Pattern forming systems

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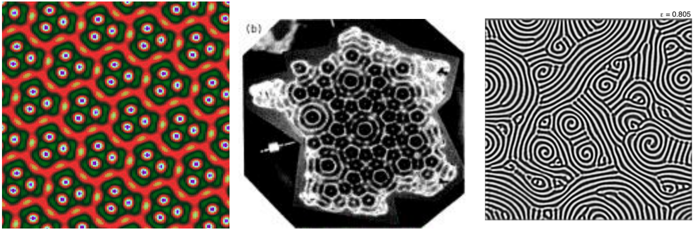
1st August 2005, Isaac Newton Institute

What are natural patterns?

- Spatial/spatiotemporal structures with symmetry
- Typically 2d
- Often periodic in space
- Tesselate the plane: stripes, squares, hexagons...
- Or not: spirals, targets, quasipatterns...
- Arise spontaneously in natural or experimental settings

A few different patterns



(a) (b) (c)

Figure: a) Super triangles ©Mary Silber, Northwestern University, 2003 b) Twelvefold quasipattern From W.S. Edwards and S. Fauve, Physical Review E, **47**, R788, 1993. Copyright (1993) by the American Physical Society. c) Spiral defect chaos in a Rayleigh–Bénard convection experiment ©Nonlinear Phenomena Group, LASSP, Cornell University, August 2004

Different systems, same features

(a) (b) (c)

Figure: Stripe patterns showing dislocations, where two stripes merge into one: a) segregation in a layer of horizontally shaken sugar and hundreds and thousands (otherwise known as sprinkles or cake decorations) From Mullin, T., *Science* **295**, 1851 (2002). Copyright (2002) AAAS.; b) sand ripples in the Sahara desert; c) on zebras ©Ed Webb, 2004

Symmetry-based approach

- Use symmetries and observable features of the pattern and its environment to determine behaviour
- Advantage: one analysis applies to multiple systems
- Disadvantage: generic approach does not fix system parameters - additional calculations needed for each system
- We will look at a couple of archetypal pattern-forming systems before going on to consider symmetry

Convection

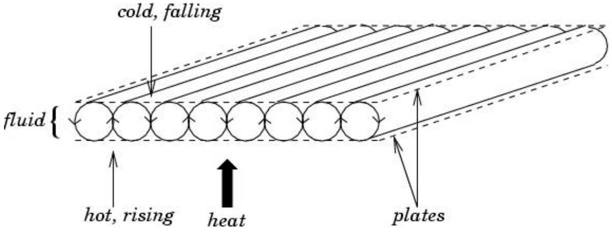


Figure: Arrows show rise and fall of fluid. Looks like stripes from the top.

- Heat at bottom of container causes fluid to expand, become less dense, more buoyant and rise through colder fluid above
- Cools as it rises, becoming denser than fluid below
- Falls back down under gravity
- Rising and falling fluid forms patterns: typically stripes (convection rolls) or hexagons
- Shadowgraph technique used for visualisation: shine a light down onto cell with reflective bottom plate and transparent top plate
- Light focused towards cold regions, which appear bright (warm regions dark)

Convection in nature

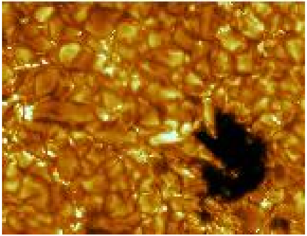


Figure: Convection cells in the photosphere of the Sun (solar granulation). ©Dr. Tom Berger, Lockheed Martin Solar and Astrophysics Lab, Palo Alto, California, 2003

- Convection often studied through carefully designed lab experiments, but...
- ... also very important in nature
- In the Earth's mantle, it leads to movement of tectonic plates
- In the oceans, it drives circulations such as the Gulf Stream
- In the atmosphere, it creates thunderclouds
- In stars, transports energy from the core to the surface (c.f. solar granulation)

How to cook hexagons in your own kitchen

Warning: Involves hot oil! Should only be attempted by competent adult cooks. Do not let any water get into the oil. If the oil starts to smoke remove pan from heat immediately.

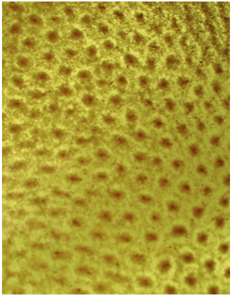


Figure: Irregular hexagonal patterns in heated cooking oil with turmeric ©Nick Safford, 2004

- Put 0.5-1.0mm cooking oil into a flat-bottomed pan
- Mix some finely ground black pepper or coloured spice into the oil - enough to finely coat the bottom of the pan
- Put the pan on a flat even heat source - e.g. Aga plate
- Heat very gently for a few seconds - **do not let the oil get very hot**
- Look sideways at the surface of the oil: you will see hexagon-shaped dimples, and little heaps of spice on the bottom of the pan in a roughly hexagonal pattern
- If it goes wrong, remove pan from heat, cool it down and start again

Boussinesq Rayleigh–Bénard convection

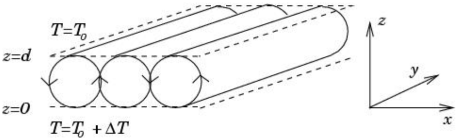


Figure: Diagram of a convection system. Fluid fills the gap between two horizontal plates at $z = 0$ and d. The top plate is maintained at a temperature $T = T_0$, while the temperature at the bottom is heated to a temperature $T = T_0 + \Delta T$, where $\Delta T > 0$.

- Takes place in a filled closed cell (no free surface)
- Fluid density, ρ, varies linearly with temperature $\rho = \rho_0[1 - \alpha(T - T_0)]$, where ρ_0 is density at $T = T_0$ and α is the (constant) coefficient of thermal expansion
- Density variation is only significant in the buoyancy force (Oberbeck–Boussinesq approximation)
- Substitute into Navier–Stokes equation for fluid flow, heat equation and continuity equation

Governing equations for Boussinesq Rayleigh–Bénard convection

$$\begin{aligned}
\rho_0\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) &= -\nabla p - \rho g \widehat{\mathbf{z}} + \rho_0 \nu \nabla^2 \mathbf{u}, \\
\frac{\partial T}{\partial t} + (\mathbf{u}\cdot\nabla)T &= \kappa \nabla^2 T, \\
\nabla\cdot\mathbf{u} &= 0,
\end{aligned}$$

$\mathbf{u}(x, y, z, t) \in \mathbb{R}^3$ is the fluid velocity

$T(x, y, z, t)$ is the fluid temperature

$p(x, y, z, t)$ is the fluid pressure

g is the (constant) acceleration due to gravity

$\widehat{\mathbf{z}}$ is a unit vector upwards

ν is the kinematic viscosity (a measure of the fluid's internal resistance to flow)

κ is the thermal diffusivity (measures rate of heat conduction through the fluid)

Conduction solution

For weak heating, the fluid does not convect, but simply conducts heat across the layer. The **conduction solution** is given by

$$
\begin{aligned}
\mathbf{u} &= \mathbf{0}, \\
T = T_c(z) &\equiv T_0 + \Delta T\left(1 - \frac{z}{d}\right), \\
p = p_c(z) &\equiv p_0 - \int_0^z \rho(T_c(z))g\ dz, \\
&= p_0 - g\rho_0 z\left[1 - \alpha\Delta T\left(1 - \frac{z}{2d}\right)\right],
\end{aligned}
$$

p_0 is the pressure at the bottom of the layer, $z = 0$
$p_c(z)$ is the hydrostatic pressure of fluid in the conducting layer

Convection near onset

When heating is just strong enough, fluid starts to convect weakly: write $p = p_c(z) + \widehat{p}$ and $T = T_c(z) + \theta$, and nondimensionalise using

$$(x, y, z) = d(\widetilde{x}, \widetilde{y}, \widetilde{z}),\ \ t = \frac{d^2}{\kappa}\widetilde{t},\ \ \mathbf{u} = \frac{\kappa}{d}\widetilde{\mathbf{u}},\ \ \theta = \frac{\nu\kappa}{g\alpha d^3}\widetilde{\theta},\ \ \widehat{p} = \frac{\rho_0\nu\kappa}{d^2}\widetilde{p}$$

giving (dropping tildes immediately)

$$\frac{1}{\sigma}\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \theta\widehat{\mathbf{z}} + \nabla^2\mathbf{u}, \qquad (1)$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{u}\cdot\nabla)\theta - Ru_z = \nabla^2\theta, \qquad (2)$$

u_z is the z-component of $\mathbf{u}$

$\sigma = \nu/\kappa$ is the Prandtl number (measures relative effects of viscous and thermal diffusion)

$R = \alpha g d^3 \Delta T/\kappa\nu$ is the Rayleigh number - nondimensionalised ΔT.

Vorticity equation

Take curl of equation (1) to get the vorticity equation

$$\frac{1}{\sigma}\left(\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\nabla\mathbf{u}\right) = \nabla\theta\times\widehat{\mathbf{z}} + \nabla^2\boldsymbol{\omega}, \tag{3}$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ is the fluid vorticity.
For stability of the conduction solution to convection linearise equations (2) and (3) around $\mathbf{u} = \boldsymbol{\omega} = \mathbf{0}$, $\theta = 0$ giving

$$\frac{1}{\sigma}\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla\theta\times\widehat{\mathbf{z}} + \nabla^2\boldsymbol{\omega}, \tag{4}$$

$$\frac{\partial \theta}{\partial t} - Ru_z = \nabla^2\theta, \tag{5}$$

$\widehat{\mathbf{z}}\cdot\nabla\times$ equation (4) gives

$$\frac{1}{\sigma}\frac{\partial}{\partial t}\nabla^2 u_z = \nabla_h^2\theta + \nabla^4 u_z, \tag{6}$$

where $\nabla_h = (\partial/\partial x, \partial/\partial y, 0)$ is the horizontal gradient operator.

Boundary conditions and eigenmodes

Solve equations (5) and (6) subject to:

$$\theta = u_z = \frac{\partial^2 u_z}{\partial z^2} = 0, \text{ at } z = 0, 1$$

(Stress-free velocity boundary conditions at top and bottom and temperature perturbation vanishes there.)
Assume convection cell horizontally infinite. Solution can be written as a superposition of Fourier eigenmodes

$$u_z^{(n)}(x, y, z, t) = u_n \sin n\pi z \; e^{i\mathbf{k}_h \cdot \mathbf{x}_h + st} + c.c., \quad (7)$$

$$\theta^{(n)}(x, y, z, t) = \theta_n \sin n\pi z \; e^{i\mathbf{k}_h \cdot \mathbf{x}_h + st} + c.c., \quad (8)$$

$c.c.$ stands for complex conjugate

$\mathbf{k}_h$ is a horizontal wavevector

$\mathbf{x}_h = (x, y, 0)$ is a horizontal position vector

s is the growth rate

u_n and θ_n are constants

Dispersion relation

Substituting $u_z^{(n)}, \theta^{(n)}$ into equations (5) and (6) gives

$$\begin{aligned} s\theta_n - Ru_n &= -(k^2 + n^2\pi^2)\theta_n, \\ -\frac{1}{\sigma}s(k^2 + n^2\pi^2)u_n &= -k^2\theta_n + (k^2 + n^2\pi^2)^2 u_n, \end{aligned}$$

where $k = |\mathbf{k}_h|$. Eliminating θ_n and u_n, gives a **dispersion relation**

$$s^2(k^2 + n^2\pi^2) + s(1 + \sigma)(k^2 + n^2\pi^2)^2 + \sigma(k^2 + n^2\pi^2)^3 - \sigma Rk^2 = 0.$$

Threshold conditions

- Growth rate, s, is zero at $R_n(k) = (k^2 + n^2\pi^2)^3/k^2$. (There is a stationary bifurcation at $R = R_n(k)$ - see next lecture.) The n^{th} eigenmode starts to grow for $R > R_n(k)$.
- Happens first for $n = 1$. $R_1(k)$ is minimum at $k = k_c \equiv \pi/\sqrt{2}$ - gives convection instability threshold

$$R_c = R_1(k_c) = \frac{27}{4}\pi^4.$$

Convection sets in for $R > R_c$ (big enough ΔT).

Just above threshold

- For $n \geq 2$, $R_n(k) - R_1(k) \geq R_g \equiv 3\pi^4(15 + 2\sqrt{63})$
- Close to threshold ($\delta k = k - k_c$ and $r = (R - R_c)/R_c$ small and $R < R_c + R_g$) only $n = 1$ mode can grow
- Growth rate is
$$s = \frac{\sigma}{\sigma + 1}\left(\frac{3}{2}\pi^2 r - \delta k^2\right)$$
- Only wavenumbers $\delta k^2 < 3\pi^2 r/2$ grow and critical wavenumber, $k = k_c$ ($\delta k = 0$), grows fastest

Selected patterns

- Close to onset patterns consist of superposed $n = 1$ modes with $k \approx k_c$. Any combination is permitted by the linear analysis, as it doesn't fix the directions of the wavevectors. Typically, nonlinear effects pick out a small number of modes.
- Convection rolls (stripes) have a single pair of wavevectors $\pm\mathbf{k}_h$:

$$\begin{aligned} u_z &= u_1 \sin \pi z e^{i\pi x/\sqrt{2}} + c.c., \\ \theta &= \theta_1 \sin \pi z e^{i\pi x/\sqrt{2}} + c.c., \end{aligned}$$

at onset. Contours of u_z and θ, in the (x, y) plane look like stripes.

Some possible convection planforms

(a) (b) (c)

Figure: a) Stripes or rolls, b) squares and c) hexagons. Constructed from filled contour plots of $u = \sum_j (e^{i\mathbf{k}_j\cdot\mathbf{x}} + e^{-i\mathbf{k}_j\cdot\mathbf{x}})$ for a) $\mathbf{k}_1 = (1,0)$, b) $\mathbf{k}_1 = (1,0)$, $\mathbf{k}_2 = (0,1)$, and c) $\mathbf{k}_1 = (1,0)$, $\mathbf{k}_2 = (-1/2, \sqrt{3}/2)$, $\mathbf{k}_3 = (-1/2, -\sqrt{3}/2)$.

Reaction-diffusion: Belousov–Zhabotinsky reaction

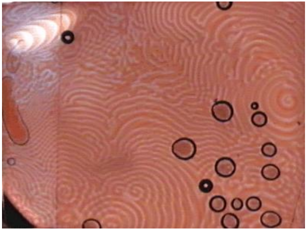


Figure: Spirals in the Belousov–Zhabotinsky reaction. ©Annette Taylor, University of Leeds, August 2004

- Patterns can form in systems of reacting and diffusing chemicals
- Canonical example is the Belousov-Zhabotinsky reaction (Belousov, 1958; Zaikin & Zhabotinsky, 1970)
- Malonic acid oxidised by bromate ions in presence of ferroin catalyst
- Reduced state of catalyst is red and oxidised state is blue
- Oscillating spiral and target patterns seen with alternating red and blue arms or rings

Diffusion

- Diffusion is the mechanism by which particles in a fluid are transported from an area of higher concentration to an area of lower concentration through jostling and bumping of the liquid or gas molecules around them
- Tends to smear out high concentrations of a substance and make the distribution more uniform
- Pattern formation requires local highs and lows of concentration to form coherent spatial structures - how can diffusion do this?

Reaction-diffusion: excitable systems and Turing patterns

- Two main types of pattern-forming system with different mechanisms: excitable and Turing
- Excitable dynamics come from the reaction terms: diffusion just couples together the behaviour of neighbouring areas
- Large diffusion coefficients lead to homogeneous oscillations of the whole system, but smaller ones allow some lag between neighbouring areas and so spatial structures - **excitation waves** - are seen
- Turing patterns (steady or oscillatory) result from differences in rates of diffusion of different chemical species - can be analysed in terms of Fourier modes as for convection (theory - Turing, 1952; practical demonstration - Castets et al, 1990)
- In this lecture we concentrate on excitability

Model reaction-diffusion system

- Two chemicals with concentrations $u(\mathbf{x}, t)$ and $v(\mathbf{x}, t)$
- Governing equations

$$\begin{aligned} \frac{\partial u}{\partial t} &= f(u, v) + D_u \nabla^2 u, \\ \frac{\partial v}{\partial t} &= g(u, v) + D_v \nabla^2 v. \end{aligned}$$

- Position vector $\mathbf{x}$ two- or three-dimensional
- $f(u, v)$ and $g(u, v)$ describe chemical reactions
- Diffusion modelled by $D_u \nabla^2 u$ and $D_v \nabla^2 v$
- System is assumed isotropic and homogeneous with D_u and D_v constant
- Can describe excitable behaviour or Turing patterns depending on form of reaction terms $f(u, v)$ and $g(u, v)$, and values of the diffusion coefficients

Excitable systems: Fitzhugh-Nagumo model

- Originally developed as a model of nerve impulse propagation (FitzHugh, 1961; Nagumo, Arimoto, Yoshizawa, 1962)
- We shall look at Barkley's variant (e.g. Barkley 1995)

$$\frac{\partial u}{\partial t} = \nabla^2 u + f(u,v), \quad (9)$$

$$\frac{\partial v}{\partial t} = g(u,v), \quad (10)$$

$$f(u,v) = \frac{1}{\epsilon} u(1-u)\left(u - \frac{v+b}{a}\right), \quad (11)$$

$$g(u,v) = u - v. \quad (12)$$

- Since ϵ is small, the reaction dynamics of the excitation variable, u, are much faster than those of the recovery variable, v.

Excitable dynamics 1

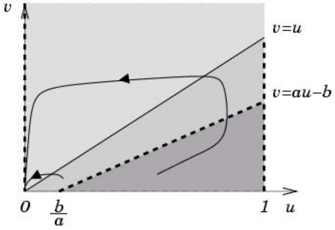


Figure: Nullclines (u dashed and v solid). Light grey: $\mathrm{d}u/\mathrm{d}t < 0$, $\mathrm{d}v/\mathrm{d}t < 0$. Medium grey: $\mathrm{d}u/\mathrm{d}t < 0$, $\mathrm{d}v/\mathrm{d}t > 0$. Dark grey $\mathrm{d}u/\mathrm{d}t > 0$, $\mathrm{d}v/\mathrm{d}t > 0$.

- Excitable dynamics come from the reaction terms, so ignore $\nabla^2 u$ for now
- $u = v = 0$ is a fixed point with $\mathrm{d}u/\mathrm{d}t = \mathrm{d}v/\mathrm{d}t = 0$. If $0 < a < 1$ and $b > 0$ it is the only stable fixed point.
- Consider the **nullclines** $f(u, v) = 0$ or $g(u, v) = 0$ for $0 \leq u \leq 1$.
- A trajectory starting to the left of $v = au - b$ decays rapidly to the origin, since $\mathrm{d}u/\mathrm{d}t$ large and negative (v may initially increase, but decays after crossing $v = u$)
- A trajectory starting to the right of $v = au - b$ grows rapidly away from origin, but crosses back to the region $\mathrm{d}u/\mathrm{d}t < 0$ as v grows, so eventually returns.

Excitable dynamics 2

- Threshold effect defines excitable system: small perturbations near to excitable stable fixed point decay quickly to zero, but disturbances greater than threshold lead to large excursions.
- System is said to be **quiescent** close to fixed point, **excited** close to the righthand nullcline and **recovering** close to lefthand nullcline, but far from fixed point.
- Recovering states are much further from threshold than quiescent ones, so an excited state must pass through recovery to quiescence before it can be excited again
- Good model for many biological processes requiring slow build-up and rapid discharge of some quantity, such as action potential in neurons.

Effect of diffusion

- Diffusion couples dynamics of neighbouring points in space: leads to propagation of excitation waves
- If excited region ($u \approx 1$) is next to quiescent region (u, v small), diffusive coupling increases u in quiescent region: tips system across threshold there and into excitation
- Newly excited region excites neighbouring quiescent areas in turn and a wave of excitation spreads outwards
- Excited area goes into recovery and then quiescence, ready to be excited again, so excitation waves can be periodic: spirals and targets (concentric rings)

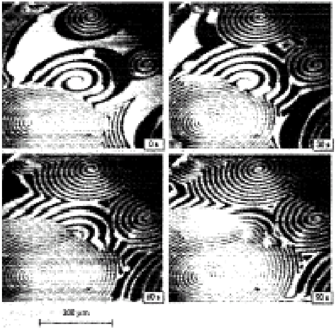


Figure: A high frequency and wavenumber spiral consumes another at lower frequency and wavenumber during the oxidation of carbon monoxide on the surface of a platinum catalyst. From Nettesheim et al (1993)

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