An Information-Based Approach to Asset-Pricing Dynamics

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work with D.C. Brody and A. Macrina
"Beyond Hazard Rates: a New Framework for Credit Risk Modelling"

8 July 2005

The "Issue"

Ad hoc modelling of asset-price dynamics.

$$dS_t = (r_t + \lambda_t \sigma_t) S_t \, dt + \sigma_t S_t \, dW_t$$

Where do we get the volatility?

(i) "Make it up as you go along" ...

(ii) Structural models (difficult)

(iii) Information-based models

Examples:

(1) Credit (defaultable discount bonds)

(2) Option on random cash-flow

The Model

* Assume <u>default-free</u> interest rates are deterministic: $P_{tT} = P_{0T} / P_{0t}$

* Work in risk-neutral measure $\mathbb{Q}$

For any cash-flow H_T at T,

$$H_t = P_{tT}\, E\left[H_T \,/\, \mathcal{F}_t \right]$$

We want to "model" $(\mathcal{F}_t)$.

Defaultable discount bond

$H_T = \{h_0, h_1\}$

h_1 full payment

h_0 partial payment (default)

Let $p_i = \mathbb{Q}(H_T = h_i)$ $i = 0, 1$

("a priori probabilities")

Noisy market information model:

Assume $\mathcal{F}_t = \mathcal{F}_t^{\xi}$, where

$$\xi_t = \sigma H_T t + \beta_{tT}$$

σ is a parameter

$\{\beta_{tT}\}$ is a $[0, T]$-Brownian bridge

Bond price:

$$B_{tT} = P_{tT}\,\mathbb{E}\left[H_T \,\middle|\, \mathcal{F}_t^{\xi}\right]$$

Solution:

$$B_{tT} = P_{tT} \sum_i h_i \pi_{it} \quad , \text{ where}$$

$$\pi_{it} = \mathbb{E}\left[\mathbb{1}\{H_T = h_i\} \,\middle|\, \mathcal{F}_t^{\xi}\right]$$

We find that:

$$\pi_{it} = \frac{p_i \exp\left[\frac{T}{T-t}\left(\sigma h_i \xi_t - \frac{1}{2}\sigma^2 h_i^2 t\right)\right]}{\sum_i p_i \exp\left[\frac{T}{T-t}\left(\sigma h_i \xi_t - \frac{1}{2}\sigma^2 h_i^2 t\right)\right]}$$

where

$$\xi_t = \sigma H_T t + \beta_{tT}$$

Bond price dynamics

$$\pi_{it} = \mathbb{E}\left[\mathbb{1}\{H_T = h_i\} \mid \mathcal{F}_t^{\xi} \right]$$

$$= \pi_i(\xi_t, t) \qquad\qquad \pi(x, y)$$

$$B_{tT} = P_{tT} \sum_i h_i \pi_i(\xi_t, t)$$

$$= P_{tT} H(\xi_t, t)$$

$$d\pi_{it} = \frac{\sigma T}{T-t}\left(h_i - H(\xi_t, t)\right)\pi_{it}\, dW_t$$

$$W_t := \xi_t + \int_0^t \frac{1}{T-s}\left(\xi_s - \sigma T H(\xi_s, s)\right) ds$$

$(\mathcal{F}_t)$ - Brownian motion

$$dB_{tT} = r_t B_{tT}\, dt + \Sigma_{tT}\, dW_t$$

$$\Sigma_{tT} = \frac{\sigma T}{T-t} P_{tT} V_{tT}$$

where $V_{tT} = \sum_i \left(h_i - H(\xi_t, t)\right)^2 \pi_{it}$

(conditional variance of H_T)

$\{B_{tT}\}$ is a diffusion process.

"One-factor model".

"Emergent" dynamics

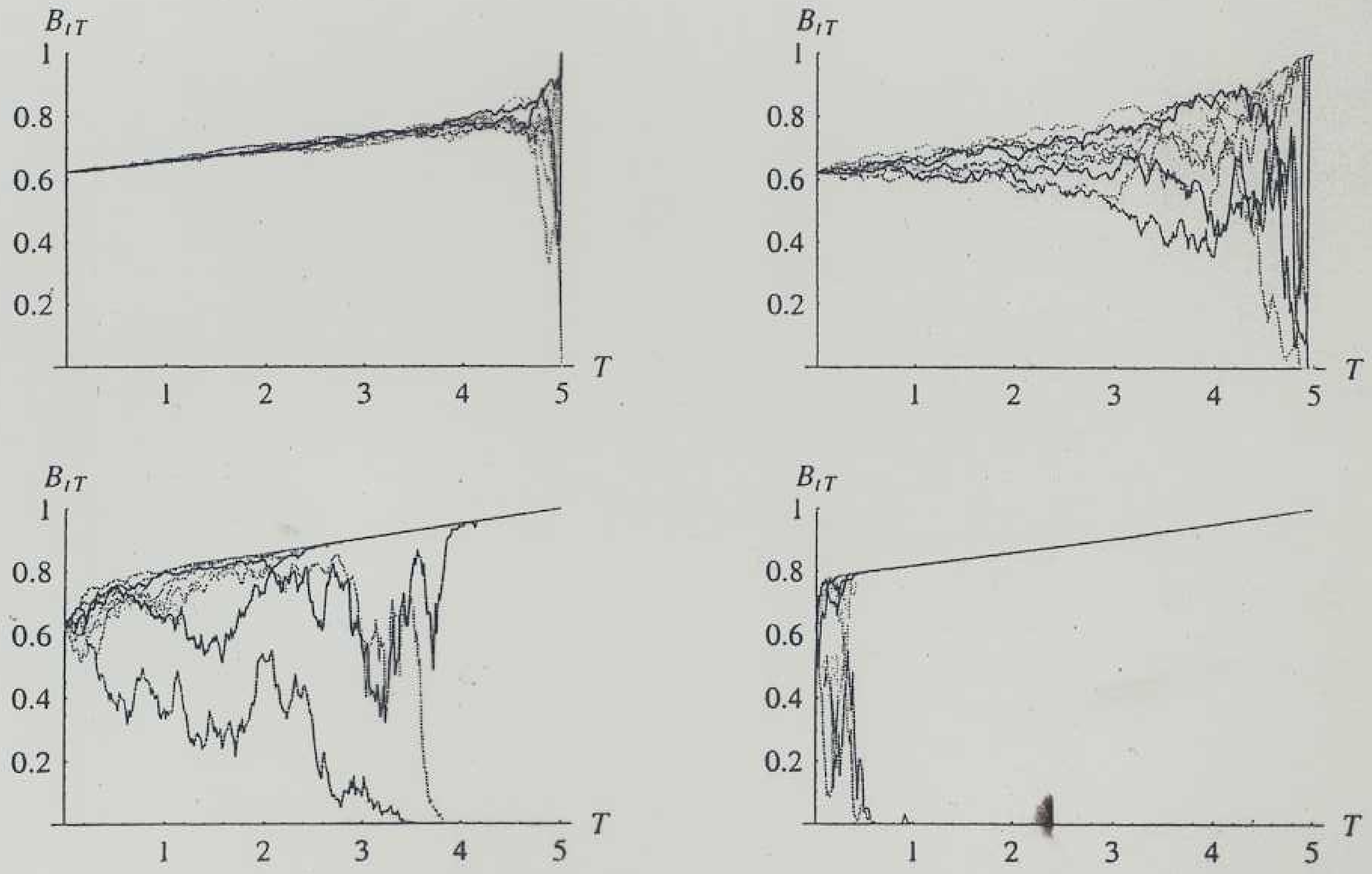


FIG. 1. *Bond price processes for various information-release rates.* The parameter σ governs the rate at which information is released to market participants concerning the payout of a defaultable discount bond. Four values of σ are illustrated. For high rates of σ default occurs only "at the last minute"

Option on a defaultable discount bond

Call option on defaultable discount bond

Exact solution!

$$C_0 = P_{0t}\,\mathbb{E}\left[\left(B_{tT} - K\right)^+\right]$$

$$= P_{0t}\,\mathbb{E}\left[\left(P_{tT}\sum_i \pi_{it} h_i - K\right)^+\right]$$

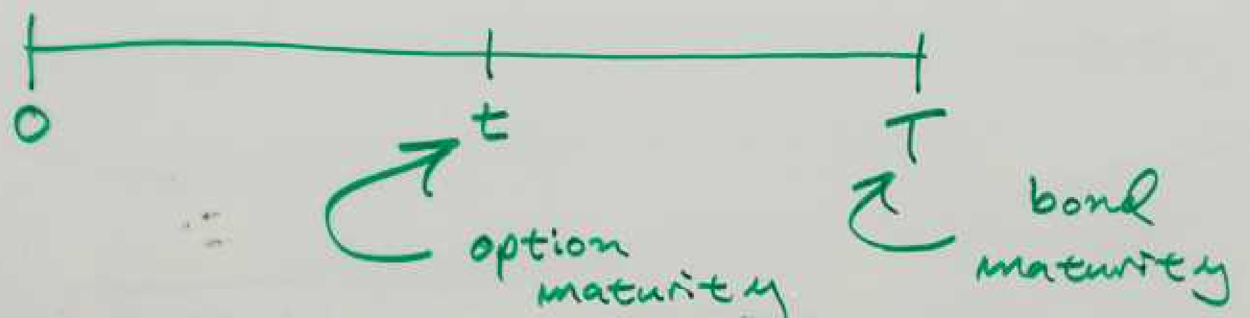
$$= P_{0t}\,\mathbb{E}\left[\frac{1}{\Phi_t}\left(\sum_i \left(P_{tT} h_i - K\right) p_{it}\right)^+\right]$$

where

$$p_{it} = p_i \exp\left[\frac{T}{T-t}\left(\sigma h_i \xi_t - \tfrac{1}{2}\sigma^2 h_i^2 t\right)\right]$$

"un-normalised" conditional probabilities

$$\Phi_t = \sum_i p_{it} \quad (\text{"normalisation factor"})$$



* Use $1/\Phi_t$ to change measure
* Call new measure $\mathbb{B}_T$ (bridge measure)
* $\{\xi_t\}$ is $\mathbb{B}_T$ - Brownian bridge.

Example: $H_T = \{h_0, h_1\}$

$C_0 =$

$$P_{0t}\, \mathbb{E}^{\mathbb{B}_T}\left[(P_{tT}h_1 - K)p_{1t} + (P_{tT}h_0 - K)p_{0t}\right]^+$$

$$\begin{cases} p_{0t} = p_0 \exp\left[\frac{T}{T-t}\left(\sigma h_0 \xi_t - \frac{1}{2}\sigma^2 h_0^2 t\right)\right] \\ p_{1t} = p_1 \exp\left[\frac{T}{T-t}\left(\sigma h_1 \xi_t - \frac{1}{2}\sigma^2 h_1^2 t\right)\right] \end{cases}$$

ξ_t is $\mathbb{B}_T$ - Gaussian

mean 0, variance $\dfrac{t(T-t)}{T}$

Expectation can be computed explicitly!

There are three cases:

(i) $P_{tT} h_0 > K \qquad C_0 = B_{0T} - P_{0t} K$

(ii) $K > P_{tT} h_1 \qquad C_0 = 0$

(iii) $P_{tT} h_1 > K > P_{tT} h_0$

$$C_0 =$$

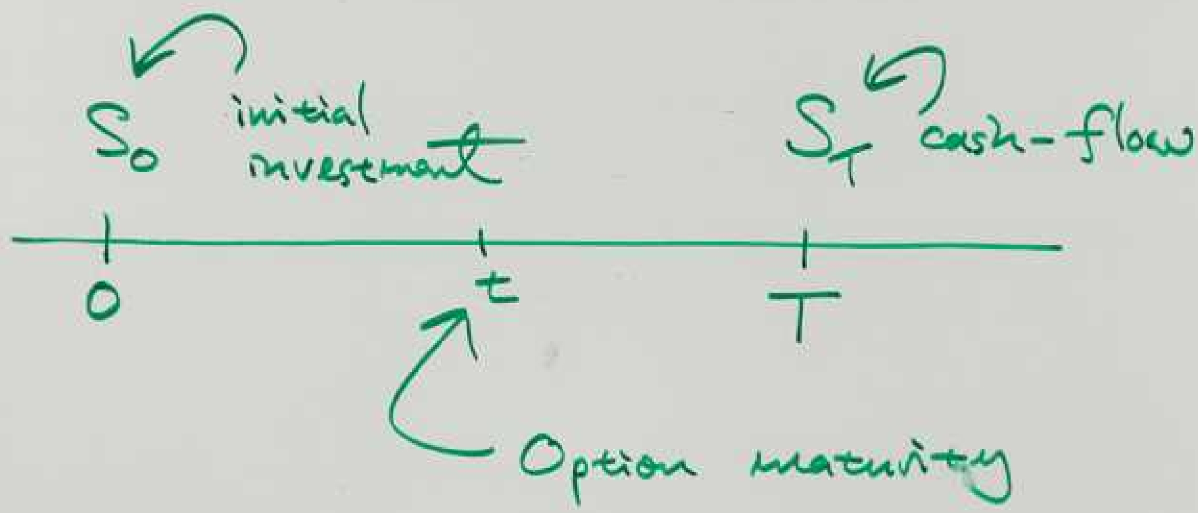
$$P_{0t}\left[p_1 \left(P_{tT} h_1 - K \right) N(d^+) - p_0 \left(K - P_{tT} h_0 \right) N(d^-) \right]$$

$$d^{\pm} = \frac{\ln\left[\dfrac{p_1 \left(P_{tT} h_1 - K \right)}{p_0 \left(K - P_{tT} h_0 \right)}\right] \pm \frac{1}{2}\sigma^2 \tau (h_1 - h_0)^2}{\sigma \sqrt{\tau} (h_1 - h_0)}$$

where $\tau = tT/(T-t)$.

* "Black-Scholes" type formula
* market information-release rate parameter σ plays role of volatility
* $\dfrac{\partial C_0}{\partial \sigma} > 0$ positive "vega" .
* Multiple payoff case can also be treated.

Option on a Random Cash-Flow



Information-process:

$$\xi_t = \sigma H_T t + \beta_{tT}$$

$$H_T = \ln(S_T/S_0)$$

$$S_T = S_0 \, e^{rT + \nu T^{1/2} Z - \frac{1}{2}\nu^2 T}$$

with $Z \sim N(0,1)$

$$S_t = S_0 P_{tT} \, E\left[e^{H_T} \,\middle|\, \mathcal{F}_t\right]$$

Find innovation process W_t ...

$$\frac{dS_t}{S_t} = r_t\,dt + \frac{\sigma\nu^2T^2}{T+(\sigma^2\nu^2T^2-1)t}\,dW_t$$

$\sigma\nu T < 1$ local vol increases

$\sigma\nu T > 1$ local vol decreases

Option pricing:
Use Black-Scholes formula
(but volatility "parameter" depends on maturity date of option)

Conclusions:

* "Pure" information-based modelling offers much scope for the development of realistic models for asset-price dynamics

* Model the flow of information about uncertain cash-flows. (E.g. payment of a bond principal.)

* Now, tractable models can be constructed.