Portfolio Credit Derivatives with Markovian Default Interaction

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1. Introduction

- **Portfolio credit derivatives** are securities whose payoff is contingent on credit events in a pool of firms (portfolio products).

- **Focus** of this talk: Pricing in portfolio credit derivatives in a Markovian model with interacting default intensities, which is an alternative to the market-standard Gauss copula model

- **Overview**
  - Introduction
  - Interacting intensities: a Markovian approach
  - Pricing (basket) default swaps in the Markov model
  - Pricing CDOs in the Markov model

Talk is based on Frey-Backhaus (2004); for background information see the forthcoming

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Reduced Form Models for Credit Portfolios

• Models with *conditionally independent* defaults and stochastic intensity

• *Copula models* such as Li (2001), Schönbucher-Schubert (2001). Market standard since they allow for default contagion and are easily calibrated to defaultable term structure. Main drawback: unintuitive parametrization of dependence. Special case: *factor copula models* such as Laurent-Gregory (2003), Schönbucher (2003, 2004), where contagion can be interpreted in terms of incomplete information.

• *Common shock models* such as Lindskog-McNeil(2001) or Kijima(2000)

• Models with *interacting intensities*
Basic Concepts and Notation

Consider \( m \) firms with default times \( \tau_i \) and default indicator process \( Y_t = (Y_t(1), \ldots, Y_t(m)) \) with \( Y_t(i) = I\{\tau_i \leq t\} \).

- \( \overline{F}_i(t) = P(\tau_i \geq t) \) survival function of obligor \( i \); joint survival function: \( \overline{F}(t_1, \ldots, t_m) = P(\tau_1 \geq t_1, \ldots, \tau_m \geq t_m) \).

- ordered default times denoted by \( T_0 < T_1 < \ldots < T_m \).
  \( \xi_n \in \{1, \ldots, m\} \) gives identity of the firm defaulting at time \( T_n \)

- Filtrations. \( \mathcal{H}_t^i = \sigma(\{Y_{s,i} : s \leq t\}) \), \( \mathcal{H}_t = \mathcal{H}_t^i \vee \cdots \vee \mathcal{H}_t^m \). \( \{\mathcal{H}_t\} \) is internal filtration of \( (Y_t) \) (only information about default history).

Default intensities. An \( \{\mathcal{H}_t\} \)-adapted process \( (\lambda_{t,i}) \) is called the default intensity of \( \tau_i \) (wrt \( \{\mathcal{H}_t\} \)) if \( Y_i(t) - \int_0^{\tau_i \wedge t} \lambda_{s,i} ds \) is an \( \{\mathcal{H}_t\} \)-martingale. Default intensities determine the law of the default indicator process \( (Y_t) \).
Copula Models - the Market Standard.

**Background on copulas** A copula is a df $C$ on $[0, 1]^m$ with uniform margins. Copulas describe the *dependence structure* of a multivariate distribution with df $F$. If $F$ has continuous margins $F_1, \ldots, F_m$ and $X \sim F$ the copula $C$ of $X$ is the df of $(F_1(X_1), \ldots F_m(X_m))$, and we have *Sklars identity*

$$F(x_1, \ldots, x_m) = C(F_1(x_1), \ldots, F_m(x_m)).$$

Similarly, the survival function of $X$ can be written as

$$\overline{F}(x_1, \ldots, x_m) = \hat{C}(\overline{F}_1(x_1), \ldots, \overline{F}_m(x_m)),$$

where $\hat{C}$ is defined by

$$\hat{C}(u_1, \ldots, u_m) = C(1 - u_1, \ldots, 1 - u_m); \hat{C} \text{ is called survival copula}$$

**Example.** Gauss copula $C^\text{Ga}_P$ is the copula of $X \sim N(\mathbf{0}, P)$ where $P$ is a correlation matrix.
Copula models.

Copula models are specified in terms of marginal distribution and survival copula (denoted $C$) of $(\tau_1, \ldots, \tau_m)$. Hence survival function of default times is given by

\[
\overline{F}(t_1, \ldots, t_m) = C\left(\overline{F}_1(t_1), \ldots, \overline{F}_m(t_m)\right),
\]

(1)

Specifying dependence structure $C$ and marginal distribution $\overline{F}_i$ separately is useful for calibration. Model is calibrated to given term structure of (single-name) CDS spreads by specifying $\overline{F}_i$; calibration of dependence structure (i.e. $C$) can then be done independently.

**Exchangeable Gauss-Copula Model.** Here $X_i = \sqrt{\rho}V + \sqrt{1-\rho}\epsilon_i$, for 'asset correlation' $\rho \in (0, 1)$ and $V, (\epsilon_i)_{1 \leq i \leq m}$ iid standard normal rvs. Set $U_i = \Phi(X_i)$, so that $U \sim C_{\text{Ga}}^\rho$. Survival function of $\tau_1, \ldots, \tau_m$ given by

\[
\overline{F}(t_1, \ldots, t_m) = P\left(U_1 \leq \overline{F}_1(t_1), \ldots, U_m \leq \overline{F}_m(t_m)\right).
\]
2. Models with Interacting Intensities

(joint work with Jochen Backhaus).

**Basic idea.** Default contagion is *explicitly* modelled. Default intensity is modelled as function $\lambda_i(t, Y_t)$ of time *and* of default-state $Y_t$ of portfolio at time $t$. (Extension to stochastic state variables possible).

**Advantage.** Intuitive and explicit parametrization of dependence between defaults; Markov process techniques available for analysis and simulation of the model.

**Disadvantage.** Calibration to term structure of defaultable bonds or CDSs more difficult than with copula models, as marginal distribution of default times typically not available in closed form.
Related work

- Jarrow-Yu(2001): only very special types of interaction; model is studied using Cox process techniques. Extensions by Yu(2004), who provides proper model construction and studies default correlations.
- Kusuoka(1999) and Bielecki - Rutkowski (2002) study mathematical aspects of the model.
Construction via Markov Chains.

A model with interacting intensities is conveniently defined as a *time-inhomogeneous Markov chain* with state space $S = \{0, 1\}^m$ and transition rate functions (from $y$ to $x$)

$$
\lambda(t, y, x) = \begin{cases} 
I_{\{y_i=0\}} \lambda_i(t, y), & \text{if } x = y^i \text{ for some } i \in \{1, \ldots, m\}, \\
0, & \text{else}, 
\end{cases}
$$

(2)

where $y^i \in S$ is obtained from $y \in S$ by flipping the $i$th coordinate.

**Interpretation.** The chain can jump only to neighbouring states, which differ from the current state $Y_t$ by exactly one default; if $Y_i(t) = 0$, the probability of a jump in $[t, t + h)$ to state $Y_t^i$ (default of firm $i$), is $\approx h\lambda_i(t, Y_t)$. 

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Model properties

- The *generator* of \((Y_t)\) equals

\[
G_{[t]} f(y) = \sum_{i=1}^{m} I_{\{y_i=0\}} \lambda_i(t, y) (f(t, y^i) - f(t, y)).
\]

- \(Y_i(t) - \int_0^{t \wedge \tau_i} \lambda_i(s, Y_s) ds\) is an \(\{H_t\}\)-martingale by the Dynkin formula, so that \(\lambda_i(s, Y_s)\) is in fact the \(\{H_t\}\)-default intensity.

- Denote by \(p(t, s, x, y) = P_{(t,x)}(Y_s = y), \ s \geq t,\) the transition probabilities of the chain. They satisfy the Kolmogorov forward- and backward equation, here an ODE system.
Kolmogorov equations

Backward equation.

\[ \frac{\partial p(t, s, x, y)}{\partial t} + \sum_{i=1}^{m} (1 - x_i) \lambda_k(t, x)(p(t, s, x_i, y) - p(t, s, x, y)) = 0, \]

\[ p(s, s, x, y) = \delta_y(x). \quad \text{(terminal condition)} \]

Forward equation.

\[ \frac{\partial}{\partial s} p(t, s, x, y) = \sum_{k=1}^{m} (y(k)) \lambda_k( s, y^k ) p(t, s, x, y^k) \]

\[ - \sum_{k=1}^{m} (1 - y(k)) \lambda_k( s, y ) p(t, s, x, y), \quad s > t, \]

\[ p(t, t, x, y) = \delta_x(y). \quad \text{(initial condition)} \]
Modelling Default Intensities

The default intensities $\lambda_i(t, y)$ are essential ingredient of the model. Some examples:

- Frey-Backhaus (04): homogenous group model with

$$
\lambda_i(t, Y_t) = h(t, M(Y_t)), \quad \text{where} \quad M(y) = \sum_{i=1}^{m} y_i. \quad (3)
$$

In exchangeable models $\lambda_i(t, Y_t)$ is necessarily of this form. Moreover, natural interpretation in terms of *mean-field interaction*. Extension to model with several groups possible.

- Yu (04) claims that $h(t, l) = 0.01 + 0.001 \cdot I_{\{l>0\}}$ is a reasonable model for European telecom bonds.
Properties of Homogenous-Group Model

- The process $M_t := M(Y_t)$ is itself a Markov chain with generator

\[ G^M_{[t]} f (l) = (1 - l) h(t, l) \left( f(l + 1) - f(l) \right) \]  

- State space of $(M_t)$ is $S^M := \{0, 1, \ldots, m\}$ so that $|S^M| = m + 1$ (instead of $2^m$).

- Kolmogorov equations for $(M_t)$ available in closed form.

- $P(Y_i(T) = 1) = 1/m E(M_T)$ and

  $P(Y_i(T) = 1, Y_j(T) = 1) = m^{-2} E(M_T(M_T - 1))$ etc.

- Limit results for large portfolios available.
Conditional Expectations

Following results useful for (semi)analytic pricing of credit derivatives.

**Proposition.** The density of $\tau_{i_0}$ equals

$$P(\tau_{i_0} \in dt) = \sum_{y:y(i_0)=0} \lambda_{i_0}(t,y) P(Y_t = y).$$

Moreover,

$$P(Y_t = y | \tau_{i_0} = t) = y(i_0) P(\tau_{i_0} \in dt)^{-1} \lambda_{i_0}(t, y^{i_0}) P(Y_t = y^{i_0}).$$

(5)

Proof is based on Markov property.

**Corollary.** In homogeneous-group model with default intensity $h(t, l)$

$$P(\tau_{i_0} \in dt) = m^{-1} \sum_{k=0}^{m-1} h(t, k) P(M_t = k)(m - k)$$

and

$$P( M_t = l \mid \tau_{i_0} = t) = \frac{(m - l + 1)h(t, l - 1)P(M_t = l - 1)}{\sum_{k=0}^{m-1} (m - k)h(t, k)P(M_t = k)}.$$

(7)
3. (Basket) Default Swaps in the Markov model

Credit Default Swap (CDS). Workhorse of market for credit derivatives. Three parties involved: reference entity C; protection buyer A; protection seller B.
Ordinary CDS.

- Premium payments are due at times $0 < t_1 < \cdots < t_N$. If $\tau_C > t_k$, A pays in $t_k$ a premium of size $x(t_k - t_{k-1})$, where $x$ denotes the swap spread; after $\tau_C$ premium payments stop. No initial payments. (accrued payments are neglected for simplicity)

- Default payment. If $\tau_C < t_N = T$ B pays A the LGD $\delta$ of C at $\tau_C$.

- Fair swap spread $x^*$. Since there are no initial payments $x^*$ is chosen such that value at $t = 0$ of default payments equals value of premium payments. $x^*$ is the quantity which is quoted on the market.

$k$th-to-default swap. Here default payment is triggered by credit events in a portfolio (the basket). Premium payments as before. If $k$th default time $T_k < t_N$ there is a default payment whose size depends on identity $\xi_k$ of defaulting firm.
Pricing Results for Default Swaps

**Goal.** Provide (semi)analytic pricing formulas which can be evaluated using Kolmogorov equations.

Throughout we assume that model has been set up under equivalent martingale measure $P$ and that risk-free interest rate $r$ and LGD $\delta_i$ are deterministic; $D(t) = \exp(-\int_0^t r(s)ds)$ is default-free discount factor.

**CDSs.**

- Premium payments are terminal value claims of the form $H = g(Y_{t_k})$ with $g(y) = x^*(t_k - t_{k-1})(1 - y_i) \Rightarrow$ pricing via backward equation.

- Price of the default payments given by

$$E \left( \delta_{i_0} D(\tau_{i_0}) I \{ \tau_{i_0} \leq T \} \right) = \delta_{i_0} \int_0^T D(t) P(\tau_{i_0} \in dt) dt, \quad (8)$$

which can be evaluated numerically using (5) or (7).
Basket Default Swaps

**Default payments** of $k$th-to-default swap. By definition

$$V^{\text{def}} := \sum_{j=1}^{m} \delta_j \mathbb{E}\left(D(\tau_j)I\{\tau_j \leq T\}I\{M_{\tau_j} = k\}\right).$$

Now

$$\mathbb{E}\left(D(\tau_j)I\{\tau_j \leq T\}I\{M_{\tau_j} = k\}\right) = \int_0^T D(t)P(M_t = k | \tau_j = t)P(\tau_j \in dt) dt.$$

Hence we get in homogeneous group model

$$V^{\text{def}} := \sum_{j=1}^{m} \delta_j \int_0^T D(t)\left(\frac{m-k+1}{m}\right)h(t, k-1)P(M_t = k - 1) dt,$$

which can be evaluated by Kolmogorov. Similar formula also for non-homogenous case.

**Premium payments** can be evaluated using $\{T_k \leq t\} = \{M_t \geq k\}$. 

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4. CDOs: Pricing and Model Calibration

Basic Structure of CDOs

Payments in a CDO structure; above arrow: asset-based structure; below arrow: *synthetic CDO*. 
Payoff of Tranches

Consider portfolio of $m$ loans with nominal $N_i$, relative LGD $\delta_i$ and default-indicator process $Y_t$. Cumulative loss of the portfolio in $t$ given by $L_t = \sum_{i=1}^{m} \delta_i N_i Y_i(t)$.

**CDO-tranches.** Maturity $T$. We have $k$ tranches, characterized by attachment points $0 = K_0 < K_1 < \cdots < K_k \leq \sum_{i=1}^{m} N_i$. The notional of tranche $\kappa$ at time $t$ is given by

$$N_\kappa(t) = f_\kappa(L_t) \text{ with } f_\kappa(l) = (K_\kappa - l)^+ - (K_{\kappa-1} - l)^+$$

Note that $f_\kappa(l)$ corresponds to a put spread.

**Stylized CDO.** Payoff of tranche $\kappa$ given by $N_\kappa(T)$, the value of the notional at maturity. Real CDOs more complicated, as there is intermediate income.
Payoff of stylized CDO with 3 tranches and attachment points 20 40 and 60 with two different loss distributions overlayed. Properties carry over to more realistic contracts.
Synthetic CDOs: Payment Description

Consider CDO with attachment points \( K_0 < \cdots < K_k \) and notional of tranche \( \kappa \) given by \( N_\kappa(t) = (K_\kappa - L_t)^+ - (K_{\kappa-1} - L_t)^+ \); define \textit{cumulative loss of tranche} \( \kappa \) as \( L_\kappa(t) := N_\kappa(0) - N_\kappa(t) \).

**Default payments of CDO.** Default payment of tranche \( \kappa \) at \( n \)th default time \( T_n < T \) given by \( \Delta L_\kappa(T_n) = (L_\kappa(T_n) - L_\kappa(T_{n-1})) \), i.e. by the part of cumulative loss at \( T_n \) which falls in the layer \([K_{\kappa-1}, K_\kappa]\).

**Protection fee or premium payments.** Holder of tranche \( \kappa \) receives periodic premium payments at \( 0 < t_1 < \cdots < t_N = T \) of size \( x_\kappa^{\text{CDO}}(t_n - t_{n-1}) N_\kappa(t_n) \). No initial payments. \( x_\kappa^{\text{CDO}} \) is called the (fair) CDO spread.
Synthetic CDOs: Pricing

Using partial integration we obtain for the value of the default payments of tranche $\kappa$

$$V^\text{def} = E\left( \int_0^T D(t) dL_\kappa(t) \right) = D(T) E(L_\kappa(T)) + \int_0^T r D(t) E(L_\kappa(t)) dt.$$ 

As $L_\kappa(t)$ is a function of $L_t$ this can be computed by one-dimensional integration if we know the distribution of $L_t$. Premium payments can also be expressed in terms of $L_t$.

For deterministic LGD $L_t$ is a function of $Y_t$ resp $M_t$. Hence in homogeneous group model computation via Kolmogorov equations possible; otherwise Monte Carlo must be used.

Computation in Gauss-copula model can also be based on above representation.
Explaining Market Quotes for CDOs

Financial industry has developed CDS-indices for a variety of sectors; spreads for CDO-tranches on these indices are available.

**Observed CDS spreads.** Consider the 5 year iTraxx EUR from August 4, 2004. The (average) index level was 42 bp. Assuming a constant recovery rate of 40% this leads to a marginal default probability \( P(\tau \leq t) = 1 - e^{-\lambda t} \) with \( \lambda = 0.007 \) and we obtain a 5-year default probability of 3.44%.

**Observed spreads for CDO tranches.** On the market we observed the following tranche prices\(^1\).

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Spread</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0, 3]*</td>
<td>27.6%</td>
</tr>
<tr>
<td>[3, 6]</td>
<td>1.68%</td>
</tr>
<tr>
<td>[6, 9]</td>
<td>0.70%</td>
</tr>
<tr>
<td>[9, 12]</td>
<td>0.43%</td>
</tr>
<tr>
<td>[12, 22]</td>
<td>0.20%</td>
</tr>
</tbody>
</table>

\(^*\)The [0,3] tranche quoted on upfront + 5% per year.

\(^1\)Taken from Hull/White(2004)
Implied Tranche- and Base Correlation

Attempts to express CDO prices in terms of correlation parameter of Gauss copula model, similarly as option prices are expressed using implied volatilities

- Implied tranche correlation is the value of $\rho$ in a homogeneous Gauss-copula model leading to the observed tranche price (generally not uniquely defined for mezzanine tranches).

- Implied base correlation is the value of $\rho$ explaining the price of an equity tranche with the corresponding attachment point ([0,3], [0,6], [0,9], ...). Base correlation is unique. Moreover, (hypothetical) prices of equity tranches can be computed recursively from observed prices of CDO tranches.
**Implied Tranche Correlation and Base Correlation (2)**

In our example we have the following values for the tranche and base correlation.

<table>
<thead>
<tr>
<th>Tranche Correlation</th>
<th>Base Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0,3]</td>
<td>21.9%</td>
</tr>
<tr>
<td>[3,6]</td>
<td>4.2%</td>
</tr>
<tr>
<td>[6,9]</td>
<td>14.8%</td>
</tr>
<tr>
<td>[9,12]</td>
<td>22.3%</td>
</tr>
<tr>
<td>[12,22]</td>
<td>30.5%</td>
</tr>
</tbody>
</table>

This is a typical pattern of tranche and base correlation, called *base correlation skew*. In particular model based on Gauss copula cannot explain all prices simultaneously.
Explaining Base Correlation Skews

**Goal.** Find a single model that explains (approximately) prices of all tranches or reproduces at least qualitative behavior of observed CDO prices. Needed for instance for pricing CDOs with non-standard attachment points.

**Idea.** Markov model does yields base correlation skews if interaction between intensities is increasing and *convex* in number of defaults $l$, leading to *infrequent but large clusters* of defaults. We use

$$h(t, l) = \lambda_0 \left\{ 1 + \lambda_1 \cdot [e^{\lambda_2 l/m} - e^{\lambda_2 \bar{\mu}_t}]^+ \right\}$$

Here $\lambda_0, \lambda_1, \lambda_2$ are parameters which should be calibrated to observed CDS and CDO spreads; $\bar{\mu}_t$ is a deterministic function giving the expected proportion of defaults.
Numerical results

We priced the observed CDO tranches by this model, increasing the parameter $\lambda_2$ and calibrating the other parameters to the observed marginal 5-year default probability and the price of the equity tranche.

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>$\lambda_1$</th>
<th>$\lambda_0$</th>
<th>[0,3]</th>
<th>[3,6]</th>
<th>[6,9]</th>
<th>[9,12]</th>
<th>[12,22]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>230</td>
<td>0.00427</td>
<td>27.6%</td>
<td>2.30%</td>
<td>1.12%</td>
<td>0.58%</td>
<td>0.16%</td>
</tr>
<tr>
<td>1</td>
<td>112</td>
<td>0.00428</td>
<td>27.6%</td>
<td>2.28%</td>
<td>1.11%</td>
<td>0.58%</td>
<td>0.16%</td>
</tr>
<tr>
<td>2</td>
<td>53</td>
<td>0.00429</td>
<td>27.6%</td>
<td>2.23%</td>
<td>1.08%</td>
<td>0.57%</td>
<td>0.18%</td>
</tr>
<tr>
<td>3</td>
<td>33.4</td>
<td>0.00430</td>
<td>27.6%</td>
<td>2.17%</td>
<td>1.04%</td>
<td>0.56%</td>
<td>0.19%</td>
</tr>
<tr>
<td>4</td>
<td>23.5</td>
<td>0.00432</td>
<td>27.6%</td>
<td>2.10%</td>
<td>1.00%</td>
<td>0.54%</td>
<td>0.20%</td>
</tr>
<tr>
<td>6</td>
<td>14</td>
<td>0.00436</td>
<td>27.6%</td>
<td>1.99%</td>
<td>0.92%</td>
<td>0.51%</td>
<td>0.21%</td>
</tr>
</tbody>
</table>

- Model prices come much closer to observed prices.

- Tentative interpretation of parameters: $\lambda_0$ responsible for default probability; product $\lambda_1\lambda_2$ responsible for default correlation/price of equity tranche; $\lambda_2$ responsible for base correlation skew.
The Resulting Base Correlations
References


