Sensitivity analysis of utility based prices and risk-tolerance wealth processes

Dmitry Kramkov, Carnegie Mellon University

Based on a paper with Mihai Sirbu from Columbia University

Outline

The prices of **non replicable** derivative securities depend on many factors:

1. risk-preferences of the investor:
   (a) reference probability measure $\mathbb{P}$
   (b) utility function $U = U(x)$
2. current portfolio of the investor
3. trading volume in the derivatives

Goal: study the **dependence** of prices on trading volume.

Model of a financial market

There are $d+1$ **traded** or **liquid** assets:

1. a **savings account** with zero interest rate.
2. d **stocks**. The price process S of the stocks is a semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$.

$\mathcal{Q}$ **:** the family of local martingale measures for S .

Assumption *(No Arbitrage)*

$$\mathcal{Q} \neq \emptyset$$

Contingent claims

Consider a family of m **non-traded** or **illiquid** European contingent claims with

1. maturity T
2. payment functions $f = (f_i)_{1 \leq i \leq m}$.

Assumption *No nonzero portfolio of* f *is replicable:*

$$\langle q, f \rangle = \sum_{i=1}^{m} q_i f_i \text{ is } \textbf{replicable} \Leftrightarrow q = 0$$

Pricing problem

Question What is the (marginal) price $p = (p_i)_{1 \leq i \leq m}$ of the contingent claims f ?

Intuitive Definition The marginal price p for the contingent claims f is the **threshold** such that given the chance to buy or sell at p^{trade} an investor will

$$\text{buy at } p^{trade} < p \;\; \& \;\; \text{sell at } p^{trade} > p$$

$$\Updownarrow$$

$$\text{do nothing at } p^{trade} = p$$

Economic agent or investor

Consider an investor with the portfolio (x, q), whose preferences are modeled by a **utility function** U:

1. $U : (0, \infty) \to \mathbf{R}$, strictly increasing and strictly concave
2. The Inada conditions hold true:

$$U'(0) = \infty \quad U'(\infty) = 0$$

Problem of optimal investment

The goal of the investor is to maximize **the expected utility of terminal wealth**:

$$u(x,q) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T + \langle q, f \rangle)],$$

where $\mathcal{X}(x)$ is the set of strategies with initial wealth x .

Order structure: a portfolio (x,q) is **better** than a portfolio (x',q') if $u(x,q) \geq u(x',q')$.

Marginal utility based price

Definition A **marginal utility based price** for the claims f given the portfolio (x, q) is a vector $p(x, q)$ such that

$$u(x, q) \geq u(x', q')$$

for any pair (x', q') satisfying

$$x + \langle q, p(x, q) \rangle = x' + \langle q', p(x, q) \rangle.$$

In other words, given the portfolio (x, q) the investor **will not trade** the options at $p(x, q)$.

Computation of $p(x) = p(x, 0)$

Define the conjugate function

$$V(y) = \max_{x>0} [U(x) - xy], \quad y > 0.$$

and consider the following **dual** optimization problem:

$$v(y) = \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E} \left[V \left(y(\frac{d\mathbb{Q}}{d\mathbb{P}}) \right) \right], \quad y > 0$$

$\mathbb{Q}(y)$ **:** the **minimal martingale measure** for y .

Computation of $p(x) = p(x, 0)$

Mark Davis gave heuristic arguments to show that if y corresponds to x in the sense that

$$x = -v'(y) \Leftrightarrow y = u'(x)$$

then

$$p(x) = \mathbb{E}_{\mathbb{Q}(y)}[f].$$

The precise mathematical results are given in a joint paper with Julien Hugonnier and Walter Schachermayer.

Computation of $p(x) = p(x, 0)$

Theorem (Hugonnier,K.,Schachermayer) *Let* $x > 0$*,* $y = u'(x)$ *and* X *be a non-negative wealth process. The following conditions are equivalent:*

1. $p(x)$ *is unique for* **any** f *such that*

$$|f| \leq K(1 + X_T) \text{ for some } K > 0$$

2. $\mathbb{Q}(y)$ *exists and* X *is a martingale under* $\mathbb{Q}(y)$*.*

Moreover, in this case $p(x) = \mathbb{E}_{\mathbb{Q}(y)}[f]$*.*

Trading problem

Assume that the investor can trade the claims at the initial time at the price p^{trade}.

Question What quantity $q = q(p^{trade})$ the investor should trade (buy or sell) at the price p^{trade}?

Using the marginal utility based prices $p(x, q)$ we can compute the optimal quantity from the "equilibrium" condition:

$$p^{trade} = p(x - qp^{trade}, q)$$

Sensitivity analysis of utility based prices

Main difficulty : $p(x, q)$ is hard to compute except for the case $q = 0$.

Linear approximation for "small" Δx and q :

$$p(x + \Delta x, q) \approx p(x) + p'(x)\Delta x + D(x)q,$$

where $p'(x)$ is the derivative of $p(x)$ and

$$D^{ij}(x) = \frac{\partial p^i}{\partial q^j}(x, 0), \quad 1 \leq i, j \leq m.$$

Quantitative questions

Question (Quantitative) How to compute $p'(x)$ and $D(x)$?

Closely related publications:

J. Kallsen (02) : formula for $D(x)$ for general semimartingale model but in a different framework of local utility maximization.

V. Henderson (02) : formula for $D(x)$ in the case of a Black-Scholes type model with basis risk and for power utility functions.

Qualitative questions

Question (Qualitative) When the following (desirable) properties hold true for **any** family of contingent claims f ?

1. The marginal utility based price $p(x) = p(x, 0)$ **does not depend** (locally) on x, that is,

$$p'(x) = 0$$

2. The sensitivity matrix $D(x)$ has **full rank**
3. The sensitivity matrix $D(x)$ is **symmetric**

Qualitative questions

4. The sensitivity matrix $D(x)$ is **negative semi-definite**: $\langle q, D(x)q\rangle \leq 0$.

5. **Stability** of the linear approximation: for any p^{trade} the linear approximation to the “equilibrium” equation:

$$p^{trade} = p(x - qp^{trade}, q)$$

that is,

$$p^{trade} \approx p(x) - p'(x)qp^{trade} + D(x)q$$

has the *“correct”* solution.

Risk-tolerance wealth process

Definition (K., Sirbu) A maximal wealth process $R(x)$ is called the **risk-tolerance wealth process** if

$$R_T(x) = -\frac{U'(\widehat{X}_T(x))}{U''(\widehat{X}_T(x))},$$

where $\widehat{X}(x)$ is the optimal solution of

$$u(x) := u(x,0) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)].$$

Risk-tolerance wealth process

Some properties of $R(x)$ (if it exists):

1. Initial value:

$$R_0(x) = -\frac{u'(x)}{u''(x)}.$$

2. Derivative of optimal wealth strategy:

$$\frac{R(x)}{R_0(x)} = X'(x) := \lim_{\Delta x \to 0} \frac{\widehat{X}(x + \Delta x) - \widehat{X}(x)}{\Delta x}.$$

Main qualitative result

Recall $p(x + \Delta x, q) \approx p(x) + p'(x)\Delta x + D(x)q$.

Theorem (K., Sirbu) *The following assertions are equivalent:*

1. *The risk-tolerance wealth process* $R(x)$ *exists.*
2. $p'(x) = 0$ *for any* f *.*
3. $D(x)$ *is symmetric for any* f *.*
4. $D(x)$ *has full rank for any (non-replicable)* f *.*
5. $D(x)$ *is negative semidefinite for any* f *.*

Existence of $R(x)$

Recall that $\mathbb{Q}(y)$ is the minimal martingale measure (the solution to the dual problem) for y .

Theorem (K., Sirbu) *The following assertions are equivalent:*

1. $R(x)$ *exists.*
2. $\frac{d}{dy}\mathbb{Q}(y) = 0$ *at* $y = u'(x)$.

In particular, $R(x)$ *exists for any* $x > 0$ *if and only if* $\mathbb{Q}(y)$ *is the same for all* y .

Second order stochastic dominance

Definition If ξ and η are nonnegative random variables, then $\xi \succeq_2 \eta$ if

$$\int_0^t \mathbb{P}(\xi \geq x)dx \geq \int_0^t \mathbb{P}(\eta \geq x)dx, \; t \geq 0.$$

We have that $\xi \succeq_2 \eta$ iff

$$\mathbb{E}[W(\xi)] \leq \mathbb{E}[W(\eta)]$$

for any *convex* and *decreasing* function W .

Existence of $R(x)$

Case 1: a utility function U is arbitrary.

Theorem (K., Sirbu) *The following assertions are equivalent:*

1. $R(x)$ *exists for any* $x > 0$ *and any utility function* U .
2. *There exists a unique* $\widehat{\mathbb{Q}} \in \mathcal{Q}$ *such that*

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \succeq_2 \frac{d\mathbb{Q}}{d\mathbb{P}} \quad \forall \mathbb{Q} \in \mathcal{Q}.$$

Existence of $R(x)$

Case 2: a financial model is arbitrary.

Theorem (K., Sirbu) *The following assertions are equivalent:*

1. $R(x)$ *exists for any* $x > 0$ *and any financial model.*
2. *The utility function* U *is*
   (a) *the power utility:* $U(x) = (x^\alpha - 1)/\alpha$, $\alpha < 1$, *if* $x \in (0, \infty)$;
   (b) *the exponential utility:* $U(x) = -\exp(-\gamma x)$, $\gamma > 0$, *if* $x \in (-\infty, \infty)$.

Computation of $D(x)$

We choose

$$R(x)/R_0(x) = X'(x)$$

as a **numéraire** and denote

$f^R = f R_0(x)/R(x)$: discounted contingent claims

$X^R = X R_0(x)/R(x)$: discounted wealth processes

$\mathbb{Q}^R$: the martingale measure for X^R :

$$\frac{d\mathbb{Q}^R}{d\widehat{\mathbb{Q}}} = \frac{R_T(x)}{R_0(x)}$$

Computation of $D(x)$

Consider the Kunita-Watanabe decomposition:

$$P_t^R = \mathbb{E}_{\mathbb{Q}^R}\left[f^R | \mathcal{F}_t\right] = M_t + N_t, \quad N_0 = 0,$$

where

1. M is $R(x)/R_0(x)$ -discounted wealth process.

   Interpretation: **hedging process**.

2. N is a martingale under $\mathbb{Q}^R$ which is orthogonal to all $R(x)/R_0(x)$ -discounted wealth processes.

   Interpretation: **risk process**.

Computation of $D(x)$

Denote $a(x) := -xu''(x)/u'(x)$ the relative risk-aversion coefficient of

$$u(x) = \max_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)].$$

Theorem (K., Sirbu) *Assume that the risk-tolerance wealth process* $R(x)$ *exists. Then*

$$D(x) = -\frac{a(x)}{x} \mathbb{E}_{\mathbb{Q}^R} \left[N_T N'_T \right]$$

Computation of $D(x)$ in practice

Inputs:

1. $\widehat{\mathbb{Q}}$. *Already implemented!*

2. $R(x)/R_0(x)$. Recall that

$$\frac{R(x)}{R_0(x)} = \lim_{\Delta x \to 0} \frac{\widehat{X}(x+\Delta x) - \widehat{X}(x)}{\Delta x}.$$

   Decide what to do with one penny!

3. Relative risk-aversion coefficient $\alpha(x)$. *Deduce from mean-variance preferences*. In any case, this is just a number!

Model with basis risk

Traded asset : $dS_t = S_t \left(\mu dt + \sigma dW_t \right)$.

Non traded asset : $d\widetilde{S} = (\tilde{\mu} dt + \tilde{\sigma} d\widetilde{W_t})$

Denote by

$$\rho = \frac{d\widetilde{W} dW}{dt}$$

the **correlation** coefficient between S and $\widetilde{S}$. In practice, we want to chose S so that

$$\rho \approx 1.$$

Model with basis risk

Consider contingent claims $f = f(\widetilde{S})$ whose payoffs are determined by $\widetilde{S}$ (maybe path dependent).

To compute $D(x)$ assume (as an example) the following choices:

1. $\widehat{\mathbb{Q}}$ is a martingale measure for $\widetilde{S}$.
2. $R(x)/R_0(x) = 1$

Then

$$D_{ij}(x) = -\frac{\alpha(x)}{x}(1-\rho^2)\mathrm{Cov}_{\widehat{\mathbb{Q}}}(f_i, f_j).$$

Assumptions

Assumption *The financial model can be* **completed** *by an addition of a finite number of securities.*

Assumption *There are strictly positive constants* c_1 *and* c_2 *such that* $c_1 < -\frac{xU''(x)}{U'(x)} < c_2, \quad x > 0$ *.*

Assumption *There is a wealth process* $X \geq 0$ *such that* $|f| \leq X_T$ *and* X *is a squire integrable martingale under the minimal martingale measure* $\mathbb{Q}(y)$ *.*

Summary

- For non replicable contingent claims prices depend on the trading volume.
- The following conditions are equivalent:
  - Approximate utility based prices have nice qualitative properties
  - Risk-tolerance wealth processes exist.
- We need to solve the mean-variance hedging problem, where the risk-tolerance wealth process plays the role of the numéraire.