An Economic Motivation for Variance Contracts

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Introduction and Motivation (I)

- Risk factors in state-of-the-art models: stock price risk, stochastic volatility, stochastic jumps, ...
  - stock and money market account only: market is incomplete
  - further derivatives needed to complete the market
- Derivatives available for trading at exchange
  - standard claims: futures, call and put options
  - recently introduced: variance contract (direct trading of realized variance)
- Research questions
  - What is special about the variance contract?
  - Why should the variance contract be introduced?

Introduction and Motivation (II)

- General argument for introduction of additional contract
  - non-redundant derivative offers additional investment opportunities
  - however: variance contract is not the most natural choice
- Argument for choice of specific non-redundant contract
  - most different from "replicating portfolio"
  - perfect hedge not possible due to transaction costs, market incompleteness, discrete trading, model mis-specification, . . .
- Focus of this paper: model mis-specification
  - improper model is used to replicate the claim
  - idea: introduce claim with largest exposure to model risk

Contributions

- Expected excess return on variance contract
  - usual explanation: pricing of volatility risk
  - alternative explanation: pricing of jump risk
- Exposure of derivatives to model mis-specification
  - traded contracts for hedging: standard call and put options
  - how large are hedging errors due to model mis-specification?
  - is there a robust hedge?
    - for further options: yes
    - for variance contract: no

Related Literature

- Variance contract
  - Static replication (in diffusion models, use of continuum of options)

    Neuberger (1994), Carr, Madan (2002), Carr, Lee (2003)
  - Evidence on risk premium

    Carr, Wu (2004), Bondarenko (2004)
- Evidence on risk premia for SV and jumps

  Pan (2002), Bakshi, Kapadia (2003), Broadie, Chernov, Johannes (2004)
- Trading of risk factors

  Liu, Pan (2003), Liu, Longstaff, Pan (2003), Branger, Schlag, Schneider (2005)

Model Setup

- Dynamics under the true measure

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW_t^{(S)} + S_{t-}\left\{\left(e^{X_t}-1\right)dN_t - E^P\left[e^{X_t}-1\right]k^P\,dt\right\} \\
dV_t &= \kappa^P\left(\theta^P - V_t\right)dt + \sigma_V\sqrt{V_t}\left(\rho dW_t^{(S)} + \sqrt{1-\rho^2}dW_t^{(V)}\right) + \left\{Y_t dN_t - E^P[Y_t]k^P dt\right\}
\end{aligned}
$$

- Realized variance of stock

$$
RV(0,T) = \int_0^T \left(d\ln S_u\right)^2 du = \int_0^T V_u du + \int_0^T X_u^2 dN_u
$$

- Risk exposure of variance contract
  - exposure to stock price: zero (by construction)
  - exposure to stochastic volatility: positive
  - exposure to jump risk: positive (irrespective of jump direction)

Variance Contract and Risk Premia

- Risk premium on variance contract

$$
\begin{aligned}
&E^P[dC_t|\mathcal{F}_t] - E^Q[dC_t|\mathcal{F}_t] \\
&= e^{-r(T-t)}\Big\{ \Big(E^P[X^2]k^P - E^Q[X^2]k^Q \Big) \\
&\quad + \Big(1 + k_1^Q E^Q[X^2]\Big) \frac{1 - e^{-\kappa^Q (T-t)}}{\kappa^Q} \lambda_V \sigma_V V_{t-} \\
&\quad + \Big(1 + k_1^Q E^Q[X^2]\Big) \frac{1 - e^{-\kappa^Q (T-t)}}{\kappa^Q} \Big(E^P[Y]k^P - E^Q[Y]k^Q \Big) \Big\} dt.
\end{aligned}
$$

- Empirical studies: risk premium is negative
  - standard explanation: negative market price of volatility diffusion risk
  - alternative 1: negative market price of squared stock jump risk
  - alternative 2: negative market price of volatility jump risk

Why Should a New Claim be Introduced?

- Claim is better than its "replicating" strategy
  - traded instruments: stock, money market account, call option
  - "replication" fails due to model mis-specification
- "Replicating" strategy
  - use of correct model: risk exposure of claim h is matched by appropriate position in hedge instrument c and in stock
  - model mis-specifiction: sensitivities are calculated in improper hedge model
    - risk exposure of claim h: $\widetilde{h}_s, \widetilde{h}_v, \Delta\widetilde{h}$ (instead of $h_s, h_v, \Delta h$)
    - risk exposure of hedge instruments: $\widetilde{c}_s, \widetilde{c}_v, \Delta\widetilde{c}$ (instead of $c_s, c_v, \Delta c$)

Structure of Hedging Error

- General structure of hedging error

$$
\begin{aligned}
&\ldots dt + \left\{h_s - \widetilde{h}_s - \widetilde{\phi}_t^{(C)}\left(c_s - \widetilde{c}_s\right)\right\} S_t \left(\sqrt{V_t} dW_t^{(S)} + \lambda^{(S)} V_t dt\right) \\
&+ \left\{h_v - \widetilde{h}_v - \widetilde{\phi}_t^{(C)}\left(c_v - \widetilde{c}_v\right)\right\} \left(\sqrt{V_t}\sigma_V \left(\rho dW_t^{(S)} + \sqrt{1-\rho^2} dW_t^{(V)}\right) + \lambda^{(V)} V_t dt\right) \\
&+ \left\{\left(\Delta h - h_s \Delta S\right) - \left(\Delta \widetilde{h} - \widetilde{h}_s \Delta \widetilde{S}\right) - \widetilde{\phi}_t^{(C)}\left[\left(\Delta c - c_s \Delta S\right) - \left(\Delta \widetilde{c} - \widetilde{c}_s \Delta \widetilde{S}\right)\right]\right\} dN_t
\end{aligned}
$$

  - $\widetilde{\phi}_t^{(C)}$: position in hedge instrument
  - position in stock: portfolio is delta-neutral in hedge model
- Robust hedge
  - errors in sensitivities offset each other
  - use of hedge instrument that is similar to claim h

Design of the Study (I)

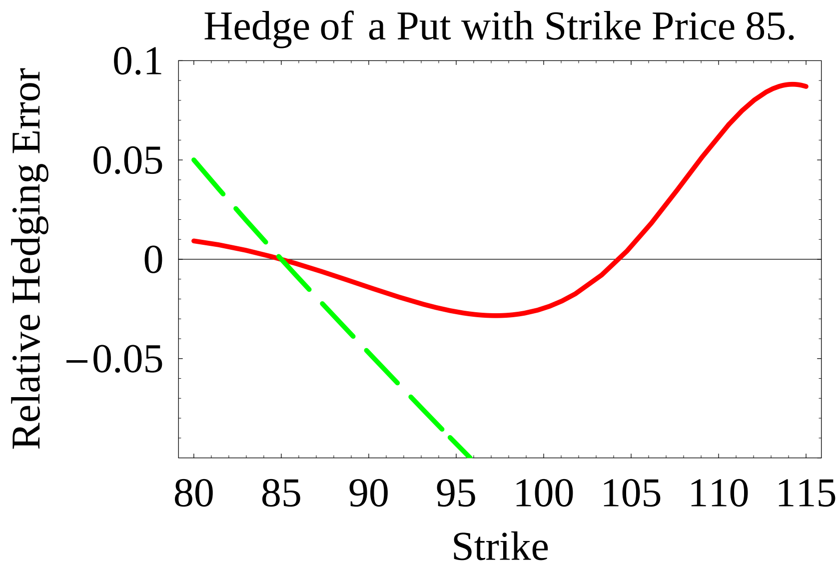
- Comparison of hedging errors under model mis-specification for
  - variance contract
  - benchmark contract: deep-OTM put (strike price: 85)
- Existence of a robust hedge?
  - natural candidates for robust hedge of variance contract
    - ATM straddle $\longrightarrow$ used to trade volatility risk
    - OTM put $\longrightarrow$ used to trade jump risk
  - natural candidate for robust hedge of deep-OTM put: OTM put

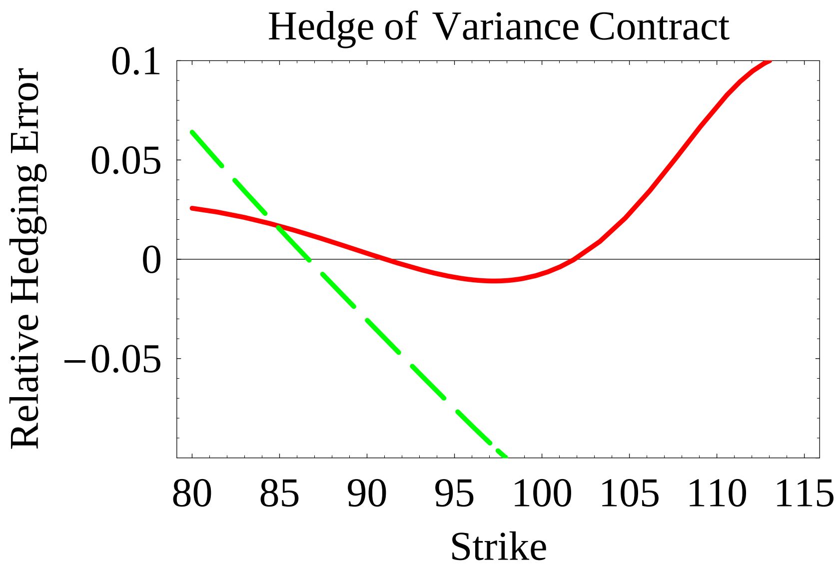
Design of the Study (II)

- Model mis-specification: true data-generating process not known
  - estimation risk: use of incorrect parameter set
  - use of incorrect risk factor: SV instead of SJ (or vice versa)
  - omission of risk factor: SV or SJ are ignored
- Hedge model: Heston (1993), Merton (1976) with deterministic jump size
  - complete with stock, money market account, option
  - hedge model is calibrated to cross section of option prices
  - hedging strategy: delta-hedge

Stochastic Volatility: Estimation Risk

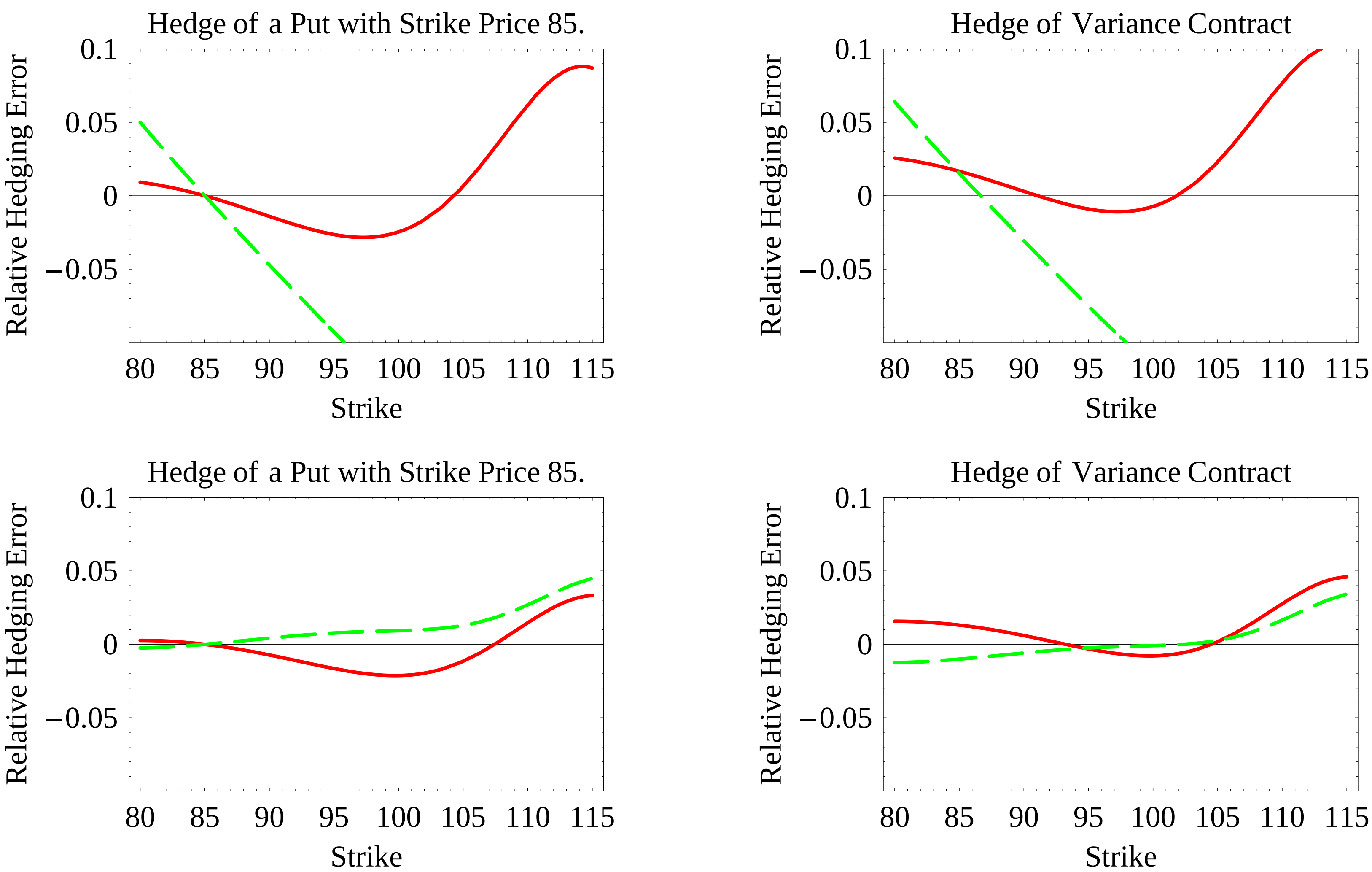
- True model: Heston (1993)
- Hedge model: Heston (1993) with different calibrated parameter set
- Hedging errors for change of $\sqrt{V}S$ in stock price and $\sigma_V\sqrt{V}$ in volatility





- ATM-straddle is not the ideal hedge instrument for SV!

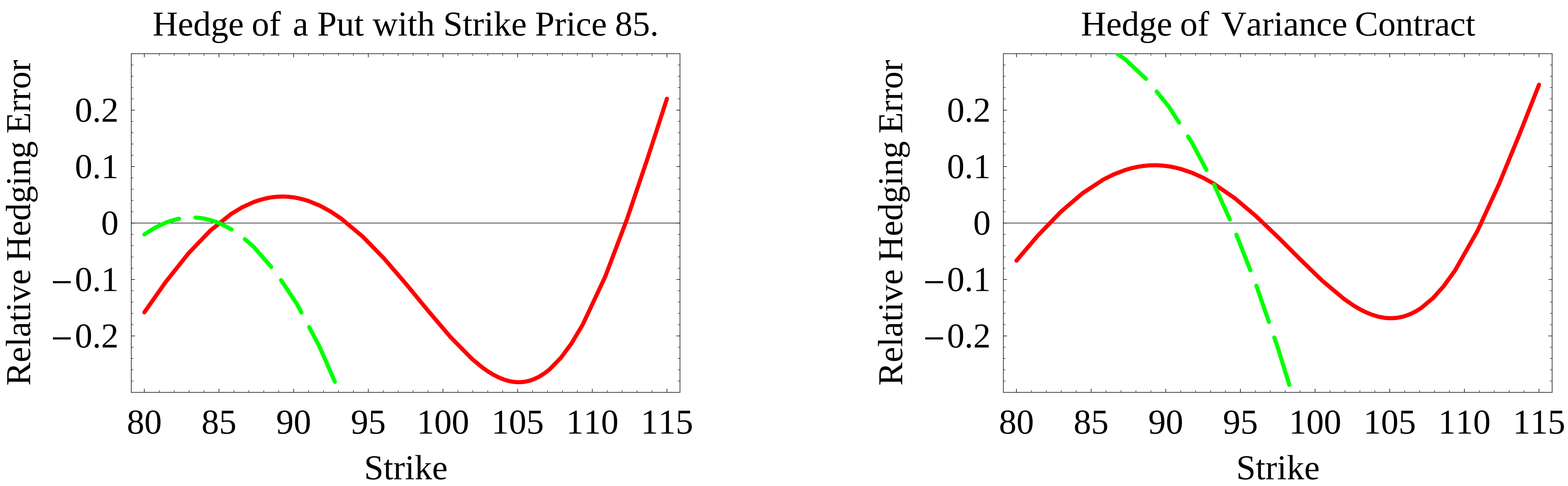
Stochastic Volatility: Estimation Risk (II)



Stochastic Volatility: Use of Incorrect Risk Factor

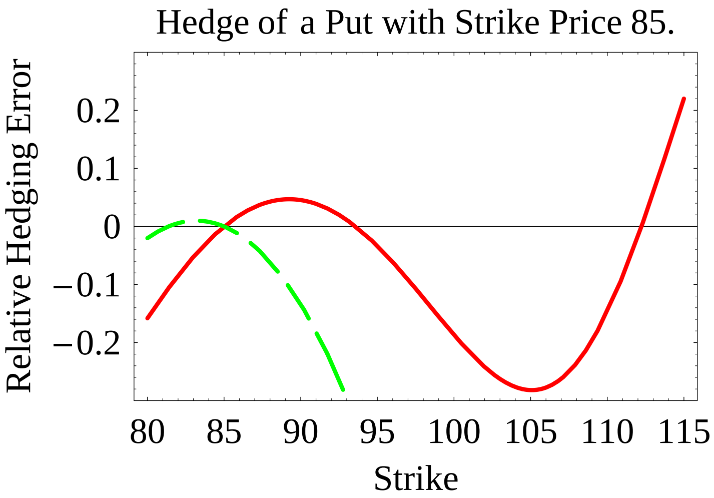
- True model: Heston (1993)
- Hedge model: Merton (1976) with deterministic jump size

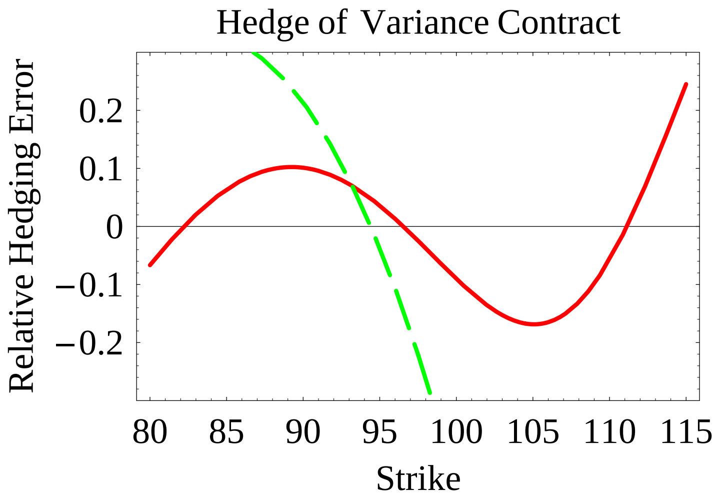
  $\longrightarrow$ improper risk factor is hedged (jumps instead of SV)
- Hedging errors for change of $\sqrt{V}S$ in stock price and $\sigma_V\sqrt{V}$ in volatility

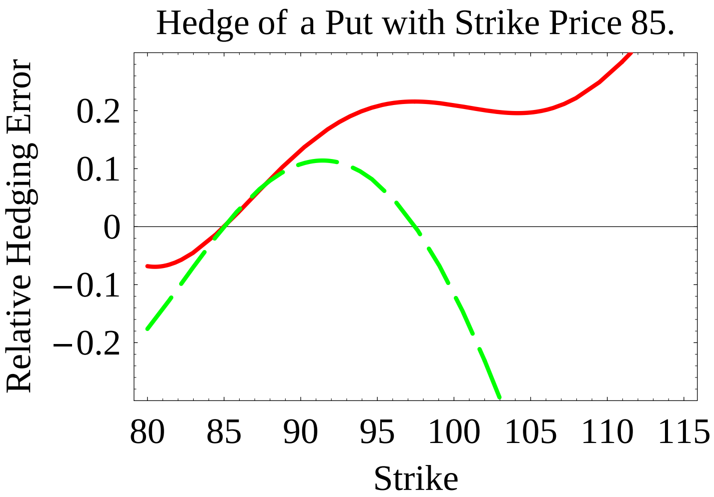


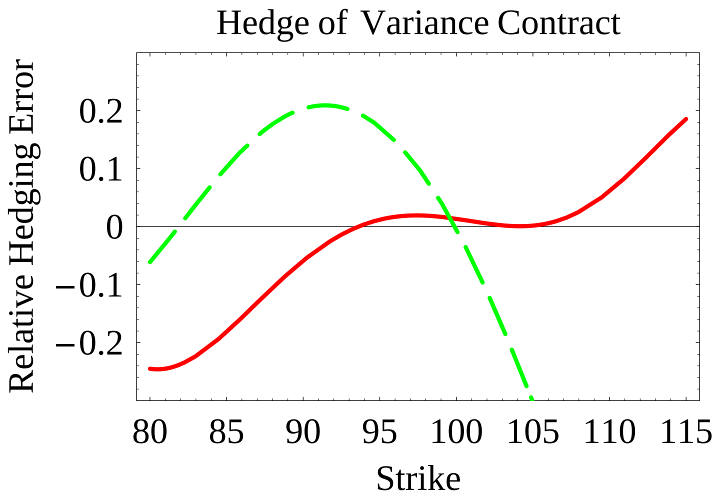
- ATM-straddle and OTM-put are not the ideal hedge instruments!

Stochastic Volatility: Use of Incorrect Risk Factor (II)





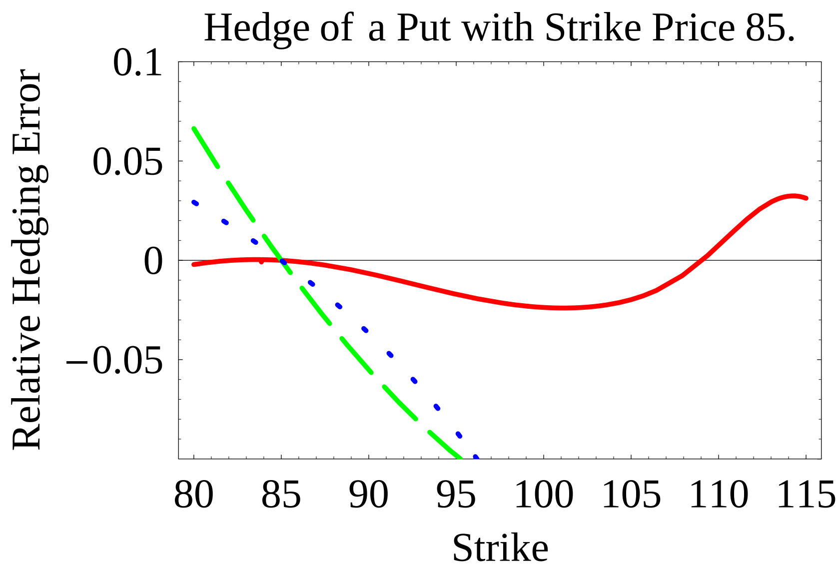


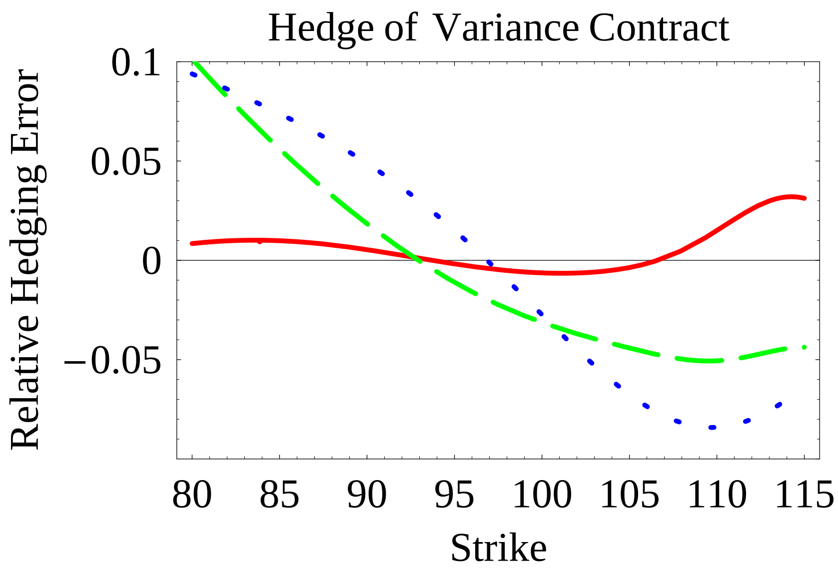


BCC: Omission of Jumps

- True model: Bakshi, Cao, Chen (1997)
- Hedge model: Heston (1993)

  $\longrightarrow$ jumps only hedged by chance (if jump exposure $\approx$ volatility exposure)
- Hedging errors for change of $\sqrt{V}S$ in stock price, ΔS in stock price, $\sigma_V \sqrt{V}$ in volatility

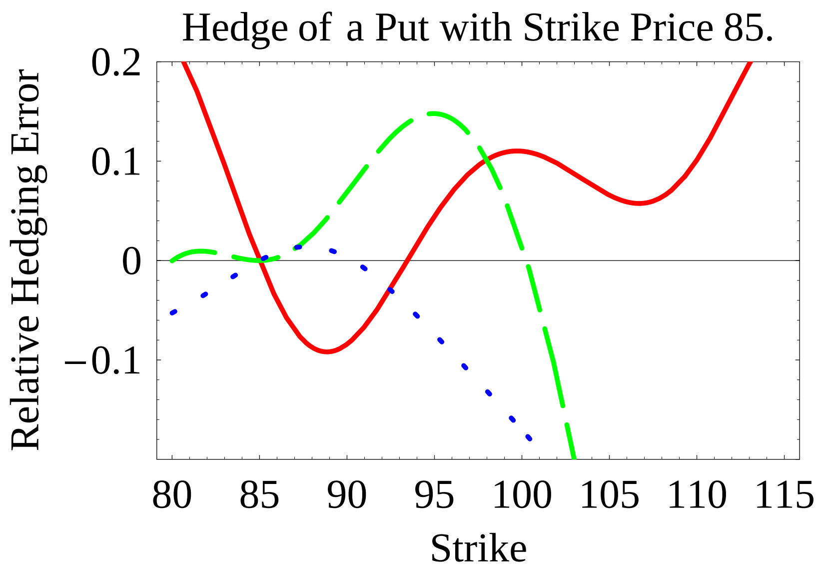


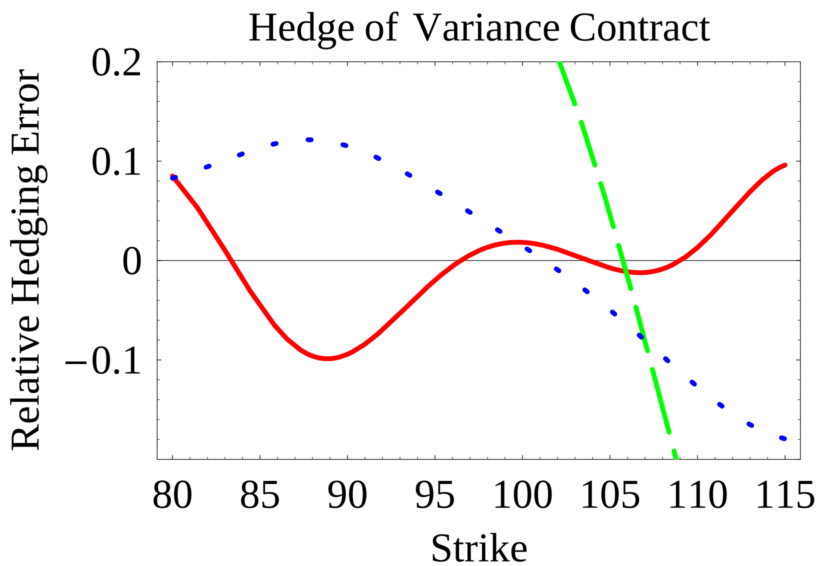


BCC: Omission of Stochastic Volatility

- True model: Bakshi, Cao, Chen (1997)
- Hedge model: Merton (1976) with deterministic jump size

  $\rightarrow$ SV only hedged by chance (if volatility exposure $\approx$ jump exposure)
- Hedging errors for change of $\sqrt{V}S$ in stock price, ΔS in stock price, $\sigma_V \sqrt{V}$ in volatility





Conclusion

- Expected excess return of variance contract
  - depends on market prices of stochastic volatility and stochastic jumps
  - negative expected return $\Rightarrow$ negative market price of volatility risk and/or negative premium for jump risk
- Economic motivation for trading variance contracts
  - investor wants to trade variance risk, i.e. volatility risk and jump risk
  - model mis-specification $\Rightarrow$ derivative is better than "replicating strategy"
  - variance contract has larger exposure to model risk than standard put