

Some ideas from dynamical systems

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Patterns and bifurcations

- ▶ Patterns typically arise at bifurcations
- ▶ Some external forcing in the system changes and a pattern appears where there was none before
- ▶ Forcing would be heating in convection, perhaps some change in chemical properties in a reaction-diffusion system
- ▶ We need to understand bifurcations, and some related ideas such as fixed points and centre manifolds

Flows

- ▶ Start with the ordinary differential equation

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n$$

where t is time and \mathbf{x} is a position vector in \mathbb{R}^n , the **phase space**. Assume f sufficiently smooth to use Taylor series expansions and so on.

- ▶ Solutions define a **flow**, $\phi(\mathbf{x}, t)$, such that $\mathbf{x}(t) = \phi(\mathbf{x}_0, t)$, where \mathbf{x}_0 is the initial value at $t = 0$, $\phi(\mathbf{x}_0, 0) = \mathbf{x}_0$
- ▶ Flow is defined for all t , positive and negative, such that the solution through \mathbf{x}_0 exists

Phase portraits

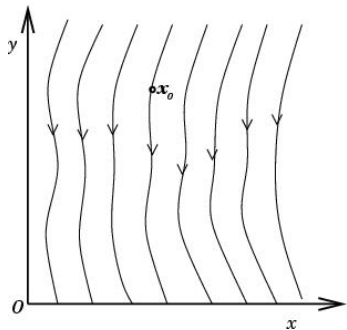


Figure: An example of a flow in the (x, y) plane. The starting point (at $t = 0$) of one of the trajectories in the flow is labelled \mathbf{x}_0 . Each trajectory has a different starting point, and so a different value of \mathbf{x}_0 .

- ▶ For a 2d phase space with $\mathbf{x} = (x, y) \in \mathbb{R}^2$ we can make a two-dimensional picture, or **phase portrait**, of a flow
- ▶ Each line on the diagram is a **trajectory**, a solution labelled by a particular starting point \mathbf{x}_0 and parameterised by time
- ▶ Arrows represent the direction of increasing time
- ▶ The $t < 0$ sections of each trajectory are drawn by running time backwards

Fixed points

- ▶ A **stationary point** (or **fixed point** or **equilibrium solution**) is a point \mathbf{x} such that

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}) = \mathbf{0}.$$

- ▶ The one-dimensional differential equation

$$\frac{dx}{dt} = x - x^3,$$

has fixed points at $x = 0, \pm 1$

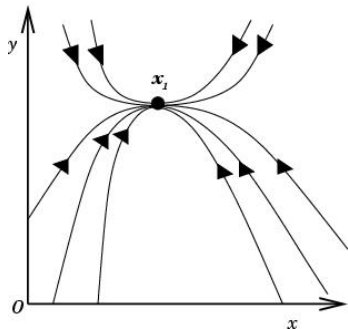


Figure: An example of a flow with a fixed point at $\mathbf{x} = \mathbf{x}_1$.

Periodic orbits

- ▶ A point, \mathbf{x} , is **periodic** with period T if and only if $\phi(\mathbf{x}, t + T) = \phi(\mathbf{x}, t)$ for all t and $\phi(\mathbf{x}, t + s) \neq \phi(\mathbf{x}, t)$ for all $0 < s < T$
- ▶ The trajectory starting at \mathbf{x} at time t first returns after an additional time T
- ▶ The closed curve $\mathcal{C} = \{\mathbf{y} | \mathbf{y} = \phi(\mathbf{x}, t), 0 \leq t < T\}$ is a **periodic orbit**, and consists of the trajectory joining the periodic point \mathbf{x} back to itself in phase space.
- ▶ The equations

$$\begin{aligned}\frac{dx}{dt} &= x(1 - x^2 - y^2) - y, \\ \frac{dy}{dt} &= y(1 - x^2 - y^2) + x,\end{aligned}$$

have a periodic orbit of the form $x = \cos t$, $y = \sin t$. Every point $\mathbf{x} = (x, y)$ on the orbit is periodic with period $T = 2\pi$.

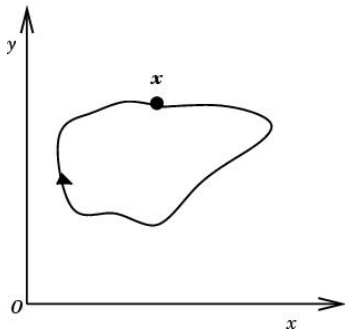


Figure: This flow has a periodic orbit passing through the periodic point \mathbf{x} . Only the trajectory making up the periodic orbit is shown.

Stability of stationary points

- ▶ Linear stability determined by the eigenvalues of the Jacobian matrix,

$$Df_{ij} = \frac{\partial f_i}{\partial x_j},$$

evaluated at the stationary point

- ▶ Perturb stationary point \mathbf{x}_0 so that $\mathbf{x} = \mathbf{x}_0 + \boldsymbol{\eta}(t)$, with $0 < |\boldsymbol{\eta}| \ll 1$:

$$\begin{aligned}\frac{d\boldsymbol{\eta}}{dt} &= f(\mathbf{x}_0 + \boldsymbol{\eta}) \\ &= f(\mathbf{x}_0) + Df|_{\mathbf{x}=\mathbf{x}_0} \boldsymbol{\eta} + O(|\boldsymbol{\eta}|^2) \\ &= Df|_{\mathbf{x}=\mathbf{x}_0} \boldsymbol{\eta} + O(|\boldsymbol{\eta}|^2)\end{aligned}$$

- ▶ If $\boldsymbol{\eta}$ is an eigenvector of $Df|_{\mathbf{x}=\mathbf{x}_0}$ with eigenvalue λ , then $d\boldsymbol{\eta}/dt = \lambda\boldsymbol{\eta}$ and $\boldsymbol{\eta} = \boldsymbol{\eta}_0 e^{\lambda t}$ to leading order
- ▶ If $\text{Re}(\lambda) < 0$ then $|\boldsymbol{\eta}| \rightarrow 0$ as $t \rightarrow +\infty$ so the perturbation dies away: stationary point \mathbf{x}_0 is linearly stable in this direction. If $\text{Re}(\lambda) > 0$ then $|\boldsymbol{\eta}| \rightarrow \infty$ as $t \rightarrow +\infty$ so the perturbation grows and \mathbf{x}_0 is linearly unstable in this direction. If $\text{Re}(\lambda) = 0$ then \mathbf{x}_0 is linearly neutrally stable in the corresponding direction.

Hyperbolic stationary points

- ▶ A stationary point, \mathbf{x}_0 , is said to be **hyperbolic** if and only if the Jacobian, $Df|_{\mathbf{x}=\mathbf{x}_0}$, has no zero or purely imaginary eigenvalues
- ▶ \mathbf{x}_0 is a **sink** if all the eigenvalues have strictly negative real part, a **source** if all the eigenvalues have strictly positive real part and a **saddle** otherwise
- ▶ Hyperbolic stationary persist under small perturbations of the governing differential equation: if the governing equation is changed slightly to

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}) + \epsilon p(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n$$

where $p(\mathbf{x})$ is a smooth vector field in \mathbb{R}^n , and $0 < \epsilon \ll 1$ is sufficiently small, then for each hyperbolic stationary point of the original equation there is a hyperbolic stationary point of the perturbed equation that lies very close to the unperturbed fixed point in phase space and is of the same stability type (sink, source or saddle). (See Glendinning 1994 or Hoyle 2005 for proof.)

Local bifurcations

- ▶ At a bifurcation there is a sudden qualitative change in the flow in response to changes in one or more parameters in the governing equation
- ▶ There will be obvious differences in the phase portrait, and typically the number and stability properties of fixed points or periodic orbits will change
- ▶ The parameters that lead to these changes are called **bifurcation parameters** and the point in parameter space at which the changes occur is called the **bifurcation point**
- ▶ If we let μ be a vector of bifurcation parameters, then $\mu = \mu_c$ say, is the bifurcation point
- ▶ Local bifurcation theory is concerned with changes in the flow in the neighbourhood of a fixed point or periodic orbit
- ▶ Global bifurcations, which affect the large-scale properties of the flow, also exist (see Glendinning, 1994)
- ▶ Stick to local bifurcations here

Local bifurcations from fixed points

- ▶ These bifurcations can only happen at parameter values for which the stationary point is nonhyperbolic
- ▶ Changing parameter values a little bit is equivalent to making a small perturbation to the governing ordinary differential equation, and hyperbolic stationary points persist under those circumstances: the perturbed equation has a hyperbolic stationary point of the same stability type
- ▶ Changes in the number or stability-type of fixed points can only happen when stationary points are nonhyperbolic
- ▶ Typically the eigenvalues of the Jacobian, $Df|_{x=x_0}$, at the stationary point x_0 will depend upon the vector of bifurcation parameters, μ : to identify the bifurcation point you find the value or values of μ for which one or more of the eigenvalues is zero or purely imaginary

Example: saddle-node bifurcation

- ▶ Consider the equations

$$\begin{aligned}\frac{dx}{dt} &= \mu - x^2, \\ \frac{dy}{dt} &= -y,\end{aligned}$$

where $\mu, x, y \in \mathbb{R}$.

- ▶ For $\mu < 0$ there are no fixed points, but for $\mu > 0$ there are stationary points at $(\pm\sqrt{\mu}, 0)$, where Df is $\begin{pmatrix} \mp 2\sqrt{\mu} & 0 \\ 0 & -1 \end{pmatrix}$ respectively
- ▶ Eigenvalues are -1 and $\mp 2\sqrt{\mu}$ at $(\pm\sqrt{\mu}, 0)$. Fixed point $(\sqrt{\mu}, 0)$ is a stable node and $(-\sqrt{\mu}, 0)$ is a saddle. Both hyperbolic, except at $\mu = 0$.
- ▶ As μ passes through zero from negative to positive, there is a change in the flow, from zero to two fixed points, one stable and one unstable
- ▶ At $\mu = 0$ the points $(\pm\sqrt{\mu}, 0)$ coalesce into one nonhyperbolic fixed point at the origin
- ▶ This is a saddle-node bifurcation: the phase portraits show the birth of a saddle and a node at $\mu = 0$

Phase portraits for saddle-node bifurcation

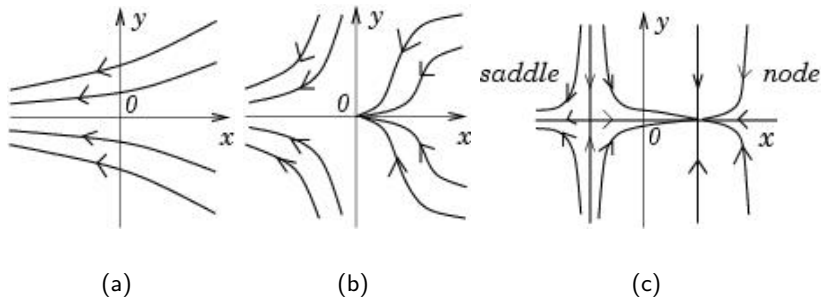


Figure: Phase portraits for the cases a) $\mu < 0$, b) $\mu = 0$ and c) $\mu > 0$

The centre manifold 1

- ▶ Typically we want to investigate the case when a stable stationary point first loses stability, so Df has eigenvalues with negative real part (stable directions) and one or more eigenvalues with zero real part (directions in which the stationary point is just losing stability)
- ▶ Separate out the linear and nonlinear parts of the equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + g(\mathbf{x}),$$

where $|g(\mathbf{x})| = O(|\mathbf{x}|^2)$ as $|\mathbf{x}| \rightarrow 0$. The nonhyperbolic stationary point is at $\mathbf{x} = \mathbf{0}$. Matrix A is the Jacobian, $Df|_{(0,\mu)}$, with eigenvalues depending on μ . The nonlinear function $g(\mathbf{x})$ may also depend on μ .

- ▶ Want to get rid of the uninteresting directions, so separate out the linearly stable and linearly neutral directions:

$$\begin{aligned}\frac{d\mathbf{y}}{dt} &= B\mathbf{y} + g_y(\mathbf{y}, \mathbf{z}), \\ \frac{d\mathbf{z}}{dt} &= C\mathbf{z} + g_z(\mathbf{y}, \mathbf{z}).\end{aligned}$$

All eigenvalues of B have zero real part; all eigenvalues of C have negative real part; $g_y(\mathbf{y}, \mathbf{z})$ and $g_z(\mathbf{y}, \mathbf{z})$ are the \mathbf{y} and \mathbf{z} components of $g(\mathbf{x})$.

How to find the centre manifold

- ▶ Locally, CM can be written $\mathbf{z} = h(\mathbf{y})$, where $h(\mathbf{0}) = \mathbf{0}$ and $Dh|_{\mathbf{y}=\mathbf{0}} = 0$
- ▶ It is invariant, so must satisfy $d\mathbf{z}/dt = Dh d\mathbf{y}/dt$, which gives

$$Ch(\mathbf{y}) + g_z(\mathbf{y}, h(\mathbf{y})) = Dh[B\mathbf{y} + g_y(\mathbf{y}, h(\mathbf{y}))].$$

- ▶ To find CM, solve for $h(\mathbf{y})$. Centre manifold theorem guarantees a solution near $\mathbf{y} = \mathbf{0}$.
- ▶ Assume a polynomial approximation to $h(\mathbf{y})$ near $\mathbf{y} = \mathbf{0}$ and find it by expanding $h(\mathbf{y})$ in powers of \mathbf{y} and equating the coefficients of powers of \mathbf{y} on either side of the equation
- ▶ After transients have died away, evolution on the CM is governed by

$$\frac{d\mathbf{y}}{dt} = B\mathbf{y} + g_y(\mathbf{y}, h(\mathbf{y}))$$

- ▶ Procedure reduces original system of n equations to one of much lower order, say m equations if $\mathbf{y} \in \mathbb{R}^m$. A good thing!
- ▶ Centre manifold reduction can be carried out even as $n \rightarrow \infty$, provided that the eigenvalues, λ_i , of C satisfy $Re(\lambda_i) \leq -\delta < 0$, where $\delta > 0$ does not depend on n

The extended centre manifold

- ▶ Near a bifurcation point the real part of one or more eigenvalues passes from negative to positive. How can we separate decaying and growing directions?
- ▶ Choose μ such that bifurcation point is at $\mu = \mathbf{0}$. Declare μ to be a variable vector like \mathbf{y} and \mathbf{z} , and add in the equation $d\mu/dt = \mathbf{0}$.

$$\frac{d\mu}{dt} = \mathbf{0}$$

$$\frac{d\mathbf{y}}{dt} = B'\mathbf{y} + g'_y(\mathbf{y}, \mathbf{z}, \mu) \equiv B\mathbf{y} + g_y(\mathbf{y}, \mathbf{z}),$$

$$\frac{d\mathbf{z}}{dt} = C'\mathbf{z} + g'_z(\mathbf{y}, \mathbf{z}, \mu) \equiv C\mathbf{y} + g_z(\mathbf{y}, \mathbf{z}).$$

- ▶ Now $\mu = \mathbf{y} = \mathbf{z} = \mathbf{0}$ is nonhyperbolic with all eigenvalues zero or negative
- ▶ Note that μ -dependent entries in B and C give rise to nonlinear terms in g'_y and g'_z , while B' and C' must be independent of μ
- ▶ Have extended system into parameter space to follow flow near stationary point for parameter values close to bifurcation point on either side
- ▶ Find **extended centre manifold**, $\mathbf{z} = h(\mathbf{y}, \mu)$, by expanding in powers of \mathbf{y} and μ , and then have $d\mathbf{y}/dt = B\mathbf{y} + g_y(\mathbf{y}, h(\mathbf{y}, \mu))$, where μ is constant

Example: extended centre manifold

- ▶ Consider the system

$$\begin{aligned}\frac{dx}{dt} &= \mu x - xy, \\ \frac{dy}{dt} &= -y + x^2,\end{aligned}$$

where $\mu, x, y \in \mathbb{R}$

- ▶ Stationary point $x = y = 0$ exists $\forall \mu$. Jacobian has eigenvalues μ and -1 there. At $\mu = 0$ the stationary point is nonhyperbolic and there is a bifurcation.
- ▶ Add in the equation $d\mu/dt = 0$ and look for extended CM
- ▶ μ has zero growth rate. Now that μ is a variable, x also has a zero eigenvalue. The linear eigenvalue for y is -1 .
- ▶ Look for extended CM in the form $y = h(x, \mu)$. Invariance of the CM gives

$$\frac{dy}{dt} = \frac{\partial h}{\partial x} \frac{dx}{dt} + \frac{\partial h}{\partial \mu} \frac{d\mu}{dt}$$

Example continued

- ▶ Substituting from governing equations gives

$$-h(x, \mu) + x^2 = \frac{\partial h}{\partial x}(\mu x - xh(x, \mu)).$$

- ▶ Expand $h(x, \mu)$ as quadratic (at least) polynomial in x and μ

$$h = ax^2 + bx\mu + c\mu^2 + \dots$$

where a , b and c are constants and substitute to get

$$x^2 - ax^2 - bx\mu - c\mu^2 - \dots = (2ax + b\mu)x(\mu - ax^2 - bx\mu - c\mu^2 - \dots).$$

- ▶ Equating coefficients of x^2 gives $a = 1$, of $x\mu$ gives $b = 0$ and of μ^2 gives $c = 0$
- ▶ To leading order the extended CM is $y = x^2$, and the flow on it evolves according to

$$\frac{dx}{dt} = \mu x - x^3,$$

where μ is a constant

Codimension-one stationary bifurcations

- ▶ Codimension-one bifurcations are the generic bifurcations that occur in a system where a single bifurcation parameter is varied
- ▶ Will first consider bifurcations that correspond to a real eigenvalue passing through zero: **stationary** or **steady** bifurcations
- ▶ Will consider the case where the bifurcation occurs at $\mu = 0$ when there is a nonhyperbolic fixed point at $x = 0$. We are working in one dimension.

Genericity of bifurcations

- ▶ Saddle-node is the generic codimension-one stationary bifurcation: the bifurcation you would expect to find when no special conditions have been imposed on the system
- ▶ The bifurcation problem $\mathrm{d}\mathbf{x}/\mathrm{d}t = f(\mathbf{x}, \mu)$ is **generic** for its class, characterised by constraints on $f(\mathbf{x}, \mu)$, if for sufficiently small $\epsilon > 0$ the perturbed problem $\mathrm{d}\mathbf{x}/\mathrm{d}t = f(\mathbf{x}, \mu) + \epsilon v(\mathbf{x}, \mu)$ has the same type of bifurcation at a nearby value of μ for all perturbations $v(\mathbf{x}, \mu)$ such that it remains in the same class
- ▶ For the saddle-node bifurcation the only constraint is that $f(x, \mu)$ be a smooth function of $x, \mu \in \mathbb{R}$

Genericity of saddle-node

- ▶ Consider the smooth perturbation

$$\frac{dx}{dt} = \mu - ax^2 + \epsilon v(x, \mu) + \dots$$

with $a > 0$ (wlog). Close to $\mu = 0$, $x = 0$

$$\frac{dx}{dt} = \mu(1 + \epsilon v_1 + \epsilon v_2 \mu) + \epsilon(v_3 + v_4 \mu)x - (a + \epsilon v_5)x^2 + \dots$$

up to quadratic order in x and μ , for some constants v_i .

- ▶ Stationary solutions are

$$x = \frac{1}{2(a + \epsilon v_5)} \left\{ \epsilon(v_3 + v_4 \mu) \pm \sqrt{\epsilon^2(v_3 + v_4 \mu)^2 + 4\mu(a + \epsilon v_5)(1 + \epsilon v_1 + \epsilon v_2 \mu)} \right\},$$

and exist only for

$$\mu > \mu_c \equiv -\frac{\epsilon^2 v_3^2}{4a} + O(\epsilon^3)$$

- ▶ There is a saddle-node bifurcation at $\mu = \mu_c \sim \epsilon^2$, close to $\mu = 0$. Thus saddle-node is generic for one-dimensional steady-state bifurcation problems.

Transcritical bifurcation

- ▶ Generic for codimension-one stationary bifurcations with a fixed point that persists for all μ ($f(0, \mu) = 0, \quad \forall \mu \in \mathbb{R}$)

$$\frac{dx}{dt} = \mu x + ax^2 + \dots,$$

$x, \mu \in \mathbb{R}, a$ a real constant

- ▶ Fixed points $x = 0$ and $x = -\mu/a \forall \mu$
- ▶ Growth rate eigenvalue at $x = 0$ is μ , so $x = 0$ stable for $\mu < 0$ and unstable for $\mu > 0$
- ▶ Growth rate eigenvalue at $x = -\mu/a$ is $-\mu$, so $x = -\mu/a$ unstable for $\mu < 0$ and stable for $\mu > 0$
- ▶ At the bifurcation point $\mu = 0$, the stability of the two fixed points is exchanged
- ▶ Not generic within wider class where f is only smooth: typically breaks up into two saddle-nodes or no bifurcation at all

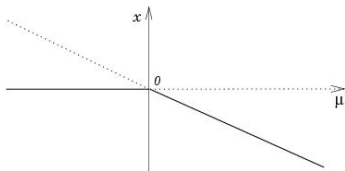


Figure: Bifurcation diagram for the transcritical bifurcation. Stable fixed points are shown as bold lines, and unstable fixed points as dotted lines.

Broken transcritical bifurcation

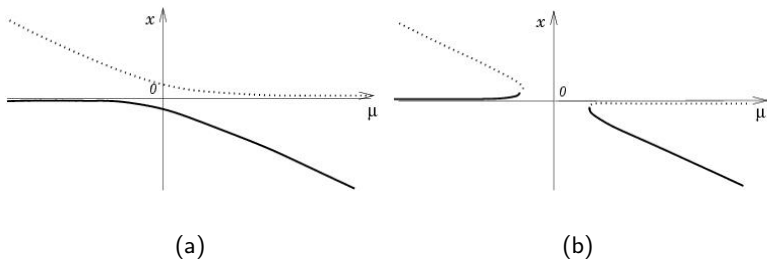


Figure: Bifurcation diagrams for a perturbed transcritical bifurcation. Stable fixed points are marked with bold lines and unstable fixed points with dotted lines.

Pitchfork bifurcation

- ▶ Generic codimension-one bifurcation where $x = 0$ is a fixed point $\forall \mu$ and the system is symmetric under $x \rightarrow -x$ ($f(-x, \mu) = -f(x, \mu)$).



$$\frac{dx}{dt} = \mu x + ax^3 + \dots,$$

$x, \mu \in \mathbb{R}$, a a real constant

- ▶ If $a > 0$ pitchfork is **subcritical** and if $a < 0$ it is supercritical
- ▶ Fixed points are $x = 0 \forall \mu$ and $x = \pm\sqrt{-\mu/a}$ for $\mu < 0$ (subcritical case) or for $\mu > 0$ (supercritical case)
- ▶ Growth rate eigenvalue at $x = 0$ is μ , so $x = 0$ is stable for $\mu < 0$ and unstable for $\mu > 0$
- ▶ Growth rate eigenvalue at $x = \pm\sqrt{-\mu/a}$ is -2μ , so these points are unstable in the subcritical case and stable in the supercritical case
- ▶ In both cases there is a bifurcation at $\mu = 0$ where the zero solution loses stability

Bifurcation diagrams for pitchfork bifurcations

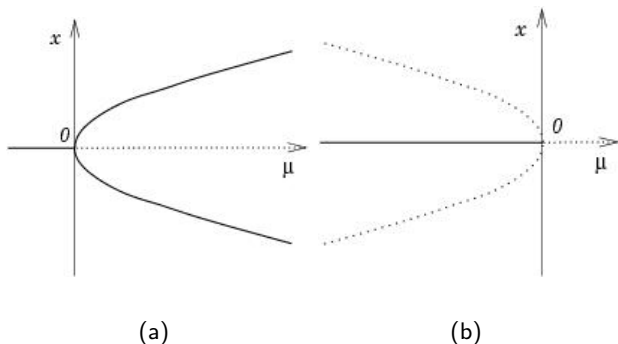


Figure: Bifurcation diagrams for a) the supercritical and b) the subcritical pitchfork bifurcations. Stable fixed points are marked with bold lines and unstable fixed points with dotted lines.

Turnaround of subcritical branch

- ▶ For a supercritical pitchfork there is a smooth transition from the zero solution to the stable bifurcating branch as μ passes from negative to positive
- ▶ For a subcritical pitchfork there are no small-amplitude stable solutions in $\mu > 0$, so what happens?
- ▶ Typically the bifurcating branch turns round at a saddle-node bifurcation, and there is a large-amplitude stable solution in $\mu > 0$
- ▶ Transition from the zero solution to the large-amplitude branch happens in a big jump
- ▶ If μ then decreases, return to zero solution is delayed until the saddle-node bifurcation: **hysteresis**

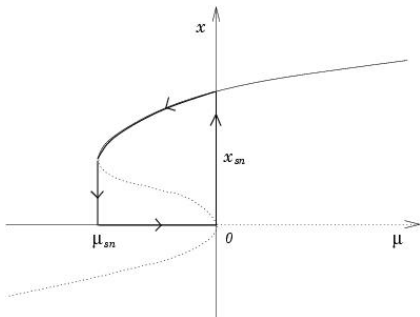


Figure: Bifurcation diagram showing the turnaround of a subcritical solution branch. There is a subcritical pitchfork bifurcation at $\mu = 0$ and a saddle-node bifurcation at $\mu = \mu_{sn}$. The hysteresis loop is followed in the direction of the arrows as μ is increased and decreased.

Broken pitchfork bifurcation

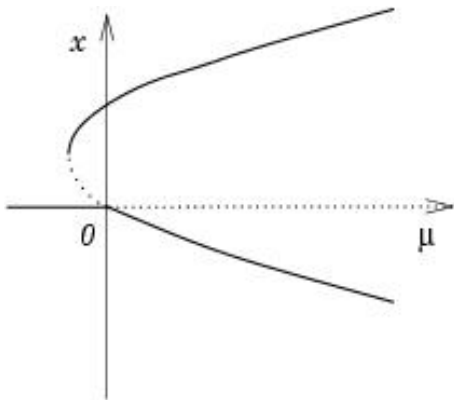


Figure: Bifurcation diagram for a pitchfork with a perturbation satisfying $v(0, \mu) = 0, \forall \mu$ (transcritical class). Stable solutions are shown by bold lines and unstable solutions by dotted lines.

Hopf bifurcation 1

- ▶ Can also have an oscillatory codimension-one bifurcation, where the real part of a complex conjugate pair of eigenvalues passes through zero, the imaginary part remaining nonzero

▶

$$\begin{aligned}\frac{dx}{dt} &= \mu x - \omega y + a(x^2 + y^2)x - b(x^2 + y^2)y, \\ \frac{dy}{dt} &= \mu y + \omega x + a(x^2 + y^2)y + b(x^2 + y^2)x,\end{aligned}$$

$x, y, \mu \in \mathbb{R}$, ω, a, b real constants

- ▶ $x = y = 0$ is a fixed point with $Df|_0 = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$
- ▶ Eigenvalues are $\mu \pm i\omega$, so Hopf bifurcation at $\mu = 0$ for $\omega \neq 0$

Hopf bifurcation 2

- ▶ In polars $x = r \cos \theta$ and $y = r \sin \theta$

$$\begin{aligned}\frac{dr}{dt} &= \mu r + ar^3, \\ \frac{d\theta}{dt} &= \omega + br^2.\end{aligned}$$

- ▶ Hopf bifurcation is a pitchfork bifurcation in r (subcritical if $a > 0$, supercritical if $a < 0$) with rotation in the θ direction
- ▶ Stationary solutions for r are $r = 0$ for all μ (fixed point at origin) and $r = \sqrt{-\mu/a}$ for $\mu/a < 0$ (periodic orbit, as long as $d\theta/dt = \omega - b\mu/a \neq 0$)
- ▶ $r = 0$ stable for $\mu < 0$ and unstable for $\mu > 0$
- ▶ Periodic orbit unstable in r if $a > 0$ (subcritical case) and stable if $a < 0$ (supercritical case)
- ▶ Bifurcation is at $\mu = 0$ where $r = 0$ loses stability

Bifurcation diagrams for Hopf bifurcations

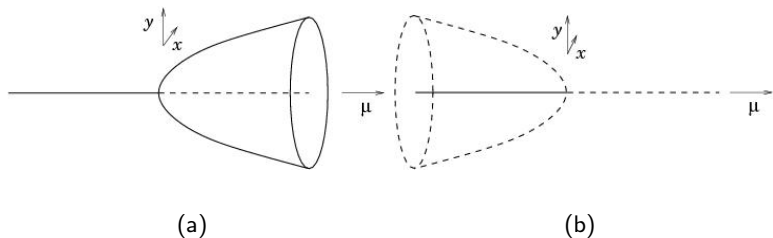


Figure: Bifurcation diagrams for the a) supercritical and b) subcritical Hopf bifurcations. Stable solutions are shown as bold lines, and unstable solutions as dotted lines. (The axes are drawn with the origin shifted.)

Phase portraits for subcritical Hopf bifurcation

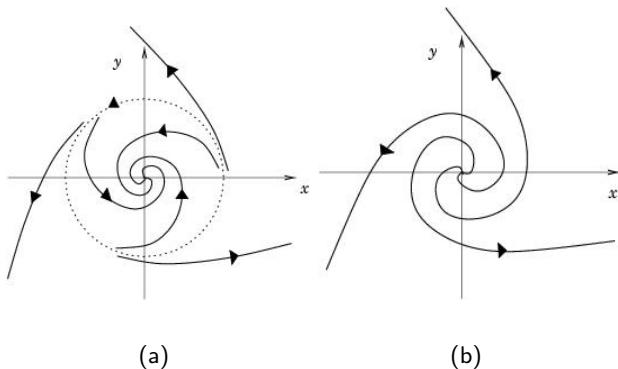


Figure: Phase plane diagrams for the subcritical Hopf bifurcation when a) $\mu < 0$ and b) $0 < \mu < a\omega/b$, for $\omega > 0$, $b > 0$ and $a > 0$. The limit cycle is shown as a dotted line because it is unstable.

Phase portraits for supercritical Hopf bifurcation

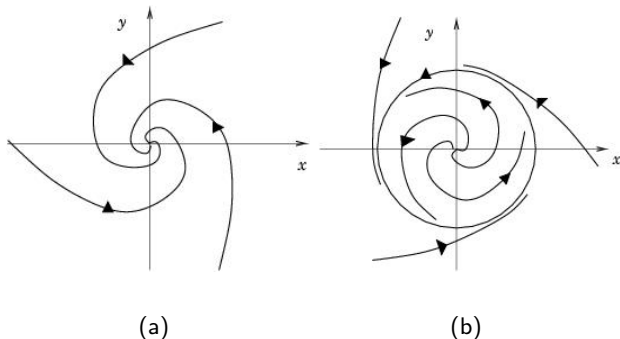


Figure: Phase plane diagrams for the supercritical Hopf bifurcation when a) $a\omega/b < \mu < 0$ and b) $\mu > 0$, for $\omega > 0$, $b > 0$ and $a < 0$.

References

- Carr, J. (1981), *Applications of centre manifold theory* (Springer-Verlag, New York).
- Glendinning, P. (1994). *Stability, instability and chaos: an introduction to the theory of nonlinear differential equations* (Cambridge University Press, Cambridge).
- Hoyle, R.B. (2005). *Pattern formation: an introduction to methods* (Cambridge University Press, Cambridge). (In press)