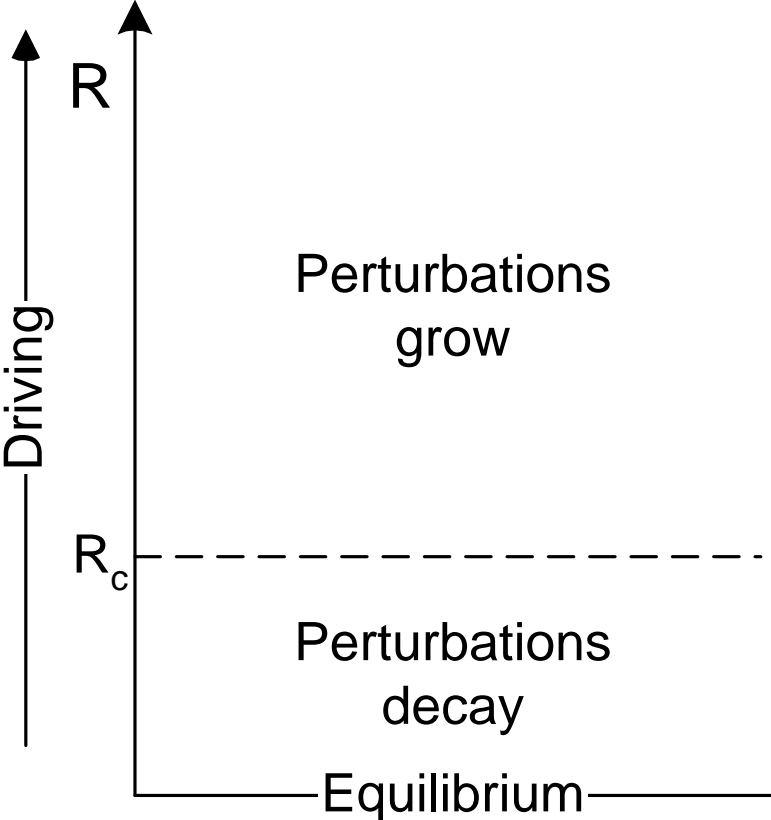
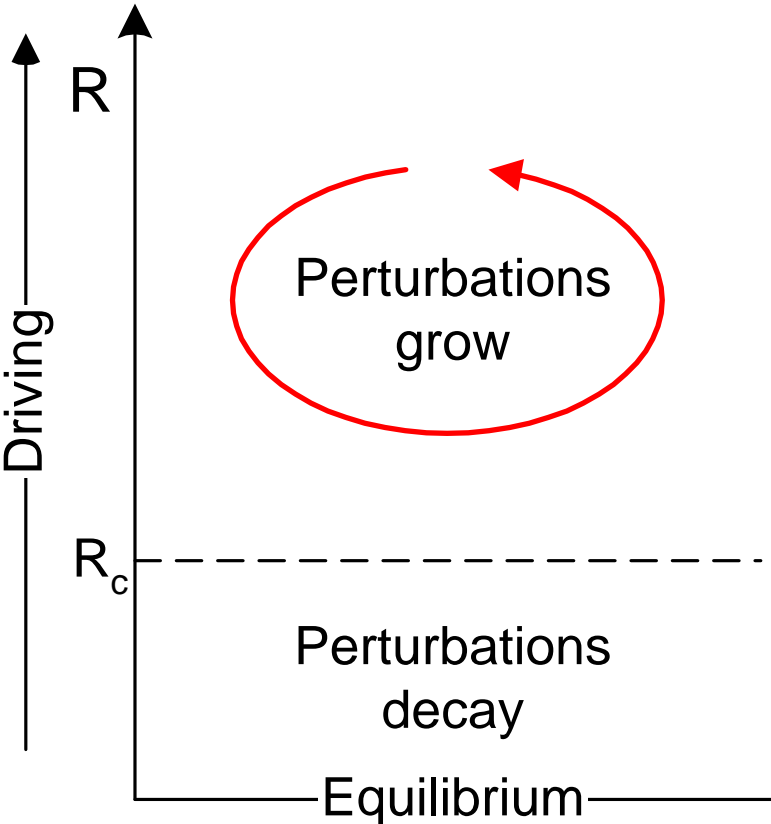


Pattern Formation in Spatially Extended Systems

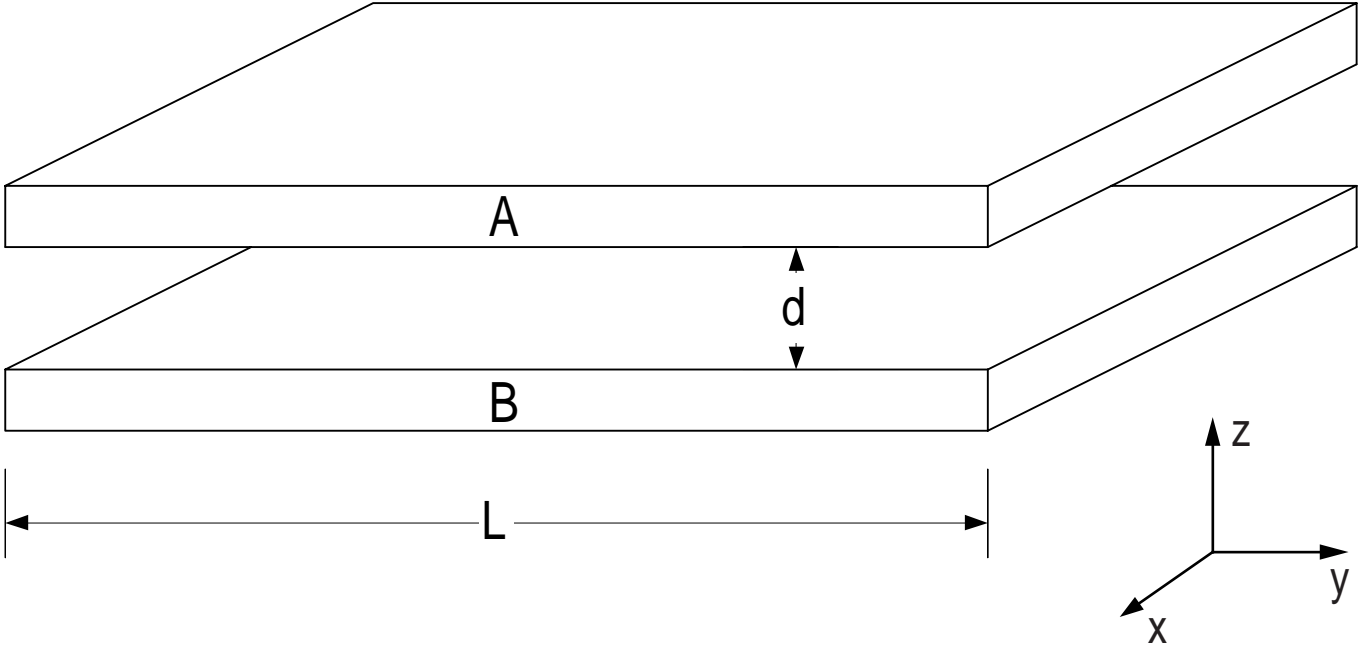
Lecture 1

- linear instability
- nonlinear saturation
- stability balloon
- amplitude equation





Pattern formation occurs in a spatially extended system when the growing perturbation about the spatially uniform state has spatial structure (a mode with nonzero wave vector).



A first approach to patterns: linear stability analysis

1. Find equations of motion of the physical variables $\mathbf{u}(x, y, z, t)$
2. Find the *uniform* base solution $\mathbf{u}_b(z)$ independent of x, y, t
3. Focus on deviation from \mathbf{u}_b

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_b(z) + \delta\mathbf{u}(\mathbf{x}, t)$$

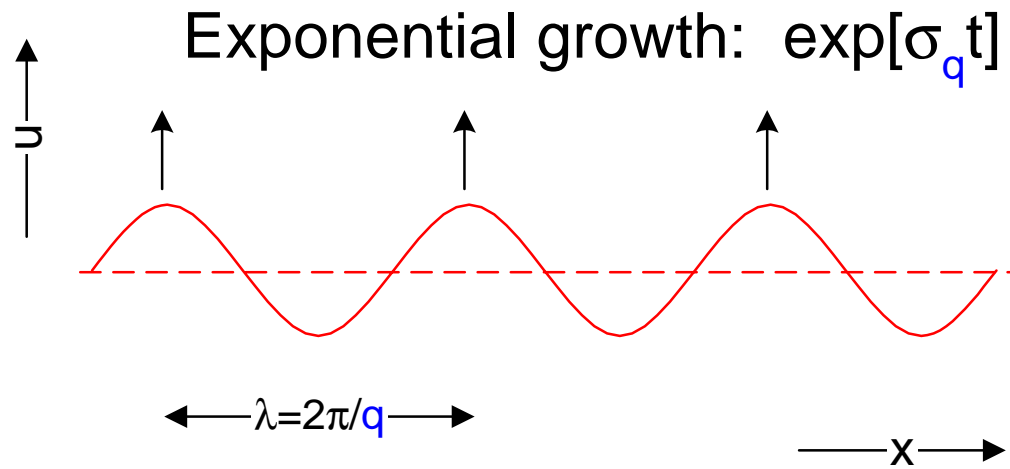
4. Linearize equations about \mathbf{u}_b , i.e. substitute into equations of part (1) and keep all terms with just one power of $\delta\mathbf{u}$. This will give an equation of the form

$$\partial_t \delta\mathbf{u} = \hat{\mathbf{L}} \delta\mathbf{u}$$

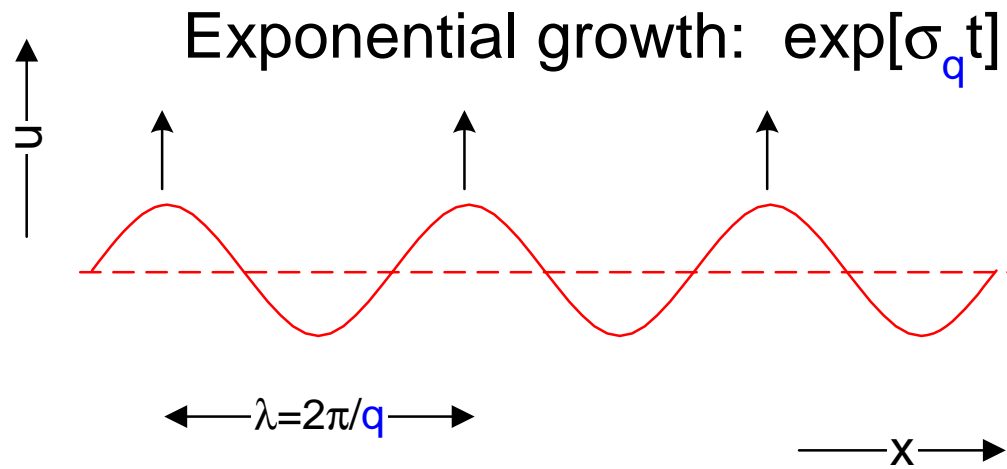
where $\hat{\mathbf{L}}$ may involve \mathbf{u}_b and include spatial derivatives acting on $\delta\mathbf{u}$

5. Since $\hat{\mathbf{L}}$ is independent of x, y, t we can find solutions

$$\delta\mathbf{u}_{\mathbf{q}}(\mathbf{x}_{\perp}, z, t) = \mathbf{u}_{\mathbf{q}}(z) e^{i\mathbf{q}\cdot\mathbf{x}_{\perp}} e^{\sigma_{\mathbf{q}}t}$$

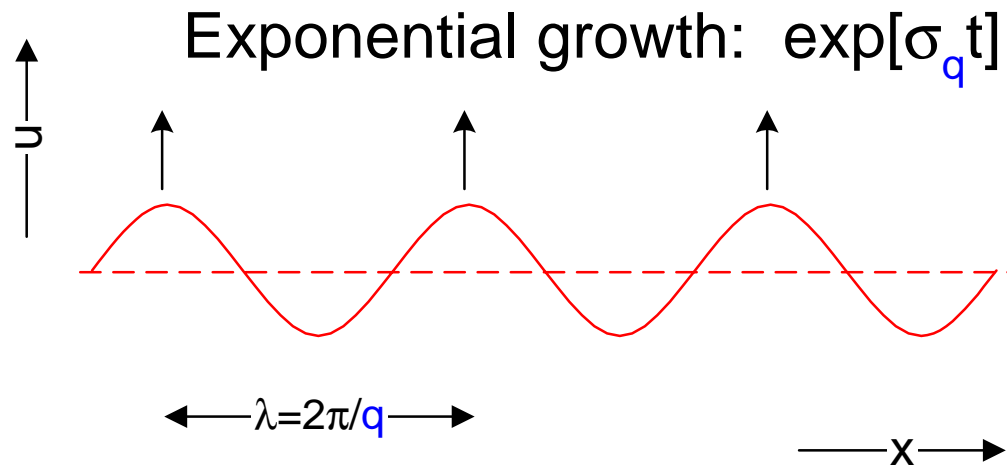


$$\delta \mathbf{u}_{\mathbf{q}}(\mathbf{x}_{\perp}, z, t) = \mathbf{u}_{\mathbf{q}}(z) e^{i\mathbf{q} \cdot \mathbf{x}_{\perp}} e^{\sigma_{\mathbf{q}} t}$$



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Re $\sigma_{\mathbf{q}}$ gives exponential growth or decay



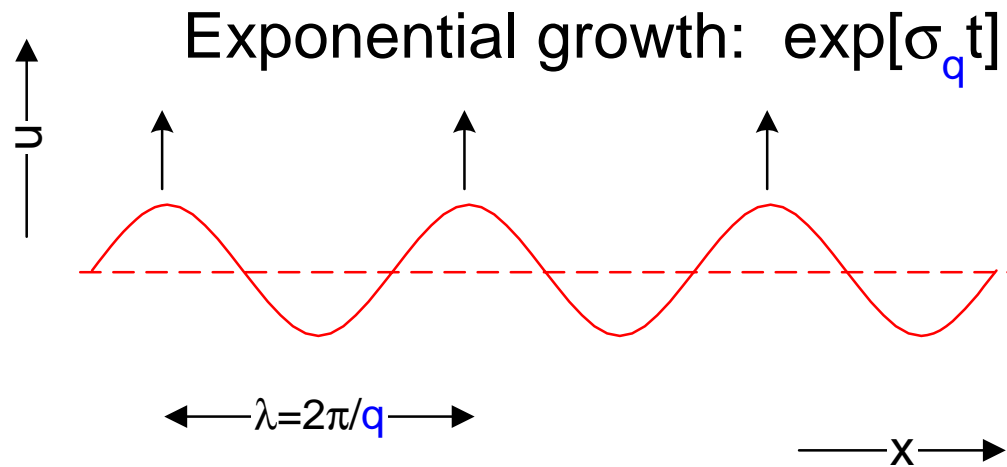
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Re $\sigma_{\mathbf{q}}$ gives exponential growth or decay

Im $\sigma_{\mathbf{q}} = -\omega_{\mathbf{q}}$ gives oscillations, waves $e^{i(\mathbf{q} \cdot \mathbf{x}_{\perp} - \omega_{\mathbf{q}} t)}$

Im $\sigma_{\mathbf{q}} = 0 \implies$ Stationary instability

Im $\sigma_{\mathbf{q}} \neq 0 \implies$ Oscillatory instability



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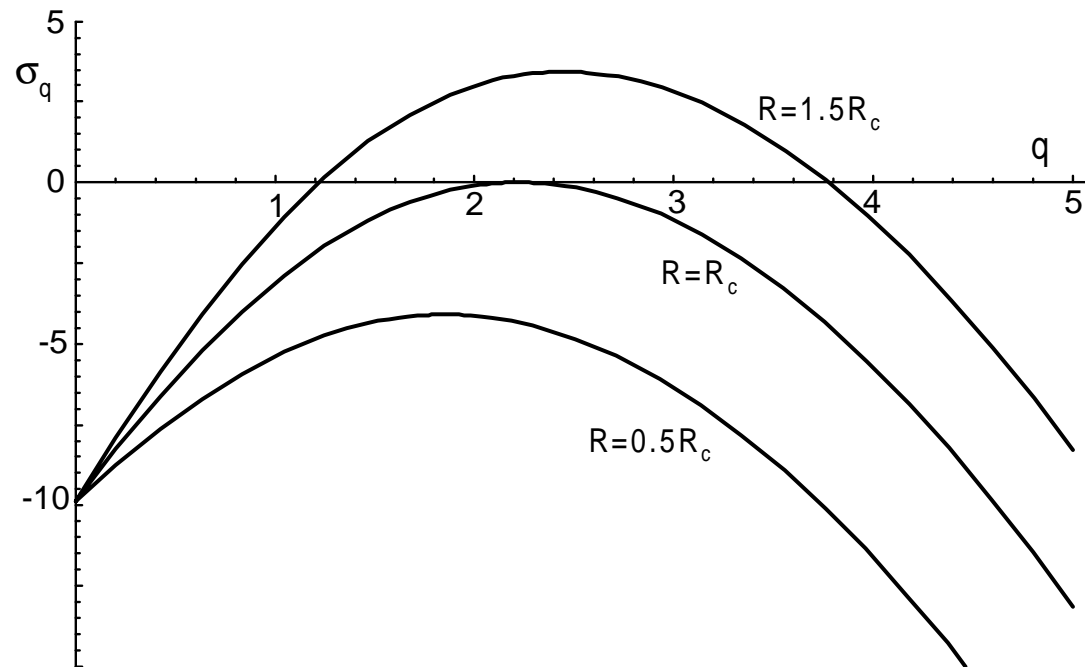
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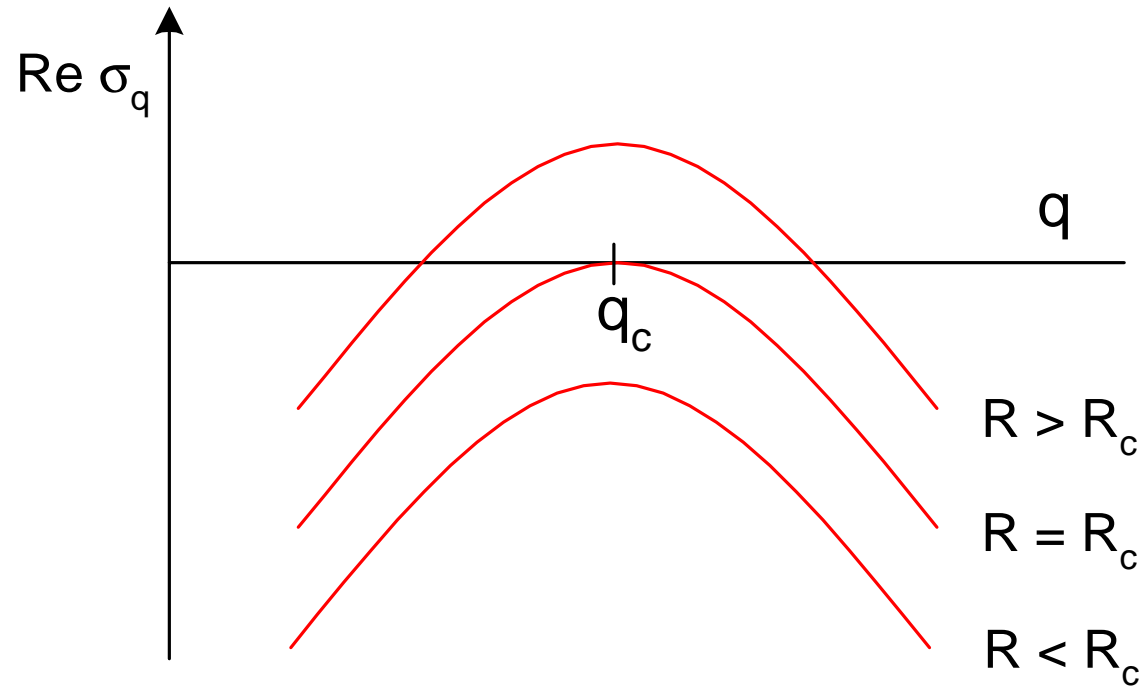
For this lecture I will look at the case of **stationary instability**

Rayleigh's calculation



$$(\sigma^{-1}\sigma_q + \pi^2 + q^2)(\sigma_q + \pi^2 + q^2) - Rq^2/(\pi^2 + q^2) = 0$$

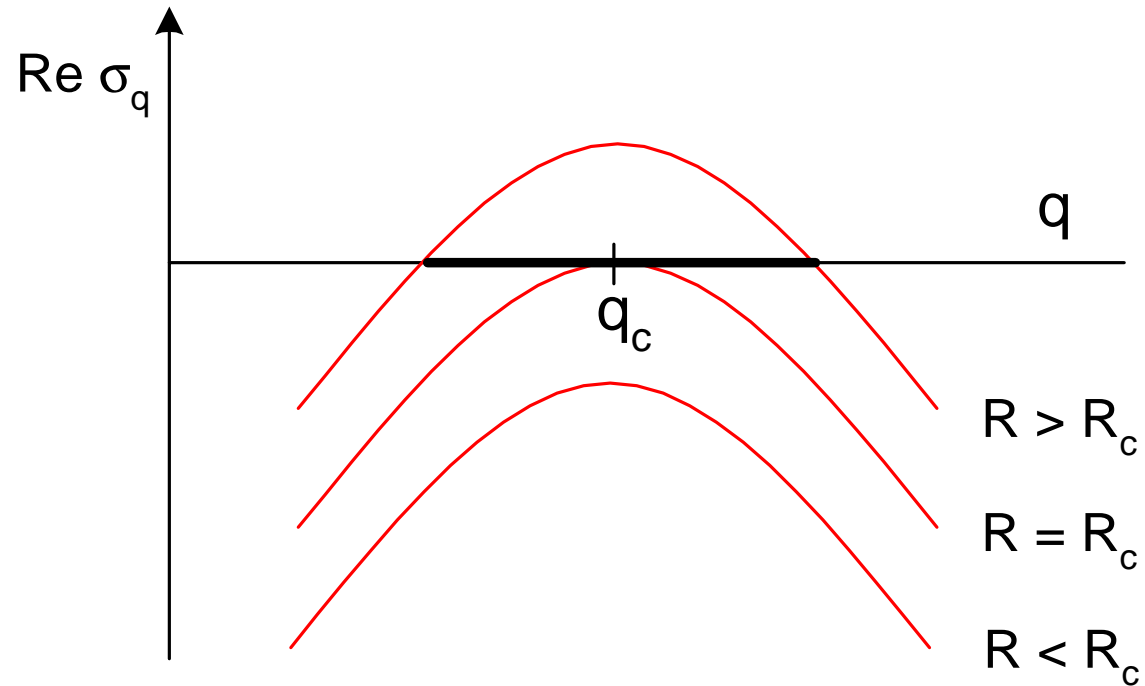
Parabolic approximation near maximum



For R near R_c and q near q_c

$$\text{Re } \sigma_q = \tau_0^{-1} [\varepsilon - \xi_0^2 (q - q_c)^2] \quad \text{with} \quad \varepsilon = \frac{R - R_c}{R_c}$$

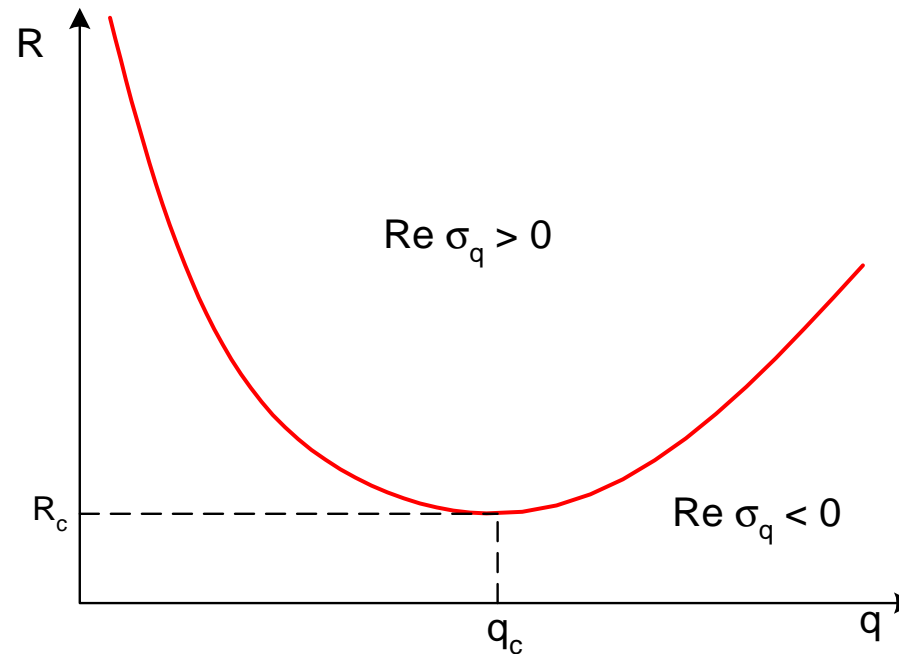
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Neutral stability curve



Setting $\text{Re } \sigma_q = 0$ defines the neutral stability curve $R = R_c(q)$

$$\text{Rayleigh : } R_c(q) = \frac{(q^2 + \pi^2)^3}{q^2} \Rightarrow R_c = \frac{27\pi^4}{4}, \quad q_c = \frac{\pi}{\sqrt{2}}$$

Linear stability theory is often a useful first step in understanding pattern formation:

- Often is quite easy to do either analytically or numerically
- Displays the important physical processes
- Gives the length scale of the pattern formation $1/q_c$

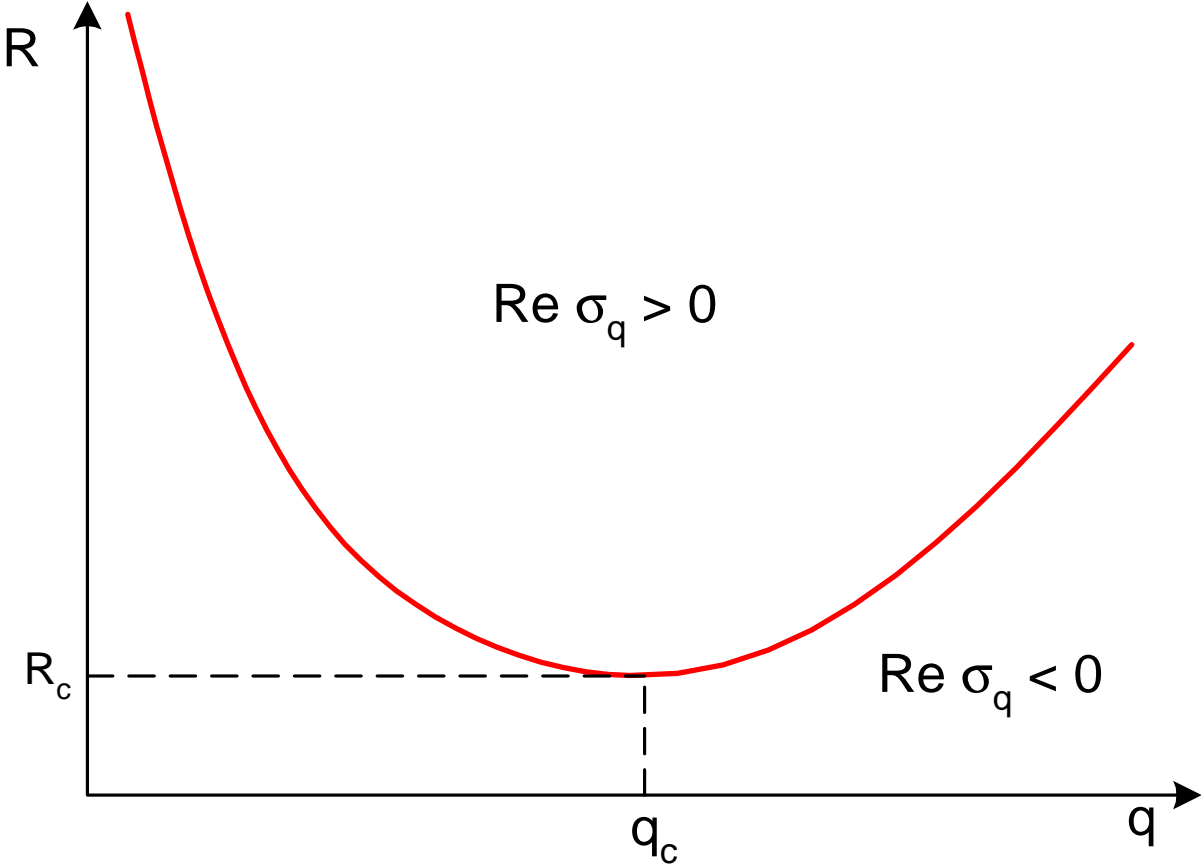
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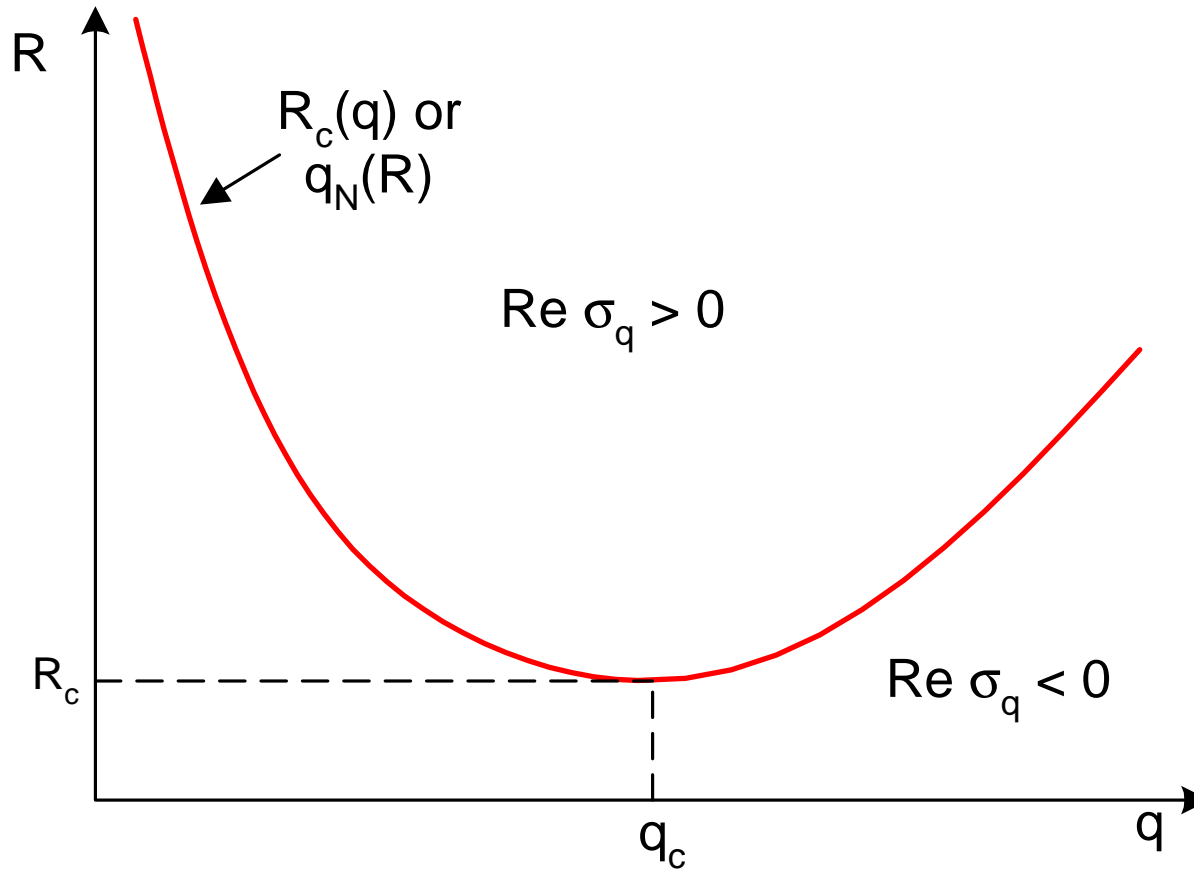
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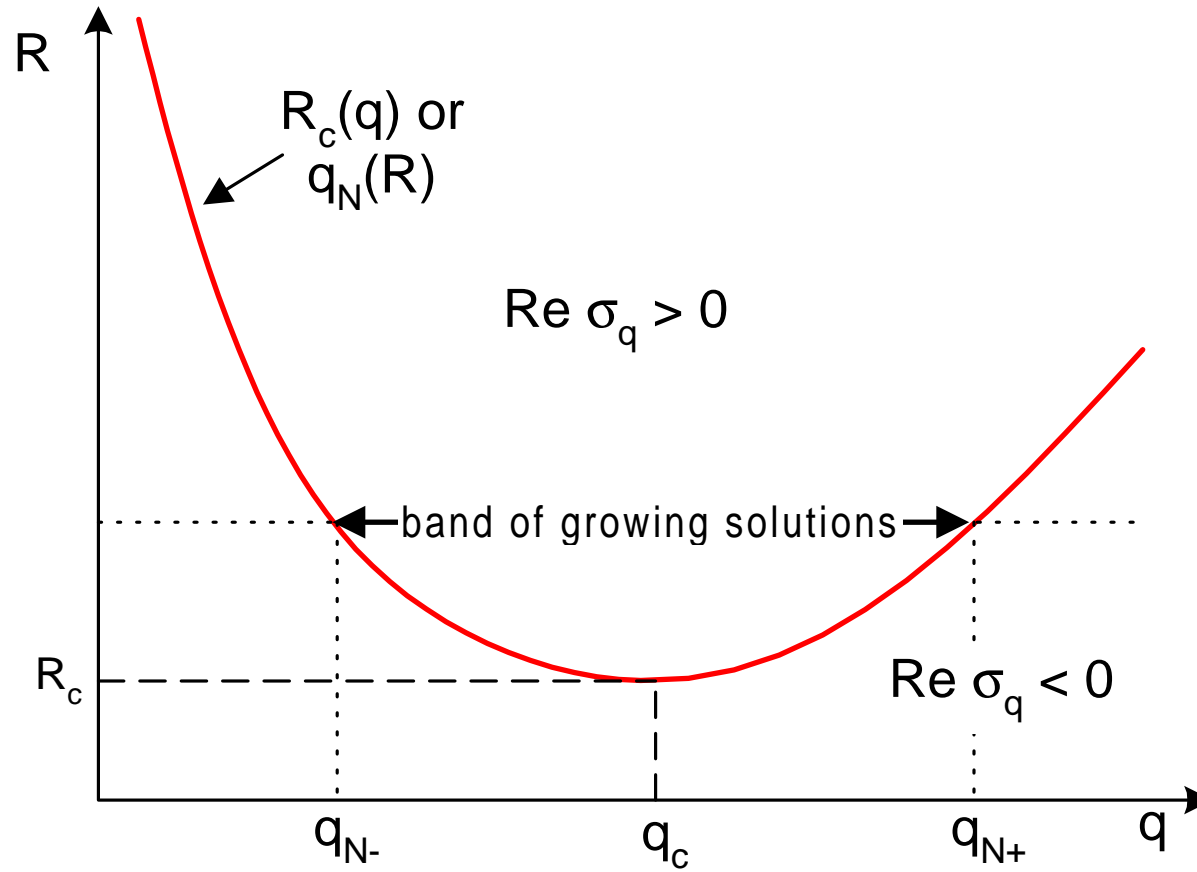
But:

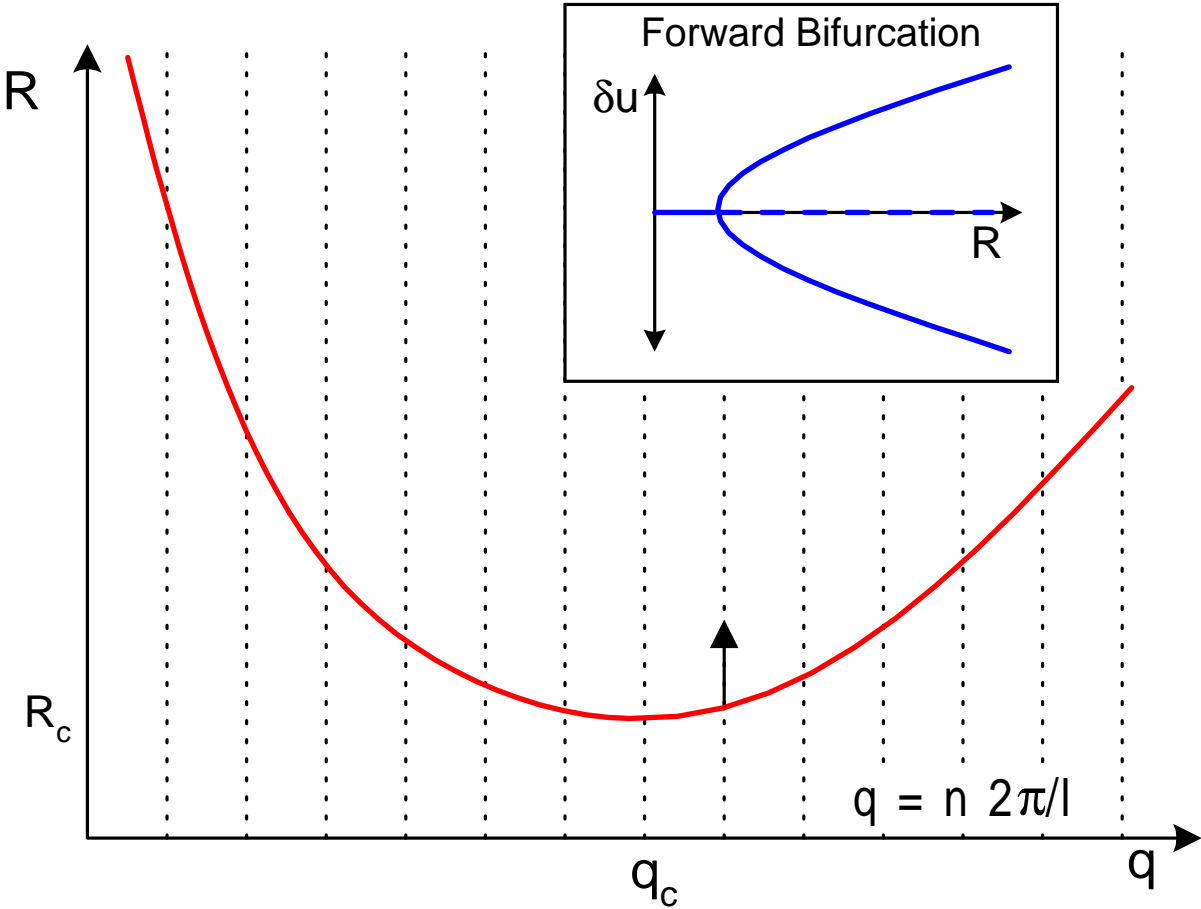
- Leaves us with unphysical exponentially growing solutions

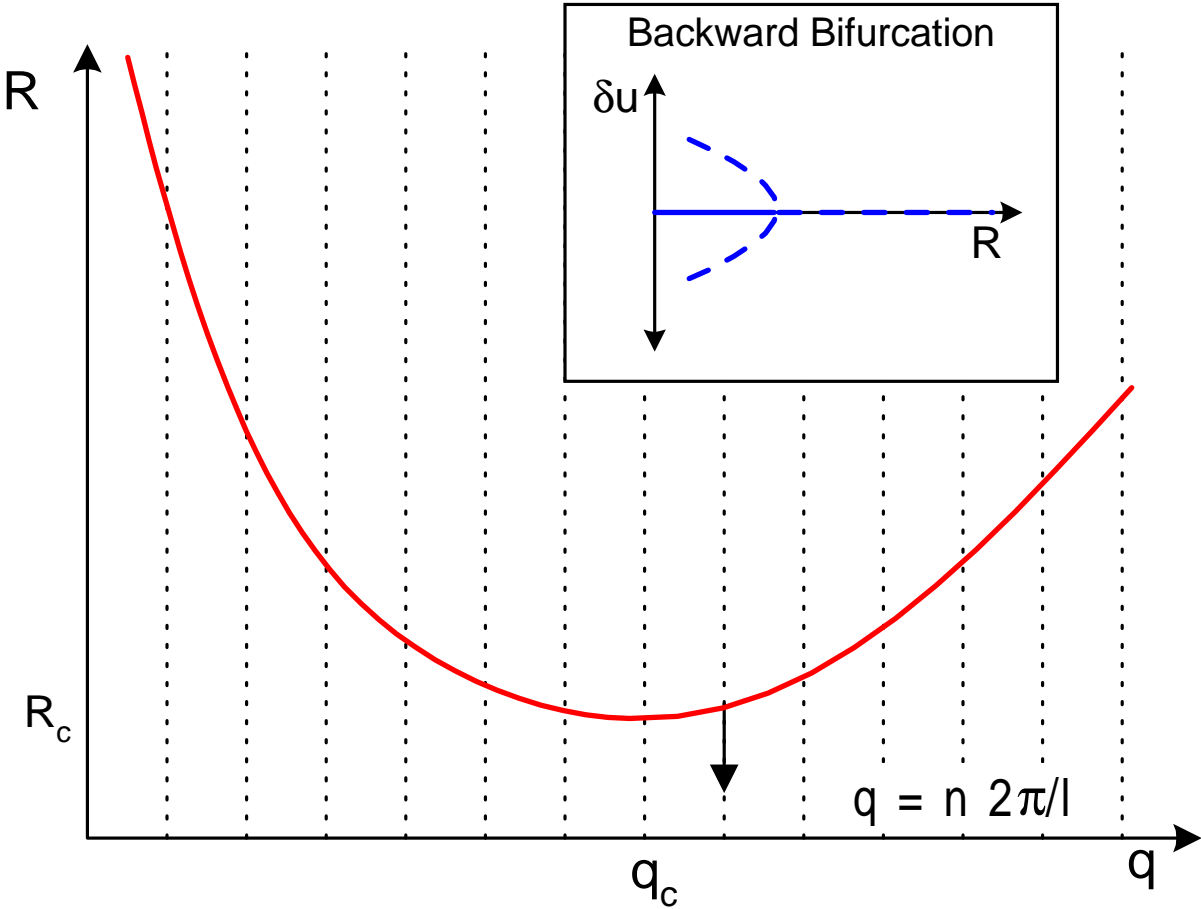
Nonlinearity

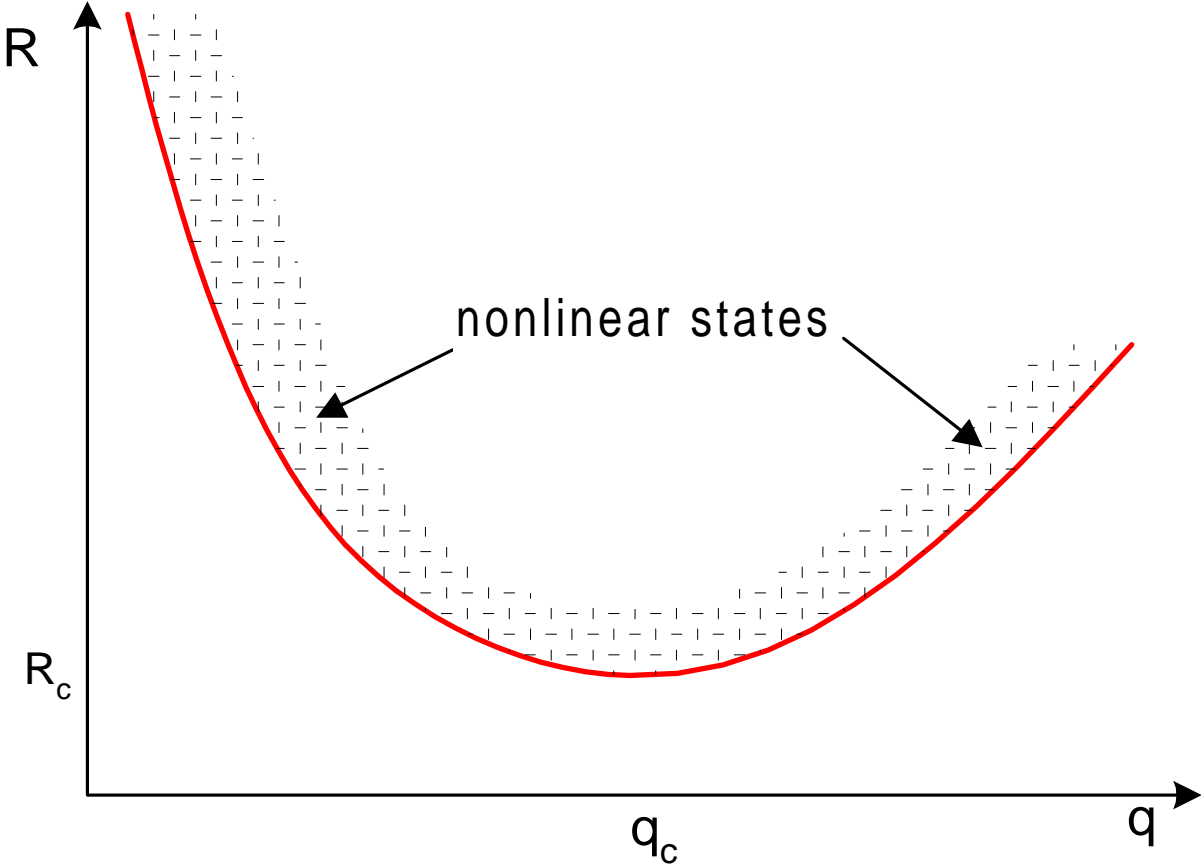


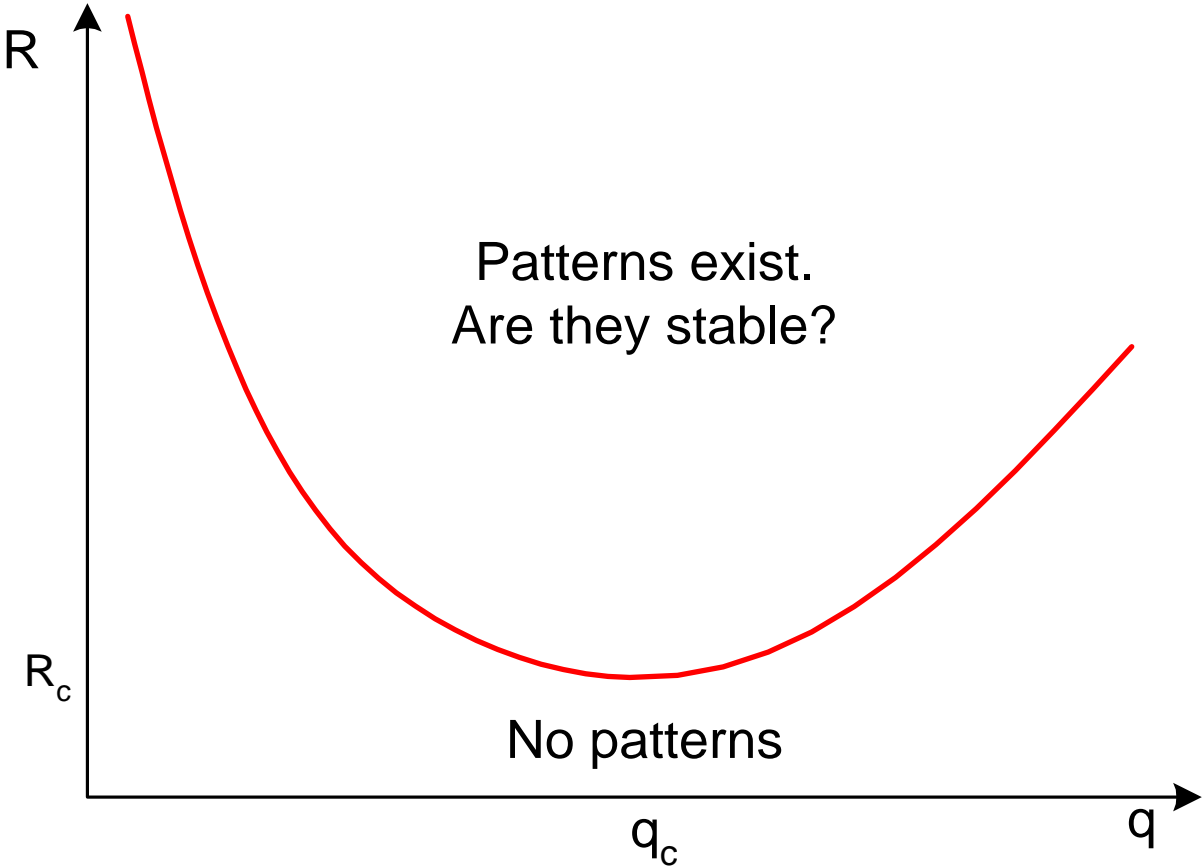


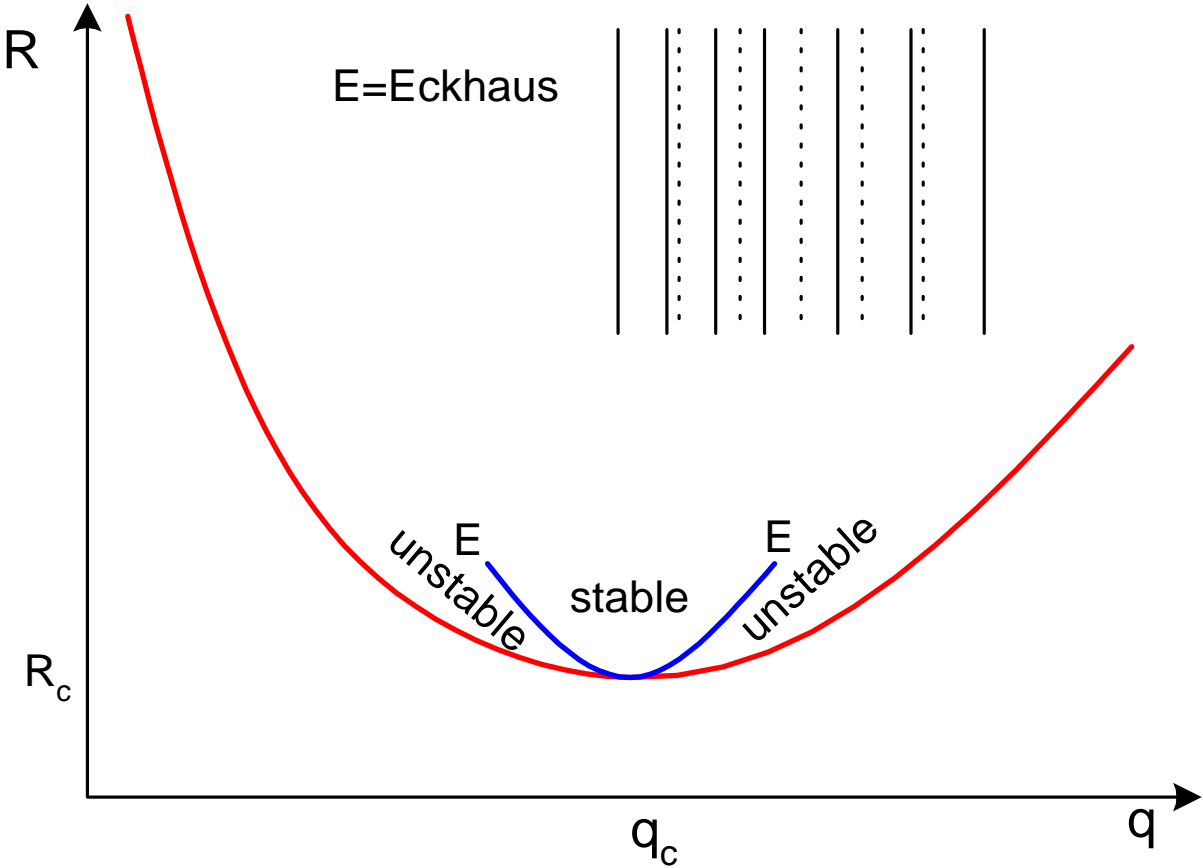


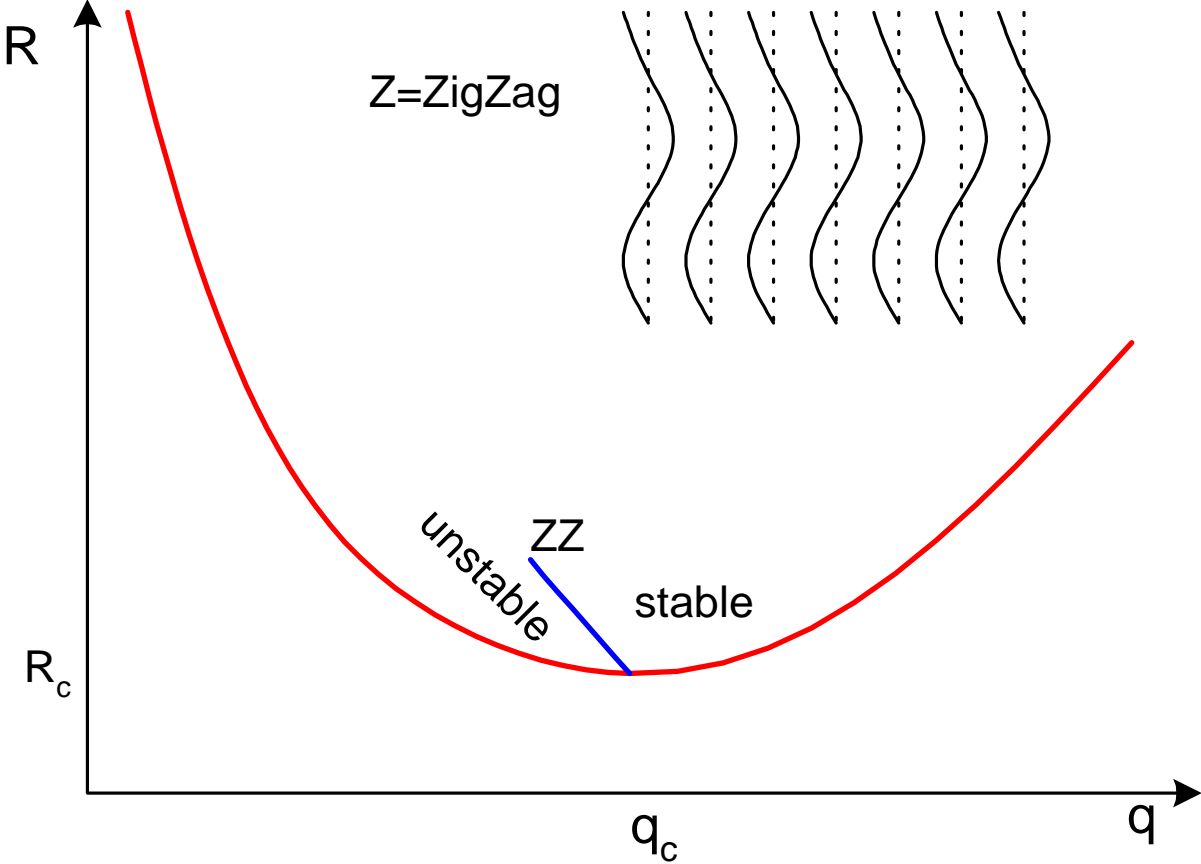


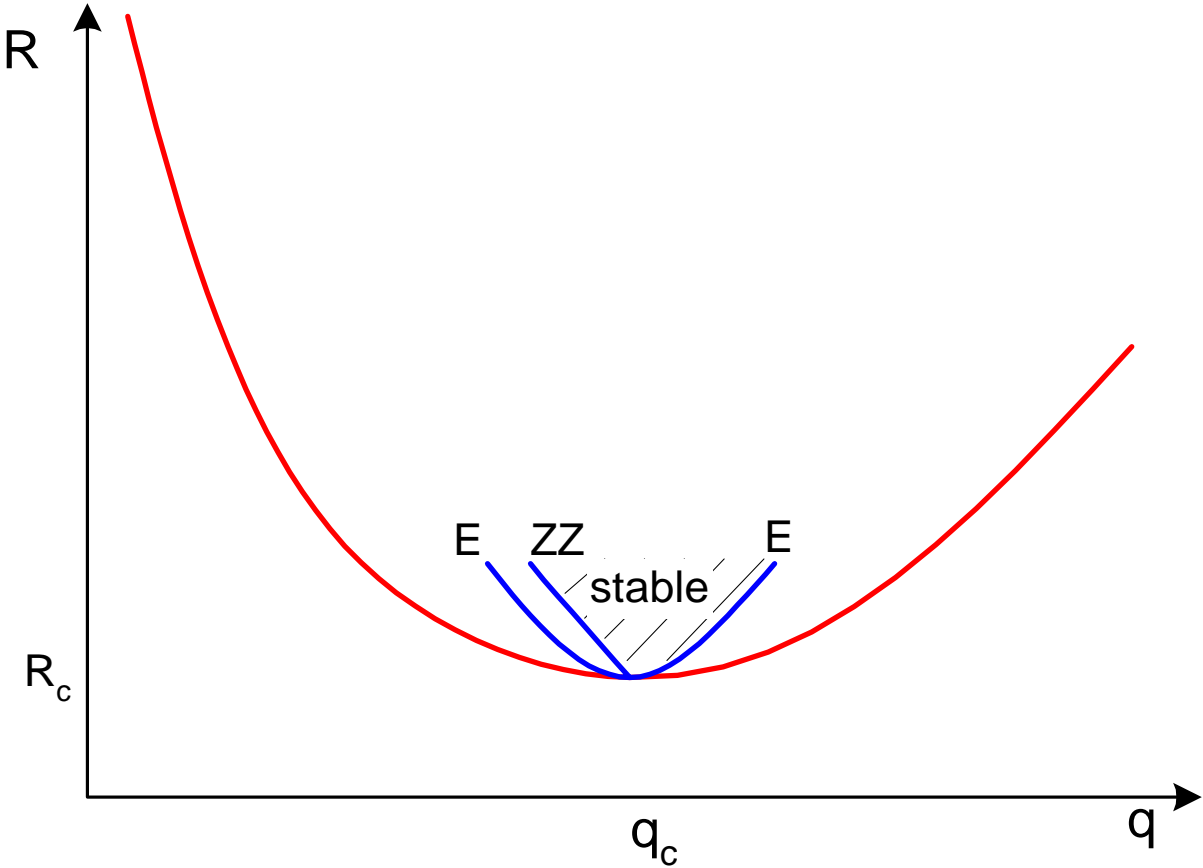


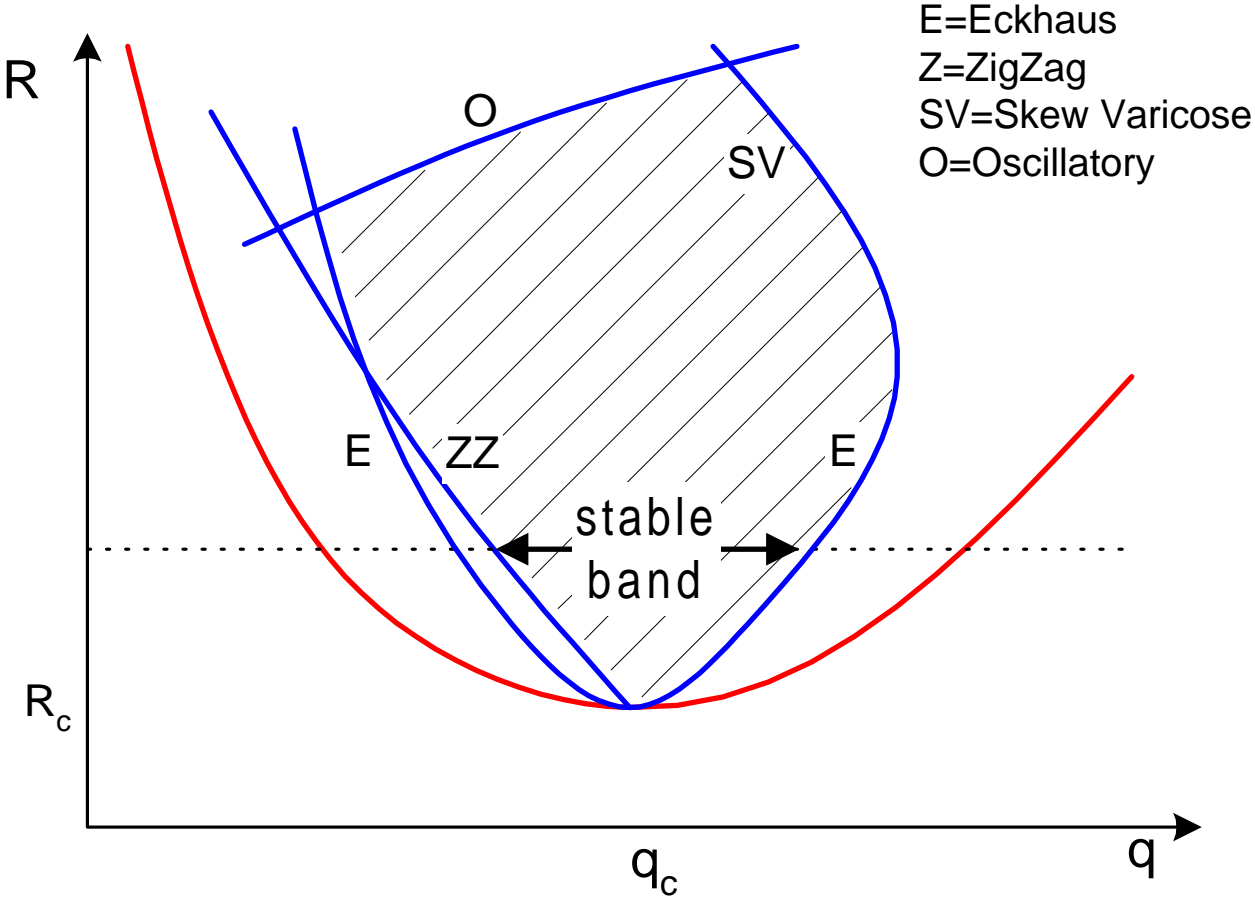


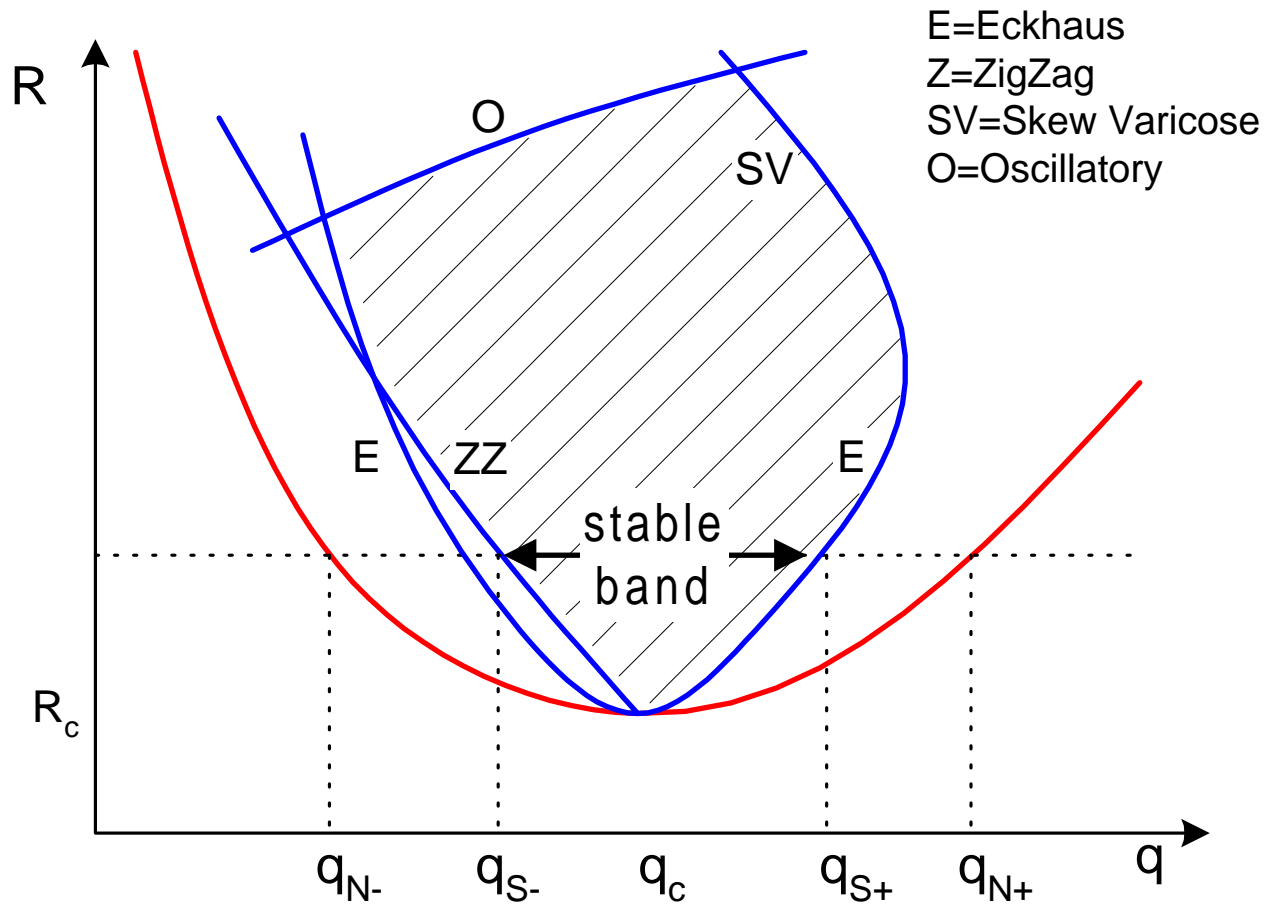


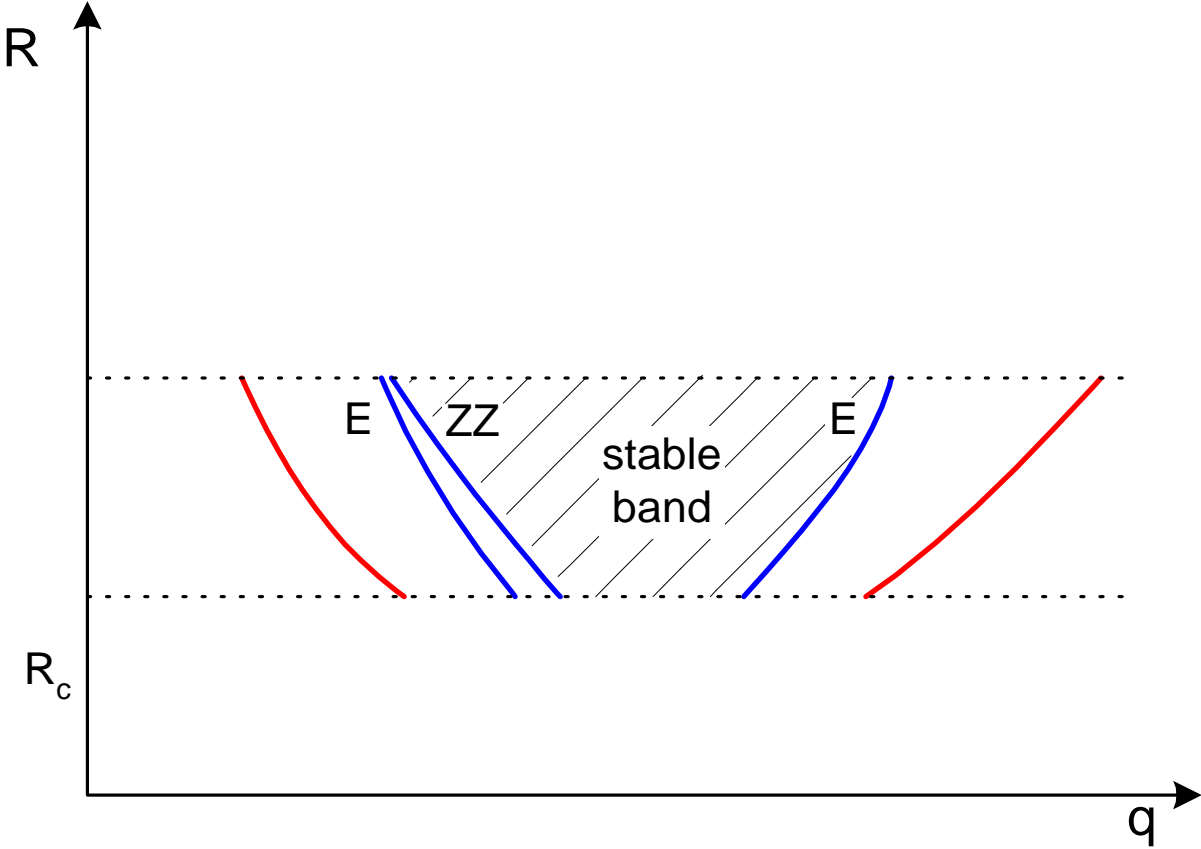








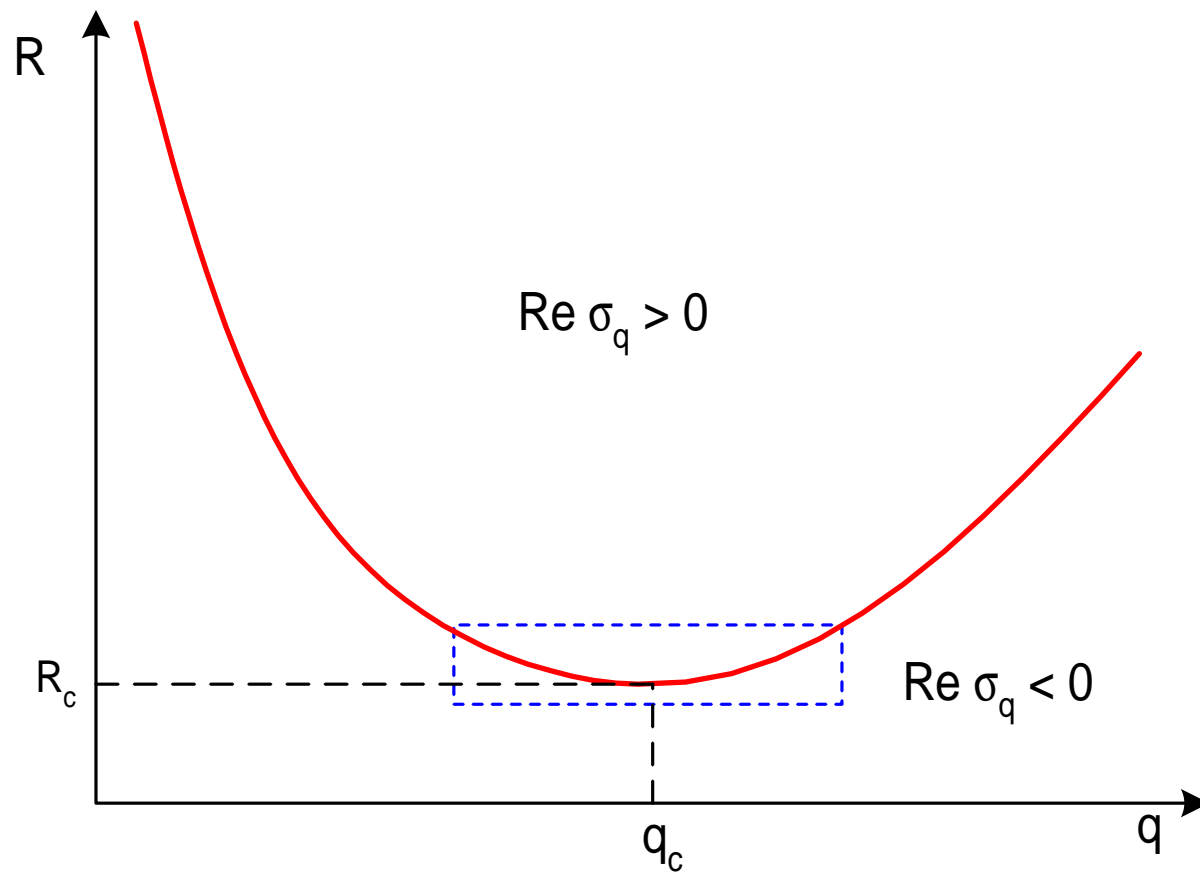




Tools for the Nonlinear Problem

Amplitude Equations

Systematic approach for describing weakly nonlinear solutions near onset



Linear onset solution

$$\delta \mathbf{u}_{\mathbf{q}}(\mathbf{x}_{\perp}, z, t) = \left[a_0 e^{i(\mathbf{q} - \mathbf{q}_c) \cdot \mathbf{x}_{\perp}} e^{\text{Re} \sigma_{\mathbf{q}} t} \right] \times \left[\mathbf{u}_{\mathbf{q}}(z) e^{i \mathbf{q}_c \cdot \mathbf{x}_{\perp}} \right] + c.c.$$

Small terms near onset Onset solution

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Small terms near onset
Onset solution

Weakly nonlinear, slowly modulated, solution

$$\delta \mathbf{u}(\mathbf{x}_{\perp}, z, t) \approx \mathbf{A}(\mathbf{x}_{\perp}, t) \times \left[\mathbf{u}_{\mathbf{q}_c}(z) e^{i\mathbf{q}_c \cdot \mathbf{x}_{\perp}} \right] + c.c.$$

Complex amplitude
Onset solution

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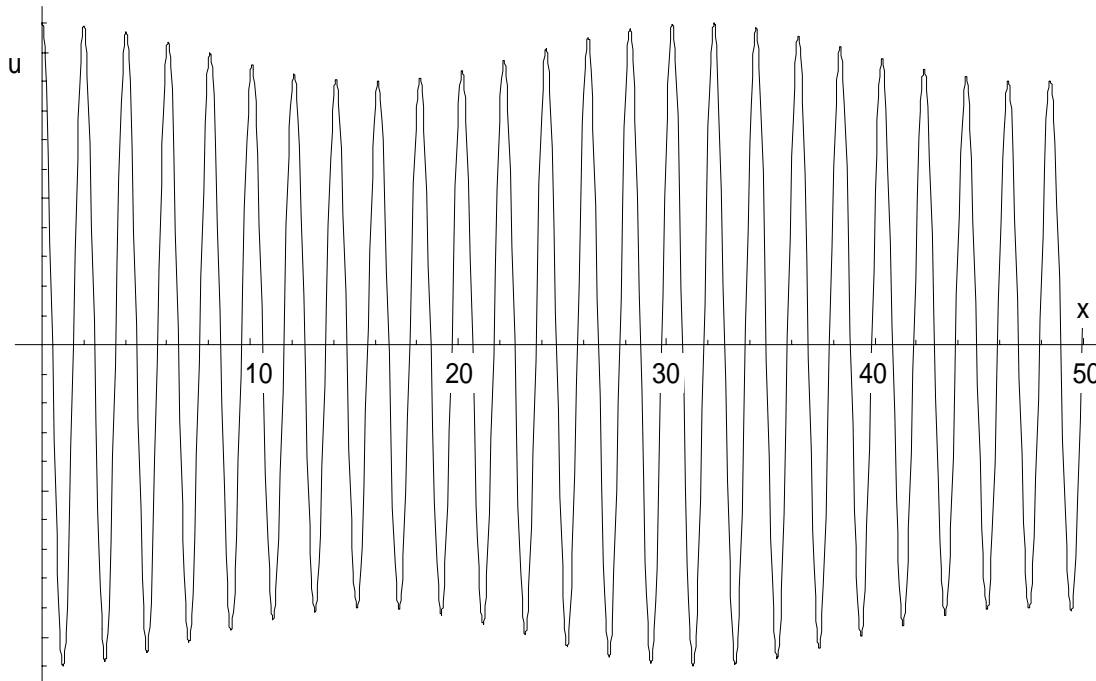
Complex amplitude
Onset solution

Substituting into the dynamical equations gives the amplitude equation, which in 1d [$\mathbf{q}_c = q_c \hat{\mathbf{x}}$, $A = A(x, t)$] is

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \quad \varepsilon = \frac{R - R_c}{R_c}$$

Pictorially

A convection pattern that varies **gradually** in space



$$u \propto \text{Re}[A(x)e^{iq_c x}]$$

$$q_c = 3.117; \quad A(x) = 1 + 0.1 \cos(0.2x)$$

Complex Amplitude

Magnitude and phase of A play very different roles

$$A(x, y, t) = a(x, y, t)e^{i\theta(x, y, t)}$$

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$$A = ae^{ikx} \text{ corresponds to } q = q_c + k$$

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 $A = ae^{ikx}$ corresponds to $q = q_c + k$
- y-gradient $\partial_y\theta$ gives rotation of wave vector through angle $\partial_y\theta/q_c$
 (plus $O[(\partial_y\theta)^2]$ change in wave number)

The amplitude equation describes

$$\tau_0 \partial_t A = \underbrace{\varepsilon A}_{\text{growth}} + \underbrace{\xi_0^2 \partial_x^2 A}_{\text{dispersion/diffusion}} - \underbrace{g_0 |A|^2 A}_{\text{saturation}}$$

Parameters

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A,$$

- control parameter $\varepsilon = (R - R_c)/R_c$

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 - ◇ τ_0, ξ_0 fixed by matching to linear growth rate
 $A = a e^{i\mathbf{k} \cdot \mathbf{x}_\perp} e^{\sigma_{\mathbf{q}} t}$ gives pattern at $\mathbf{q} = \mathbf{q}_c \hat{x} + \mathbf{k}$)

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$$\sigma_{\mathbf{q}} = \tau_0^{-1} [\varepsilon - \xi_0^2 (q - q_c)^2]$$

- ◇ g_0 by calculating nonlinear state at small ε and $q = q_c$.

Scaling

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \quad \varepsilon = \frac{R - R_c}{R_c}$$

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Introduce scaled variables

$$x = \varepsilon^{-1/2} \xi_0 X$$

$$t = \varepsilon^{-1} \tau_0 T$$

$$A = (\varepsilon/g_0)^{1/2} \bar{A}$$

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This reduces the amplitude equation to a *universal* form

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Since solutions to this equation will develop on scales $X, Y, T, \bar{A} = O(1)$ this gives us scaling results for the physical length scales.

Derivation

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \quad \varepsilon = \frac{R - R_c}{R_c}$$

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 - ◇ $A(\mathbf{x}_\perp) \rightarrow A(\mathbf{x}_\perp) e^{i\Delta}$ with Δ a constant, corresponding to a physical translation;
 - ◇ $A(\mathbf{x}_\perp) \rightarrow A^*(-\mathbf{x}_\perp)$, corresponding to inversion of the horizontal coordinates (parity symmetry);

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- Mode projection (MCC 1980)

Amplitude Equation = Ginzburg Landau equation

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A,$$

Familiar from other branches of physics:

- Good: take intuition from there
- Bad: no *really* new effects

e.g. equation is relaxational (potential, Lyapunov)

$$\tau_0 \partial_t A = -\frac{\delta V}{\delta A^*}, \quad V = \int dx \left[-\varepsilon |A|^2 + \frac{1}{2} g_0 |A|^4 + \xi_0^2 |\partial_x A|^2 \right]$$

This leads to

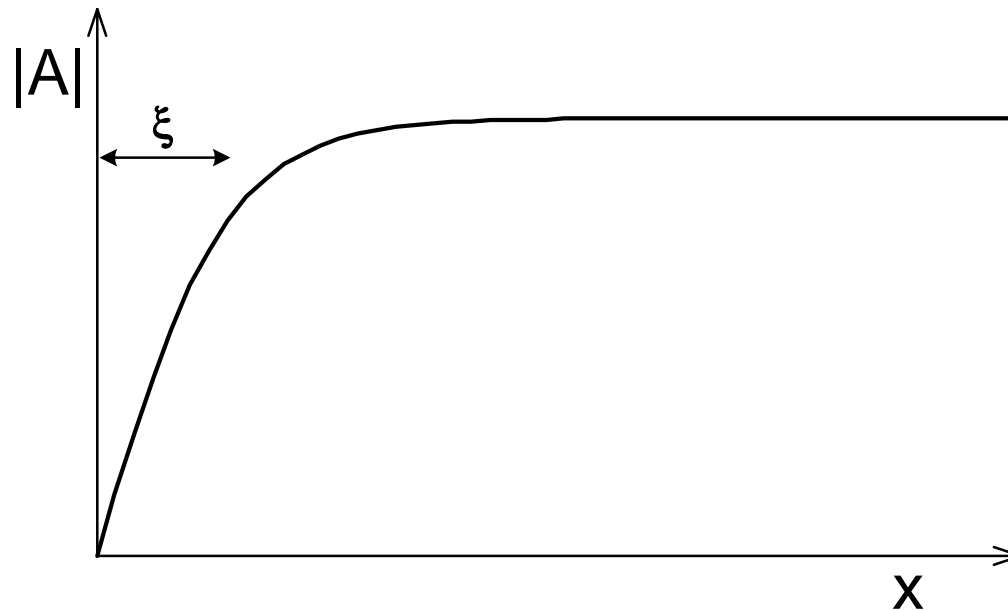
$$\frac{dV}{dt} = -\tau_0^{-1} \int dx |\partial_t A|^2 \leq 0$$

and dynamics runs “down hill” to a minimum of V — no chaos!

Example: one dimensional geometry with boundaries that suppress the pattern (e.g. rigid walls in a convection system)

First consider a single wall

$$\partial_T \bar{A} = \bar{A} + \partial_X^2 \bar{A} - |\bar{A}|^2 \bar{A} \quad \bar{A}(0) = 0$$



$$\bar{A} = e^{i\theta} \tanh(X/\sqrt{2})$$

$$A = e^{i\theta} (\varepsilon/g_0)^{1/2} \tanh(x/\xi) \quad \text{with} \quad \xi = \sqrt{2}\varepsilon^{-1/2}\xi_0$$

$$A = e^{i\theta} (\varepsilon/g_0)^{1/2} \tanh(x/\xi)$$

- arbitrary position of rolls
- asymptotic wave number is $k = 0$, giving $q = q_c$: no band of existence

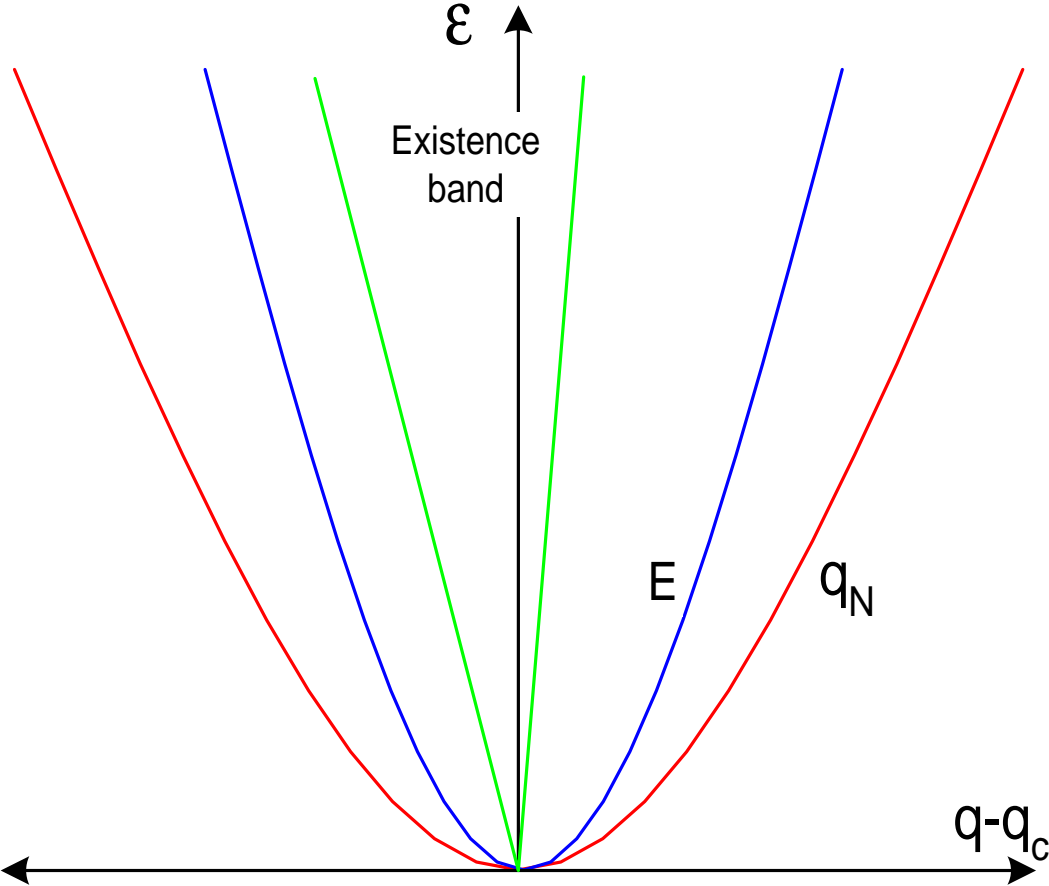
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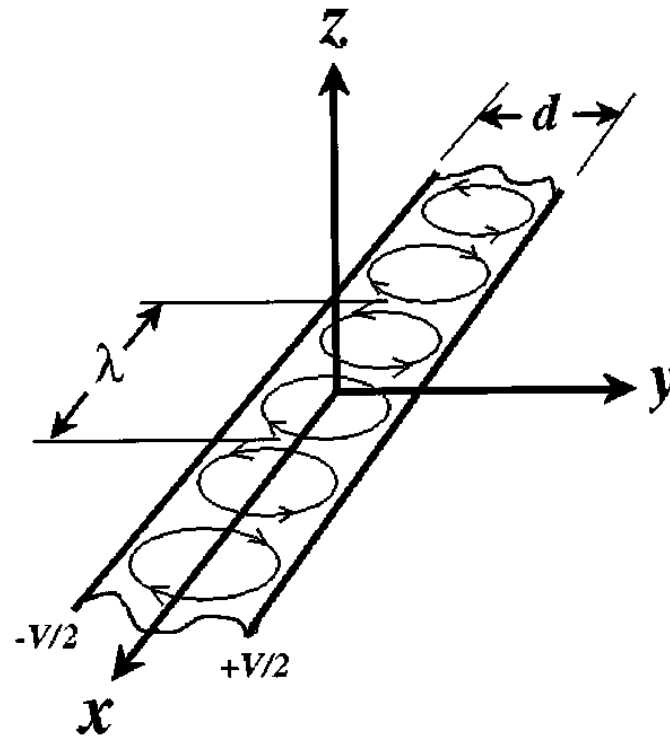
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Extended amplitude equation to next order in ε (MCC, Daniels, Hohenberg, and Siggia 1980) shows

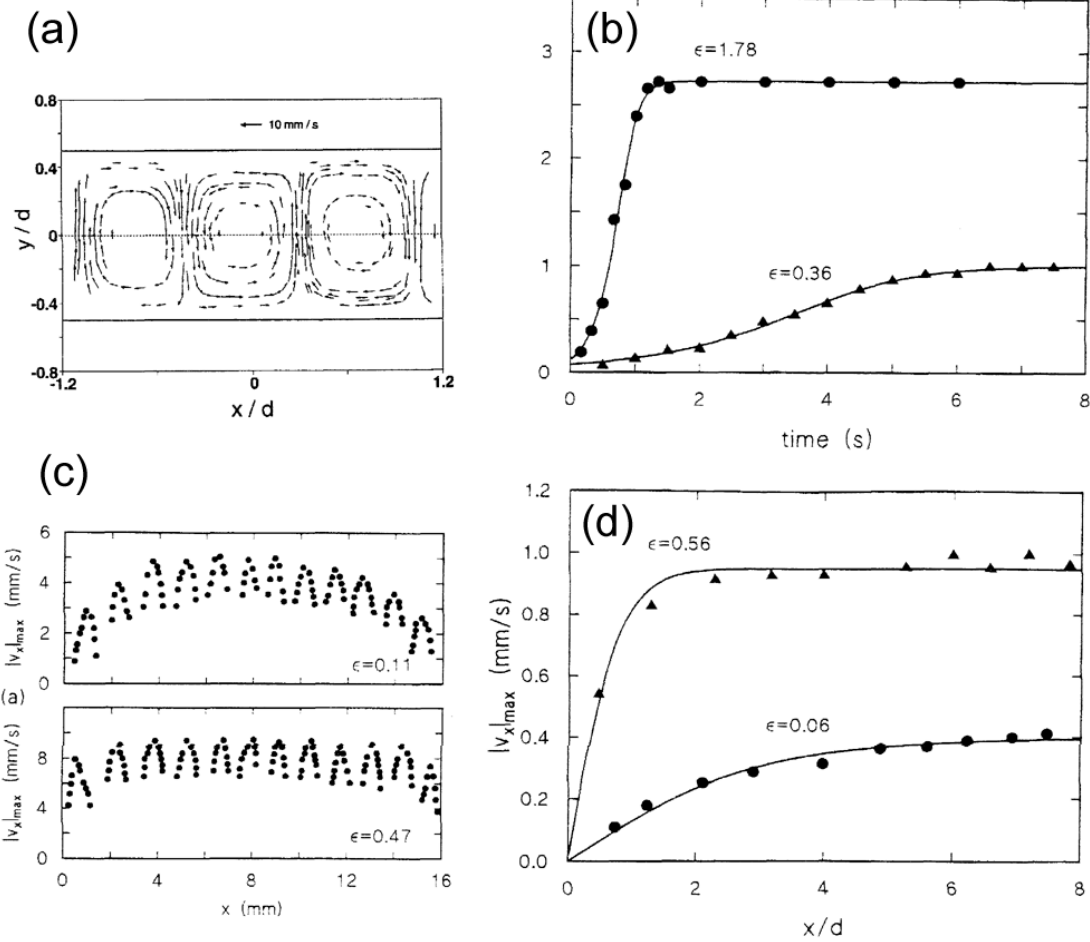
- discrete set of roll positions
- solutions restricted to a narrow $O(\varepsilon^1)$ wave number band with wave number far from the wall

$$\alpha_{-\varepsilon} < q - q_c < \alpha_{+\varepsilon}$$

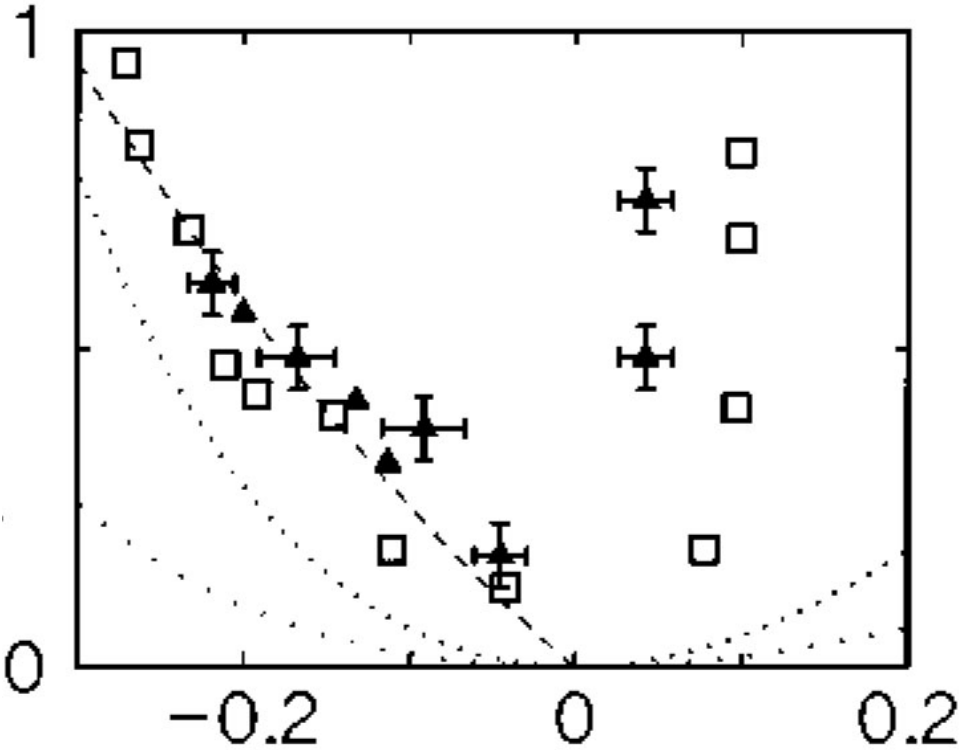




V. B. Deyirmenjian, Z. A. Daya, and S. W. Morris (1997)

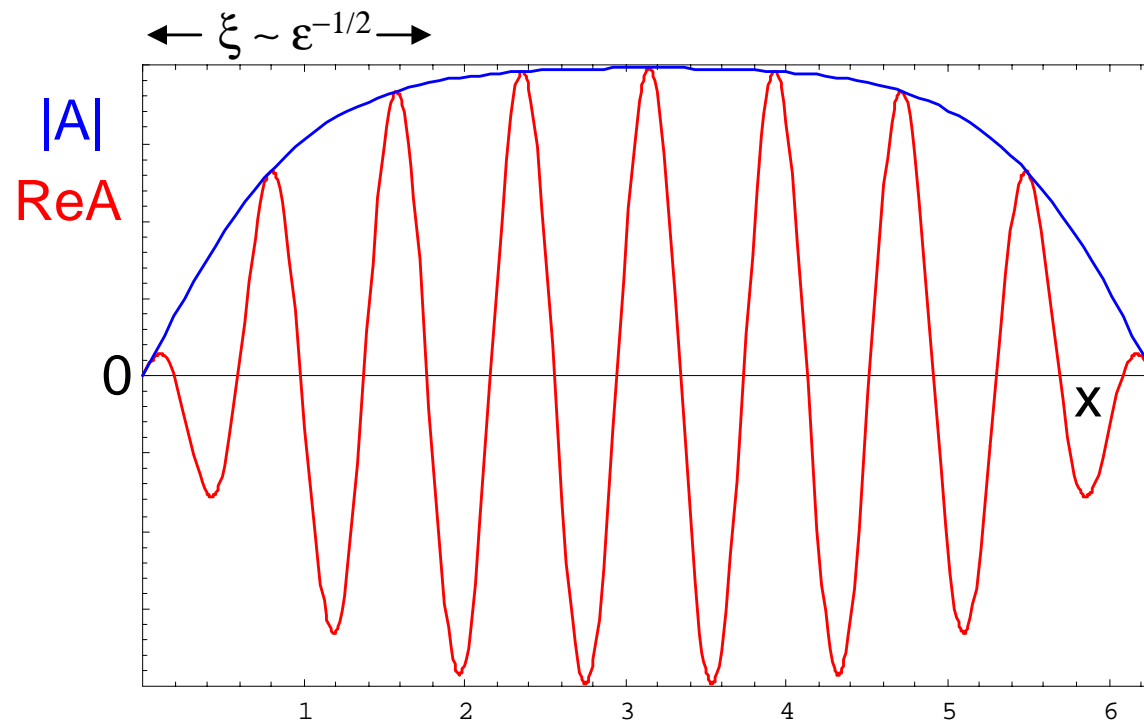


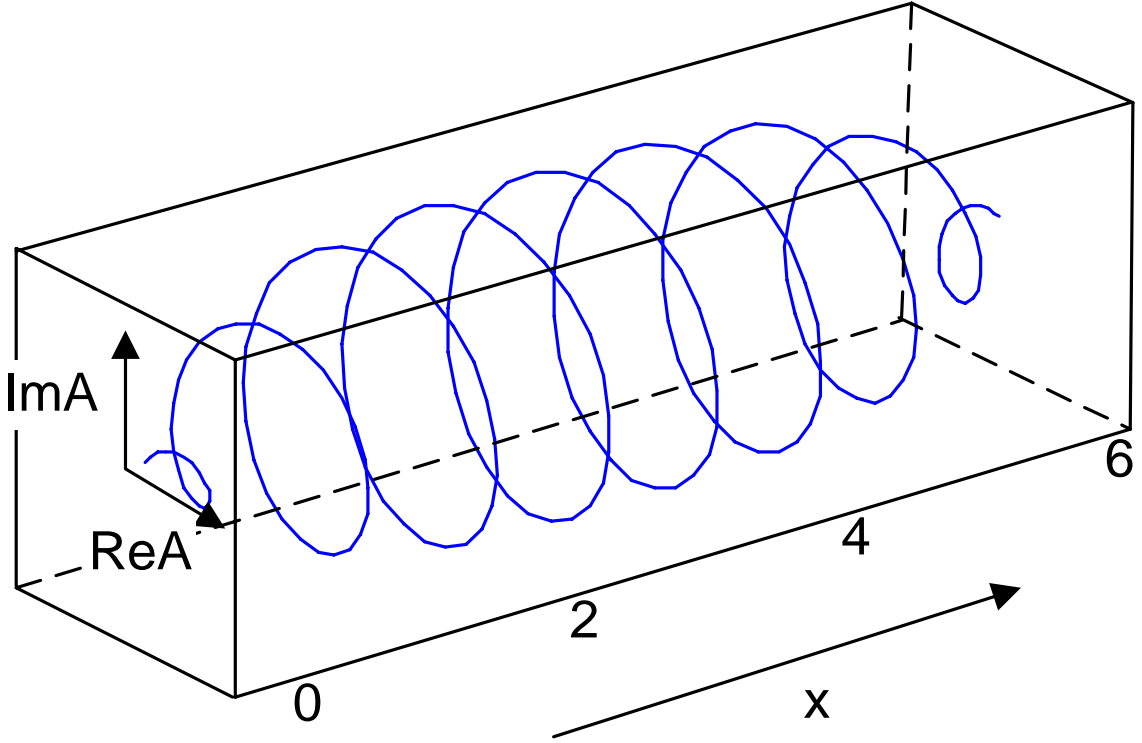
From Morris et al. (1991) and Mao et al. (1996)



Mao et al. (1996)

Two sidewalls





Conclusions

In today's lectures I introduced some of the basic ideas of pattern formation:

- linear instability at nonzero wave number;
- nonlinear saturation;
- stability balloons.

I then introduced the amplitude equation which is the simplest theoretical approach that captures the key effects in pattern formation (growth, saturation, and dispersion).

I focussed on the equation in one dimension, and on a phenomenological derivation. You can find more technical aspects in the supplementary notes.

Next lecture: the role of continuous symmetries — rotation and translation