Option pricing in the stochastic volatility model of Barndorff-Nielsen and Shephard

Indifference pricing and the minimal entropy martingale measure

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Motivation

- The indifference price of a claim C:
  - Investor indifferent between issuing a claim and investing the proceeds, or investing without issuing any claim
  - Hodges and Neuberger (1989)
- Zero risk aversion for exponential utility gives a price

$$\Lambda(t) = \mathrm{e}^{-r(T-t)}\mathbb{E}_{Q_{ME}}\left[C \,|\, \mathcal{F}_t\right]$$

- Q is the minimal entropy martingal measure (MEMM)

$$H(Q,P) = \mathbb{E}\left[\frac{dQ}{dP}\ln\left(\frac{dQ}{dP}\right)\right], Q \sim P$$

- Analyze the Barndorff-Nielsen and Shephard (2001) model

Overview of the talk

- The non-Gaussian stochastic volatility model of Barndorff-Nielsen and Shephard
- Derivation of the MEMM for this model via dynamic programming
- Verification of the MEMM
- A Black & Scholes integro-PDE for options
- Numerical examples

A non-Gaussian stochastic volatility model

- Model proposed by Barndorff-Nielsen and Shephard (2001)
- Asset price dynamics

$$\frac{dS(t)}{S(t)} = (\mu + \beta Y(t))\,dt + \sqrt{Y(t)}\,dB(t)$$

- Volatility (squared) is a non-Gaussian Ornstein-Uhlenbeck process with a stationary distribution

$$dY(t) = -\lambda Y(t)\,dt + dL(\lambda t)$$

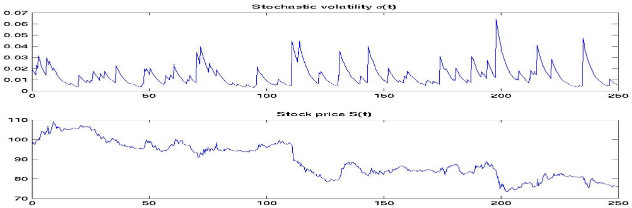
- L is a subordinator, e.g. Lévy process without continuous martingale part and only positive jumps

An example:

- Suppose $NIG(\mu, \beta, \alpha, \delta)$-logreturns
- Mean-variance mixture model

$$R|\sigma^2 \sim N(\mu + \beta\sigma^2, \sigma^2)\,, \sigma^2 \sim IG(\alpha, \delta)$$

- Choose $L(t)$ to be a IG-subordinator



MEMM via dynamic programming

- Wealth $X(t)$ when the cash amount π is invested in risky asset (assuming risk-free return is zero):

$$dX(t) = \pi(t)\left(\mu + \beta Y(t)\right)\,dt + \pi(t)\sqrt{Y(t)}\,dB(t)$$

- Optimal terminal wealth with exponential utility $1 - \exp(-\gamma x)$

$$V^0(t,x,y) = \sup_{\pi \in \mathcal{A}_t} \mathbb{E}\left[1 - \exp\left(-\gamma X^{t,x,y}(T)\right)\right].$$

- $\mathcal{A}_t$: set of admissible controls (integrability condition)

- HJB-equation for V^0

$$V_t^0 + \max_{\pi}\left\{(\mu+\beta y)\pi V_x^0 + \frac{1}{2}y\pi^2 V_{xx}^0\right\} + \mathcal{L}_Y V^0 = 0$$

- Terminal condition $V^0(T,x,y) = 1-\exp(-\gamma x)$
- Integro-operator

$$\mathcal{L}_Y V^0 = -\lambda y V_y^0 + \lambda \int_0^\infty \left\{V^0(t,x,y+z) - V^0(t,x,y)\right\}\nu(dz).$$

- Logarithmic transform of V^0

$$V^0(t,x,y) = 1-\exp(-\gamma x)H(t,y)$$

The function H

- H turns out to rescale the jumps of L under MEMM
- Integro-pde for H

$$H_t - \frac{(\mu + \beta y)^2}{2y} H + \mathcal{L}_Y H = 0$$

- Terminal data $H(T, y) = 1$

- Feynman-Kac representation of H:

$$H(t,y) = \mathbb{E}\left[\exp\left(-\frac{1}{2}\int_t^T \frac{(\mu+\beta Y^{t,y}(u))^2}{Y^{t,y}(u)}\,du\right)\right]$$

- $H \in C_b^{1,1}$, thus strong solution of the integro-pde
- If $\mu = 0$,

$$H(t,y) = \exp\left(-b(t)y + c(t)\right)$$

- A verification theorem validates the solution

- Optimal terminal portfolio with a claim issued (payoff $g(s)$)

$$V(t,x,s,y) = \sup_{\pi \in \mathcal{A}_t} \mathbb{E}\left[1 - \exp\left(-\gamma(X^{t,x,y}(T) - g(S^{t,s}(T)))\right)\right].$$

- HJB-equation

$$V_t + \max_{\pi}\left\{(\mu + \beta y)\pi V_x + \frac{1}{2} y\pi^2 V_{xx} + y\pi s V_{xs}\right\} + \mathcal{L}_S V + \mathcal{L}_Y V = 0$$

- Transformation of value function

$$V(t,x,y,s) = 1 - \exp\left(-\gamma x + \gamma \Lambda^{(\gamma)}(t,s,y)\right) H(t,y)$$

- $\Lambda^{(\gamma)}$ is the indifference price of the claim g

$$V^0(t,x,y) = V(t, x + \Lambda^{(\gamma)}(t,s,y), s, y).$$

- A Black & Scholes equation for this price

$$0 = \Lambda_t^{(\gamma)} + \frac{1}{2} y s^2 \Lambda_{ss}^{(\gamma)} - \lambda y \Lambda_y^{(\gamma)} + \frac{\lambda}{\gamma} \int_0^\infty \left\{ \exp\left(\gamma\left(\Lambda^{(\gamma)}(y+z) - \Lambda^{(\gamma)}(y)\right)\right) - 1 \right\} \frac{H(t,y+z)}{H(t,y)} \nu(dz)$$

- Derivations leading to this are all informal

MEMM price

- From general theory, $\gamma \downarrow 0$ gives the MEMM-price of the claim
- Informal limit leads to the Black & Scholes' integro-PDE

$$\Lambda_t + \frac{1}{2}\sigma^2(y)s^2\Lambda_{ss} - \lambda y \Lambda_y + \lambda \int_0^\infty \left(\Lambda(t, y+z, s) - \Lambda(t, y, s)\right) \frac{H(t, y+z)}{H(t, y)} \nu(dz) = 0.$$

- Terminal condition $\Lambda(T, s, y) = g(s)$.
- Use this equation to:
  - Read off the density process for the MEMM Q_{ME}
  - Price options (numerically)

The density process

- Consider the state dynamics

$$d\widetilde{S}(t) = \sqrt{\widetilde{Y}(t)}\,\widetilde{S}(t)\,d\widetilde{B}(t),$$
$$d\widetilde{Y}(t) = -\lambda\widetilde{Y}(t)\,dt + d\widetilde{L}(\lambda t),$$

- $\widetilde{B}(t)$ is Brownian motion and $\widetilde{L}(t)$ is a pure jump Markov process with the predictable compensating measure

$$\widetilde{\nu}(\omega, dz, dt) = \frac{H(t, \widetilde{Y}(t,\omega) + z)}{H(t, \widetilde{Y}(t,\omega))}\,\nu(dz)\,dt.$$

- Define $Z(t) := Z^B(t) \times Z^L(t)$ where

$$Z^B(t) = \exp\left(-\int_0^t \frac{\mu + \beta Y(s)}{\sqrt{Y(s)}}\, dB(s) - \frac{1}{2}\int_0^t \frac{(\mu + \beta Y(s))^2}{Y(s)}\, ds \right)$$

$$\begin{aligned} Z^L(t) = \exp\Bigg(& \int_0^t \int_0^\infty \ln \frac{H(s, Y(s) + z)}{H(s, Y(s))} N(dz, dt) \\ & + \int_0^t \int_0^\infty \left(1 - \frac{H(s, Y(s) + z)}{H(s, Y(s))} \right) \nu(dz) dt \Bigg) \end{aligned}$$

Verification of the MEMM

- Claim: $Z(t)$ is the density process for the MEMM,

$$dQ_{ME} = Z(T)\,dP$$

- The derivation of Q_{ME} informal
- Needs to verify that it indeed is the MEMM
- Use the results of Rheinländer (2003)
  - Gives sufficient conditions for a MEMM

Theorem: Suppose we have

$$\int_0^\infty \left\{ \exp\left(\frac{\beta^2}{\lambda} \left(1 - \exp(-\lambda T)\right) z \right) - 1 \right\} \nu(dz) < \infty,$$

Then $Z(t)$ is the density process of the minimal entropy martingale measure Q_{ME}.

Proof: Goes in two parts:

1. $Z(t)$ is a martingale which can be represented as

$$Z(t) = c \cdot \exp\left(- \int_0^t \frac{\mu + \beta Y(s)}{Y(s)} S^{-1}(s) dS(s) \right)$$

2. Q_{ME} induced by $Z(T)$ has finite entropy

- 1 holds by using Ito's Formula and the integro-PDE for $H(t, y)$
- Recall that $H \in C^{1,1}$.
- Exponential integrability gives martingale property from the Novikov condition.
- 2 follows from direct estimation using the exponential integrability condition

Numerical pricing of options

- A Feynman-Kac solution of the Black & Scholes integro-PDE

$$\Lambda(t,s,y) = \mathbb{E}_{Q_{ME}}\left[g\left(S^{t,s,y}(T)\right)\right]$$

- This verifies that Λ solving the integro-PDE is the MEMM price of g

$$\Lambda_t + \frac{1}{2}\sigma^2(y)s^2\Lambda_{ss} - \lambda y \Lambda_y + \lambda \int_0^\infty \left(\Lambda(t,y+z,s) - \Lambda(t,y,s)\right)\frac{H(t,y+z)}{H(t,y)}\nu(dz) = 0.$$

- We discuss a numerical solution of this integro-PDE

- Three practical problems for putting up a scheme:
  1. Asymptotics in y and s when considering a finite domain?
  2. Integral operator outside the finite domain?
  3. $y = 0$, e.g., zero initial volatility?
- Note in addition: To solve for Λ, we need to solve for H as well

Example: A call option

- Strike $K = 200$ at exercise time $T = 1$, zero interest rate.
- Using properties of H, we can show that when $y \to \infty$

$$\widetilde{Y}(t) \sim y e^{-\lambda t}$$

- Hence, $\widetilde{S}$ becomes asymptotically a geometric Brownian motion with squared volatility $y \exp(-\lambda t)$

$$\Lambda(t, s, y) \sim C(t, s, \sigma^2 = y \exp(-\lambda t))$$

- When $s \to \infty$, we find

$$\Lambda(t, s, y) \sim s - K$$

- We can collect asymptotics in s and y into

$$\Lambda(t,s,y) \sim C(t,s,\sigma^2 = y\exp(-\lambda t))$$

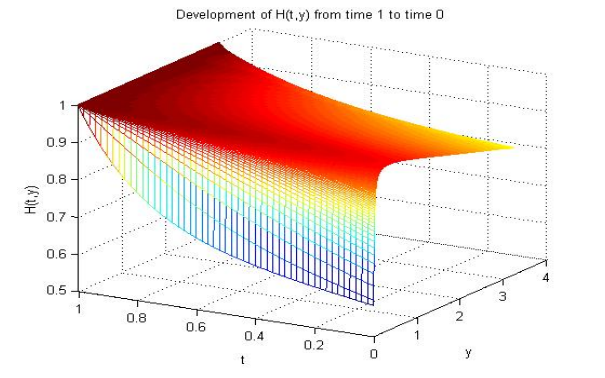
- Yields values for Λ outside the finite domain to be used in the integral term
- For zero volatility, $y = 0$, we put on a gradient condition yielded by the integro-PDE for Λ
- Asymptotics for H when $y \longrightarrow \infty$:

$$H(t,y) \sim \exp(-b(t)y + c(t))$$

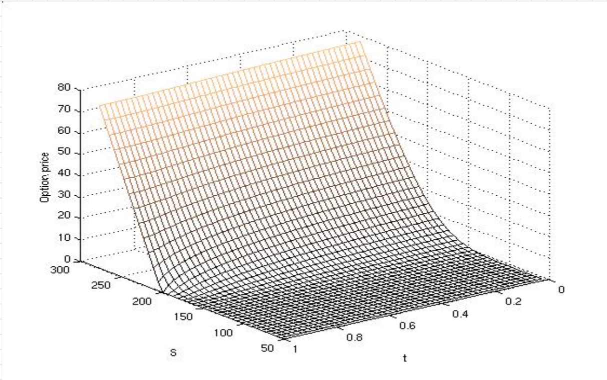
- For H, we consider the integro-PDE defined for all real y's, thus avoiding imposing a boundary condition for $y = 0$

- Implemented a Lax-Wendroff scheme for both H and Λ
  - Central differences
  - Integral evaluated using a trapezoid method
- Suppose L is inverse Gaussian subordinator, $IG(11.98, 0.0872)$
  - About 8.5% in average yearly volatility
  - Parameters collected from Nicolato and Venardos (2002)
- Consider the parameters $\mu = 0.05, \beta = 0.5$

Numerical solution of H

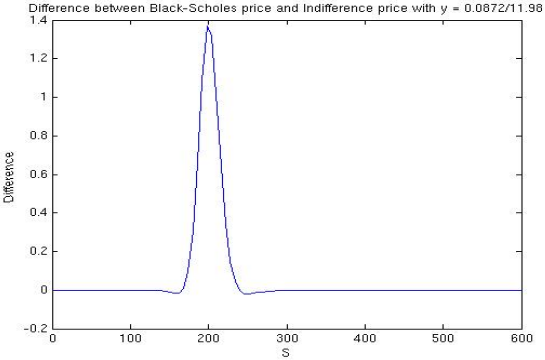


Numerical solution of Λ



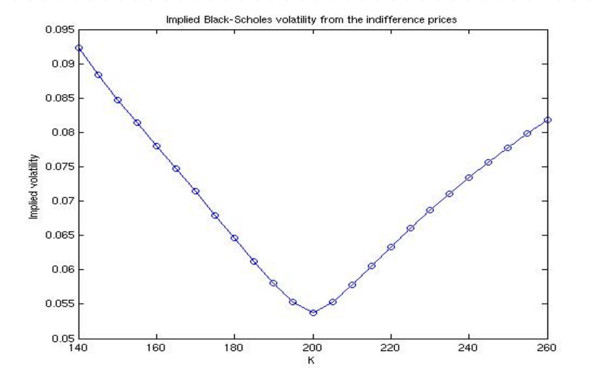
Difference with Black & Scholes price

- Black & Scholes with expected $Y(t)$ as squared volatility
  - About 8.5% in volatility



Volatility smile

- Implied volatility from Black & Scholes formula based on the MEMM prices



Further work

- Convergence analysis of the numerical scheme
- Pricing of Asian options
  - Influence of autocorrelation structure in BNS-model on price