(t) + \eta W(t)$$

→ The $CGMY_e$ characteristic function is given by

$$\log[\phi_{CGMY_e}(u)] = \log[\phi_{CGMY}(u)] - \eta^2 u^2 \frac{t}{2}$$

→ For the risk-neutral process, we employ the same type of process, set the mean equal to the interest rate and the other parameters are determined by matching option prices

Option Pricing with Lévy Processes using the Fast Fourier Transform

→ We typically model the stock price under the risk-adjusted measure by

$$S(t) = S(0)\exp[(r - q + \omega)t + X(t)]$$

where X(t) is a process with a known characteristic function

$$\phi(u) = E[\exp(iu\,X(t))] = \exp(t\psi(u))$$

→ To ensure a mean rate of return of (r-q), we need

$$\omega = -\psi(-i)$$

→ We define the Fourier transform of the call price expressed interms of the log of the strike

$$\gamma(u) = \int_{-\infty}^{+\infty} e^{iuk} e^{\alpha k} C(k)\,dk$$

where

- the term $e^{\alpha k}$ is there to ensure the convergence of the integral
- C(k) is the price of a European call with maturity T and strike e^k.

Then

$$\gamma(u) = \frac{e^{-rt}\phi_{\ln S}(\alpha + 1 + iu)}{(\alpha + iu)(\alpha + 1 + iu)}$$

Call prices may easily be recovered by inversion

$$C(k) = \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{+\infty} e^{-iuk}\gamma(u)\,du$$

→ The method is applicable whenever one has an analytical expression for the characteristic function of the log price. It may be applied uniformly across all strikes to provide a very fast surface calibration

Introducing Stochastic Volatility In,Lévy processes
CGMY, 2003, *Mathematical Finance*

→ Homogeneous Lévy process fit statistical data quite well but impose too strict conditions on the term structure of risk–neutral variances : in particular, the variance rate is constant across maturities

→ Heston (1993), Bates (1996), Barndorff-Nielsen and Shephard (2001) show that volatilities estimated from option time series are stochastic and usually clustered

→ Random changes in volatility can be produced by random changes in time.

→ The rate of time change must be positive for the transaction clock to be increasing.

→ This rate should be reverting in order for the random time change (and volatility) to persist

→ A classical example of a mean- reverting positive process is the CIR (1985) square –root process

$$dy = a\left(b - y\right)dt + \lambda\sqrt{y}\, dW_t$$

→ Hence, we consider the new clock Y(t) defined by

$$Y(t) = \int_0^t y(u)\, du$$

whose characteristic function is known (Pitman-Yor 1982)

$$E\left[\exp\left(iu\, y(t)\right)\right] = A(t,u)\exp\left[B(t,u)\, y(o)\right]$$

where A and B involve sinh, cosh, coth and the parameters of the (y_t) process

We introduce the stochastic volatility Lévy process

$$Z(t) = X\left[Y(t)\right]$$

where Y is independent of X

Local Volatility

→ Consider the risk neutral stock price evolution as described for example in Dupire (1994), Derman and Kani (1994)

$$d\,S = (r - \eta)\,S\,dt + \sigma\,(S, t)\,S\,dW\,(t)$$

where σ is supposed to be a deterministic funcion of S and t

→ This one dimensional Markov process has one dimensional marginal stock price densities q_t (S) that are reflected in the option prices C (K, T) for the continuum of strikes K > 0 and maturities T > 0 via the Breeden and Litzenberger (1978) result

$$q_t\,(K) = e^{rT}\,C_{KK}\,(K, T)$$

where C_{KK} denotes the second partial derivative with respect to k

→ The question addressed in Dupire (1994) was the recovery of what is now called the local volatility function $\sigma(S, t)$ from market option prices.
We start from the well-known partial differential equation

$$\frac{\partial C}{\partial t} + S^2 \frac{\partial^2 C}{\partial S^2} \sigma^2 (S_t, t) + r S \frac{\partial C}{\partial S} - r C = 0$$

where $C(S_t, t)$ satisfies the terminal condition
$C(S, T) = (S - K)^+$

We now consider C as a function C (K, T) to represent the collection of options available in the market. Assuming all strikes are available (which may require in practice some interpolations between quoted strikes)

- we get from Breeden-Litzenberger

$$q_t(K) = e^{r(T-t)} \frac{\partial^2 C}{\partial K^2}(K, T)$$

This one-dimensional stock price density at date T provides the prices of all European options with maturity T through

$$C(K, T) = e^{-r(T-t)} \int_0^{+\infty} \max(S - k, 0)\, q_T(S)\, dS$$

- from Dupire-Derman-Kani, we get

$$\sigma^2(K,T) = \frac{2}{K^2 C_{KK}}(C_T + \eta C + (r-\eta) K C_K)$$

which now *defines* the process (S(t))

→ *Local volatility* models are now calibrated in a variety of ways. One needs to compute second and first partials of option prices and if these are interpolated in some fashion, then one could lose positivity of the numerator or denominator

→ This can be avoided if the prices employed come from an arbitrage free model that has first been calibrated to the option prices.

→ Consider the structure of forward skews at for example the 3 month 90-110 point, which is of great interest for practitioners

Local Lévy Processes

→ Let $k(x)$ be the Lévy measure of a Lévy process that we employ to model the jumps in the logarithm of the stock price

→ For processes of finite activity we have that

$$\int_{-\infty}^{\infty} k(x)\,dx < \infty$$

→ Otherwise the process is one of infinite activity. The process has finite variation if

$$\int \left(|x| \wedge 1 \right) k(x)\,dx < \infty$$

→ Otherwise it has infinite variation. In all cases we have finite quadratic variation and

$$\int \left(|x|^2 \wedge 1 \right) k(x)\,dx < \infty$$

→ We shall primarily be considered with cases where in addition we have

$$\int \left(|x|^2 \wedge |x| \right) k(x)\, dx < \infty$$

→ The choice of the Lévy measure builds *local* skewness into the process by fixing *relative* arrival rates of negative and positive moves

→ The jump compensator has a speed function that alters the speed at which the Lévy process is running deterministically as a function of the stock price and time. We denote this speed function by a (S,t)

→ The jump compensator is then taken to be of the form

$$\nu\,(dx, dt) = a(S\,(t_-), t)\; k(x)\; dx\; dt$$

→ Consider then the local Lévy risk neutral stock price dynamics given by

$$dS = (r - \eta) S(t_-) dt + \sigma(S(t_-), t) dW +$$

$$\int_{-\infty}^{\infty} S(t_-)(e^x - 1)(\mu(dx, dt) - \nu(dx, dt))$$

where μ (dx, dt) is the Dirac measure associated with the jumps in the logarithm of the price process

→ We have the usual risk neutral drift now coupled with a continuous martingale component that we shall take to be known plus a compensated jump martingale describing shocks to the log price process

→ In keeping with our earlier research on time changed Lévy processes of infinite activity, we shall be particularly interested in the case where $\sigma = 0$ and there is no continuous martingale component

→ The objective now is to recover the local speed function *a* from the prices of options

We show that the forward speed function may be identified as

$$a(Y,T) = \frac{b(\ln Y, T)}{Y^2 C_{YY}}$$

and b satisfies the equation

$$C_T + \eta C + \left[r - y + \frac{\sigma^2(e^k, T)}{2} \right] C_k - \frac{\sigma^2(e^k, T)}{2} C_{k,k}$$
$$= \int_{-\infty}^{+\infty} b(y,T)\chi(k-y)dy$$

We identify the function χ explicitly for some examples of Lévy processes, such as the negative exponential jump process.

The function χ in the above equation is defined as the "double tail" of the Lévy measure k(x)

$$\begin{cases} \chi(z) = \int_{-\infty}^{z} \int_{-\infty}^{x} k(u)du\,dx & \text{for } z < 0 \\ \chi(z) = \int_{z}^{+\infty} \int_{x}^{+\infty} k(u)du\,dx & \text{for } z > 0 \end{cases}$$

It is important as it measures quadratic variation, from the relationship

$$\int_{-\infty}^{+\infty} \chi(z)dz = \frac{1}{2} \int_{-\infty}^{+\infty} x^2 k(x)dx \qquad \text{(integration by parts)}$$

Using the integrated form of S(t) and the Meyer-Tanaka formula, we can write

$$(S(T)-K)^{+} = (S(0)-K)^{+} + \int_0^T \mathbf{1}_{(S(u_-)>k)}\, dS(u)$$
$$+\frac{1}{2}\int_0^T \mathbf{1}_{(S_{u_-}=k)}\sigma^2(S(u),u)S^2(u)du + \sum_{u\le T}\mathbf{1}_{(S(u_-)>k)}(K-S(u))^{+}$$
$$+\mathbf{1}_{(S(u_-)<k)}(S(u)-K)^{+}$$

We introduce q (Σ, u) as the transition density that the stock price is Σ at time u given that at time 0 it is at S(0). Computing the expectation in the above equation and using the expression of v(dx, dt) in terms of the "speed" *a* and Lévy measure k, we obtain the expression of e^{rt} C(k, T).

Differentiating with respect to T this expression leads to

$$C_T = -\eta C - (r-\eta)KC_k + \frac{\sigma^2(K,T)}{2}K^2 C_{kk}$$
$$+\int_0^\infty C_{YY}\, Y a(Y,T)\chi\left(\ln\left(\frac{K}{Y}\right)\right)dY$$

which becomes the reference equation in a Lévy framework