Hyperbolicity of second-order in space systems of evolution equations

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Motivation

- ADM, BSSN, etc $\left\{ \begin{array}{l} \partial_t g \sim \partial g, K \\ \partial_t K \sim \partial K, \partial\partial g \end{array} \right.$
  GC 90s, Seidel

- 1st order symm. hyp. systems
  - well-posed IBVP
  - standard numerics (shift)
  - characteristic variables + multipatch

- Friedrich, Reula, York ... 1990s
- KST 2001, ...
- LSU 2002
  - ill-posedness cannot be ignored
  - proved well-posedness of (one form) of BSSN by reduction

- Use 1st order reduction for numerics? Probably not!
- Use 1st order system that is not a reduction? Maybe.

1st order system $\dot{u} = P^i(u)\, u_{,i} + S(u)$, u vector of variables

check first $\boxed{\dot{u} = P^i\, u_{,i}}$ with P^i constant square matrix (1)

Def: (1) symm. hyp. :$\Leftrightarrow$ $HP^i = (HP^i)^+$, $H = H^+$, $H > 0$

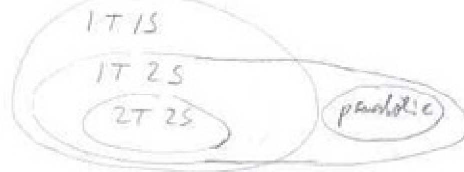
motivation $\epsilon \equiv u^+ H u$, $\phi^i \equiv u^+ H P^i u \Rightarrow \dot{\epsilon} = \phi^i{}_{,i} \Rightarrow \frac{d}{dt}\int_\Omega \epsilon\, dV = \int_{\partial\Omega} \phi^i dS_i$

Def: (1) strongly hyp. :$\Leftrightarrow$ $P^n \equiv P^i n_i = S^{-1}\Lambda S$, Λ real, diag, $S(n)$, $S^{-1}(n)$ bounded

motivation $U \equiv Su \Rightarrow \dot{U} = \Lambda\, \partial_n U$ + transv. derivs. (char. vars.)

Consider systems $u=(v,w)$

$$\left\{\begin{array}{l} \dot v = A_1^{\,i}\, v_{,i} + A_2\, w \quad + A_1 v \\ \dot w = B_1^{\,ij}\, v_{,ij} + B_2^{\,i}\, w_{,i} \quad + D_1^{\,i}\, v_{,i} + B_1 v + B_2 w \end{array}\right. \qquad (2)$$



$\Leftrightarrow$ reducible to 1st order (with $u=(v,w,d_i\equiv v_{,i})$)

$\Leftarrow$ ($\not\Rightarrow$) $\ddot v = \partial\partial v + \partial\dot v +$ l.o. (with $w\equiv\dot v$)

Many inequivalent reductions because $v_{,ij}\to d_{i,j} = d_{j,i}$. Parametrise!

Auxiliary constraints $c_i \equiv d_i - v_{,i}$, $c_{ij}\equiv d_{i,j}-d_{j,i}$ obey closed auxiliary system.

Def: (2) $\left.\begin{array}{c}\text{strongly}\\ \text{symm.}\end{array}\right\}$ hyp. $:\Leftrightarrow$ $\exists$ $\left.\begin{array}{c}\text{strongly}\\ \text{symm.}\end{array}\right\}$ hyp. reduction (1) with $u=(v,w,d_i)$

Task: Find necessary and sufficient criteria in terms of $A_1^{\,i}, A_2, B_1^{\,ij}, B_2^{\,i}$ alone.

Benefit: use 2nd order formulations analytically and numerically

Thm 1 (2) strongly hyp $\Leftrightarrow$ $\mathcal{A} \equiv \begin{pmatrix} B_2^{\,n} & B_1^{\,nn} \\ A_2 & A_1^{\,n} \end{pmatrix} = S^{-1} \Lambda S$ for all n;

Λ real, diagonal, $S(n)$, $S^{-1}(n)$ bounded

Thm 2 (2) symm. hyp. $\Leftrightarrow$ $\mathcal{H}\mathcal{A} = (\mathcal{H}\mathcal{A})^{*}$ for all n;

$$\mathcal{H} \equiv \begin{pmatrix} K & L^{n} \\ L^{+n} & M^{nn} \end{pmatrix} \quad \text{for some} \quad H \equiv \begin{pmatrix} K & L^{j} \\ L^{+m} & M^{mj} \end{pmatrix} > 0$$

Note $\epsilon = u^{+}Hu = v^{+}v + w^{+}Kw + (w^{+}L v_{,j} + v_{,m}^{+} L^{+m} w) + v_{,m}^{+} M^{mj} v_{,j}$

Idea of Thm 1

2+1 split $\partial_i = (\partial_n, \partial_A)$, $d_i = (d_n, d_A)$

$$\partial_t \begin{pmatrix} w \\ d_n \\ v \\ d_A \end{pmatrix} = \mathcal{P}\, \partial_n \begin{pmatrix} w \\ d_n \\ v \\ d_B \end{pmatrix} + \partial_A \dots \qquad \mathcal{P} \equiv \begin{pmatrix} A & B \\ 0 & \mathcal{E} \end{pmatrix}$$

$\mathcal{P}$ diagonalisable $\Longrightarrow$ $A, \mathcal{E}$ diagonalisable

" $\Longleftarrow$ " and $A, \mathcal{E}$ have no common eigenvalues

$\exists$ reduction with

$$\mathcal{P} u = \begin{pmatrix} A & \begin{matrix} 0 & * \\ 0 & * \end{matrix} \\ 0 & \begin{matrix} 0 & 0 \\ 0 & i\mu \epsilon_A{}^B \end{matrix} \end{pmatrix} \begin{pmatrix} w \\ d_n \\ v \\ d_B \end{pmatrix}$$

μ a parameter

$\mathcal{E}$ is diagonalisable with eigenvalues $\pm\mu$. Choose μ large enough,

$\mathcal{P}$ diag. $\Longleftrightarrow$ A diag

Idea of Thm 2

HP^i hermitian on $u \equiv (v, w, d_i)$

$\Longleftrightarrow$ restriction of HP^n to (w, d_n) hermitian for all n_i

same i

$\Longrightarrow$ obvious, $\Longleftarrow$ check each block in $HP^i = \begin{pmatrix} * & * \\ * & * \end{pmatrix}\begin{pmatrix} w \\ d_i \end{pmatrix}$ in turn and solve for some of the reduction parameters

Note $\quad \epsilon(v, w, d_i) = \epsilon(v, w, v_{ii})$