Formation of singularities for nonlinear wave equations

(equivariant wave maps and Yang-Mills equations)

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Outline:

- Introduction; scaling and regularity
- Supercritical case; self-similar blowup
- Critical case; concentration of energy and static solutions

Motto:

In dealing with mathematical problems, specialization plays, as I believe, a still more important part than generalization. Perhaps in most cases where we seek in vain the answer to a question, the cause of the failure lies in the fact that problems simpler and easier than the one in hand have been either not at all or incompletely solved. All depends, then, on finding out these easier problems, and on solving them by means of devices as perfect as possible and of concepts capable of generalization.

David Hilbert (1900)

Wave maps

(M, g) spacetime M with metric g,
(N, G) Riemannian manifold N with metric G.
Map $U : M \to N$ is a wave map if it extremizes the action

$$S(U) = \int_M G_{AB} \frac{\partial U^A}{\partial x^a} \frac{\partial U^B}{\partial x^b} g^{ab} dv_g.$$

$$\Box_g U^A + \Gamma^A_{BC}(U) \frac{\partial U^B}{\partial x^b} \frac{\partial U^C}{\partial x^c} g^{bc} = 0.$$

Let $M = \mathbb{R}^{N+1}$ and $N = S^N$

$$g = -dt^2 + dr^2 + r^2 d\omega^2_{N-1}, \quad G = du^2 + \sin^2 u \, d\Omega^2_{N-1}.$$

Equivariant ansatz:

$$u = u(t, r), \quad \Omega = \omega$$

Then

$$u_{tt} = u_{rr} + \frac{N-1}{r} u_r - \frac{N-1}{2r^2} \sin(2u)$$

Yang-Mills equations in $n+1$ dimensions

$A_\alpha : \mathbb{R}^{n+1} \to g$ Lie algebra of a Lie group G

Let $G = SO(n)$, so $g = so(n)$

A_α – $n \times n$ skew-symmetric matrices

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]$$

$$\partial_\alpha F^{\alpha\beta} + [A_\alpha, F^{\alpha\beta}] = 0$$

Spherically-symmetric ansatz

$(i, j = 1, ..n,\ \alpha = 0, 1, ...n)$

$$A_\alpha^{ij}(x) = \left(\delta_\alpha^i x^j - \delta_\alpha^j x^i\right) \frac{1 - u(t,r)}{r^2}$$

Then

$$u_{tt} = u_{rr} + \frac{n-3}{r} u_r + \frac{n-2}{r^2} u(1 - u^2)$$

Effective spatial dimension $N = n - 2$

($5+1$ Yang-Mills as a toy-model for $3+1$ Einstein)

Semilinear radial wave equations

$$u_{tt} - \Delta_r u + \frac{f(u)}{r^2} = 0 \qquad (1)$$

$u = u(t, r)$, $\Delta_r = \partial_{rr} + \frac{N-1}{r}\partial_r$

$$f(u) = \begin{cases} \frac{N-1}{2}\sin(2u) & \text{wave maps } R^{N+1} \to S^N \\ Nu(u^2-1) & \text{Yang-Mills in } R^{(N+2)+1} \end{cases}$$

Central question: can solutions starting from smooth initial data

$$u(0, r) = f(r), \quad u_t(0, r) = g(r)$$

become singular in future?

Equation (1) is scale invariant:

$$u(t,r) \to u_\lambda(t,r) = u(t/\lambda, r/\lambda).$$

The conserved energy

$$E(u) = \int_0^\infty \left[u_t^2 + u_r^2 + \frac{F(u)}{r^2} \right] r^2 dr, \qquad F'(u) = f(u).$$

scales as

$$E[u_\lambda] = \lambda^{N-2} E[u].$$

Classification:

$N = 1$ subcritical (expect global regularity)

$N = 2$ critical (?)

$N \geq 3$ supercritical (expect singularities)

Supercritical case $N = 3$

Self-similar solutions

$$u(t,r) = U(\rho), \qquad \rho = \frac{r}{T-t}$$

Boundary value problem inside the past light cone of the singularity ($0 \leq \rho \leq 1$)

$$U'' + \frac{2}{\rho}U' - \frac{f(U)}{\rho^2(1-\rho^2)} = 0$$

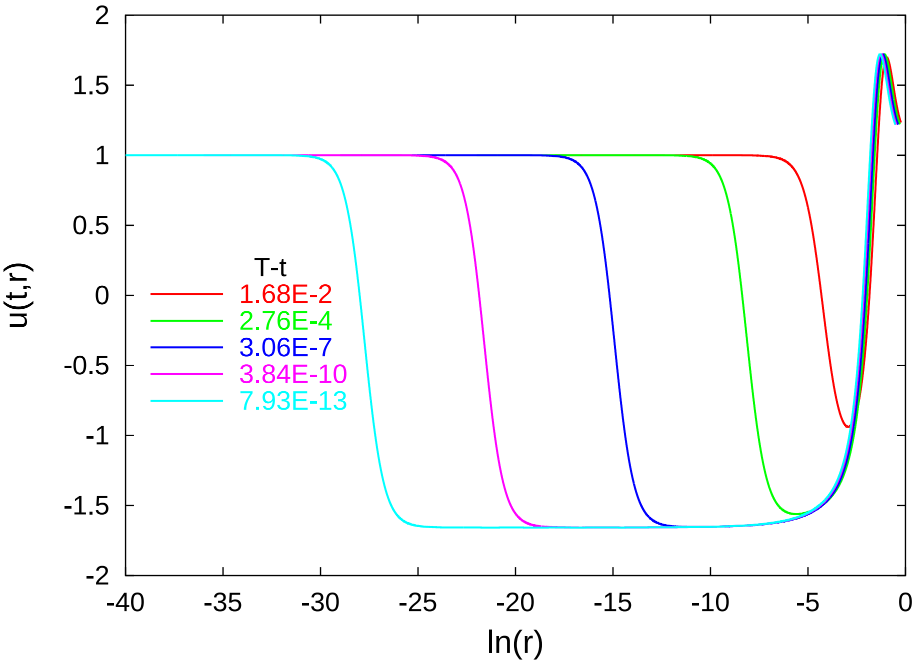
with regularity conditions at the endpoints.

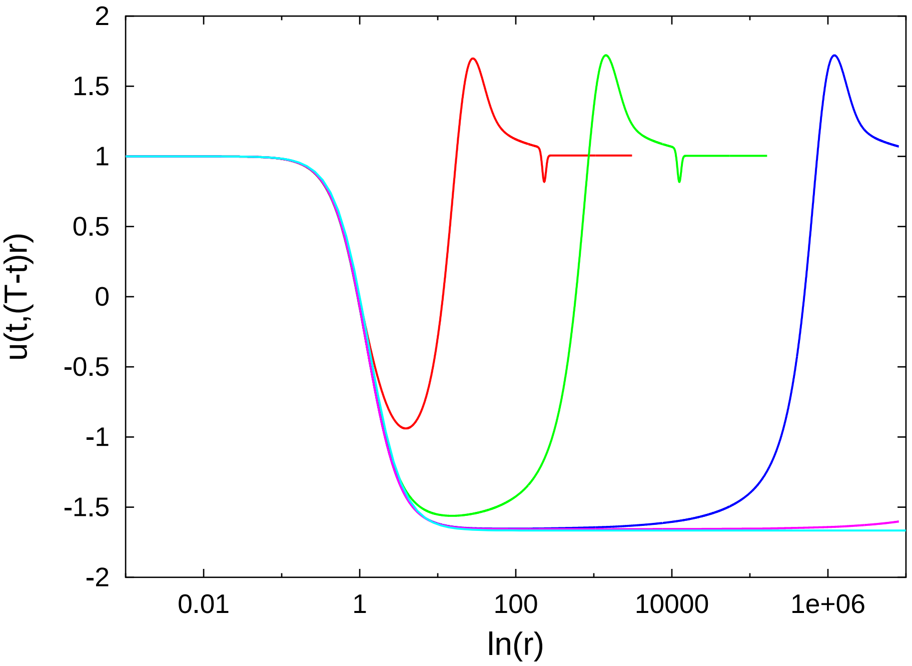
Theorem 1. *(PB, 2002) There exists a countable family of self-similar solutions U_n which are analytic in the interval $0 \leq \rho \leq 1$. The index $n = 0, 1, 2, \ldots$ denotes the number of oscillations of $U_n(\rho)$. Moreover, the solution U_n has exactly n unstable modes.*

Each self-similar solution U_n is an example of a singularity which develops from smooth initial data. The $n = 0$ solutions are known explicitly

$$U_0(\rho) = \begin{cases} 2\arctan(\rho) & \text{wave maps (Shatah 1988)} \\ \frac{1-\rho^2}{1+\frac{3}{5}\rho^2} & \text{Yang-Mills (PB 2002)} \end{cases}$$

Blowup for Yang-Mills in $5+1$ dimensions





PB&Tabor, 2001

Conjecture 1 (On supercitical blowup in $N = 3$). *The solutions of equation*

$$u_{tt} - u_{rr} - \frac{2}{r} u_r + \frac{f(u)}{r^2} = 0$$

corresponding to sufficiently large initial data do blow up in finite time. The asymptotic profile of blowup is given (generically) by the stable self-similar solution:

$$\lim_{t \to T} u(t, (T-t)r) = U_0(r).$$

Idea of proof: dissipation by dispersion

Define $\tau = -\ln(T-t)$ to get

$$u_{\tau\tau} + u_\tau + 2\rho u_{\rho\tau} = (1-\rho^2)\left(u_{\rho\rho} + \frac{2}{\rho} u_\rho\right) + \frac{f(u)}{\rho^2}$$

Is there a quasilocal Lyapunov functional?

Asymptotics of blowup
(linear stability of the self-similar solutions)

$$u_{\tau\tau} + u_\tau + 2\rho\, u_{\rho\tau} - (1-\rho^2)(u_{\rho\rho} + \frac{2}{\rho} u_\rho) + \frac{f(u)}{\rho^2} = 0.$$

Linear perturbations about U_n

$$u(\tau,\rho) = U_n(\rho) + w(\tau,\rho)$$

$$w_{\tau\tau} + w_\tau + 2\rho w_{\rho\tau} - (1-\rho^2)(w_{\rho\rho} + \frac{2}{\rho} w_\rho) + \frac{f'(U_n)}{\rho^2} w = 0.$$

Substitute $w(\tau,\rho) = e^{\lambda\tau} v(\rho)/\rho$

$$-(1-\rho^2)v'' + 2\lambda\rho v' + \lambda(\lambda-1)v + \frac{V(U_n)}{\rho^2} v = 0.$$

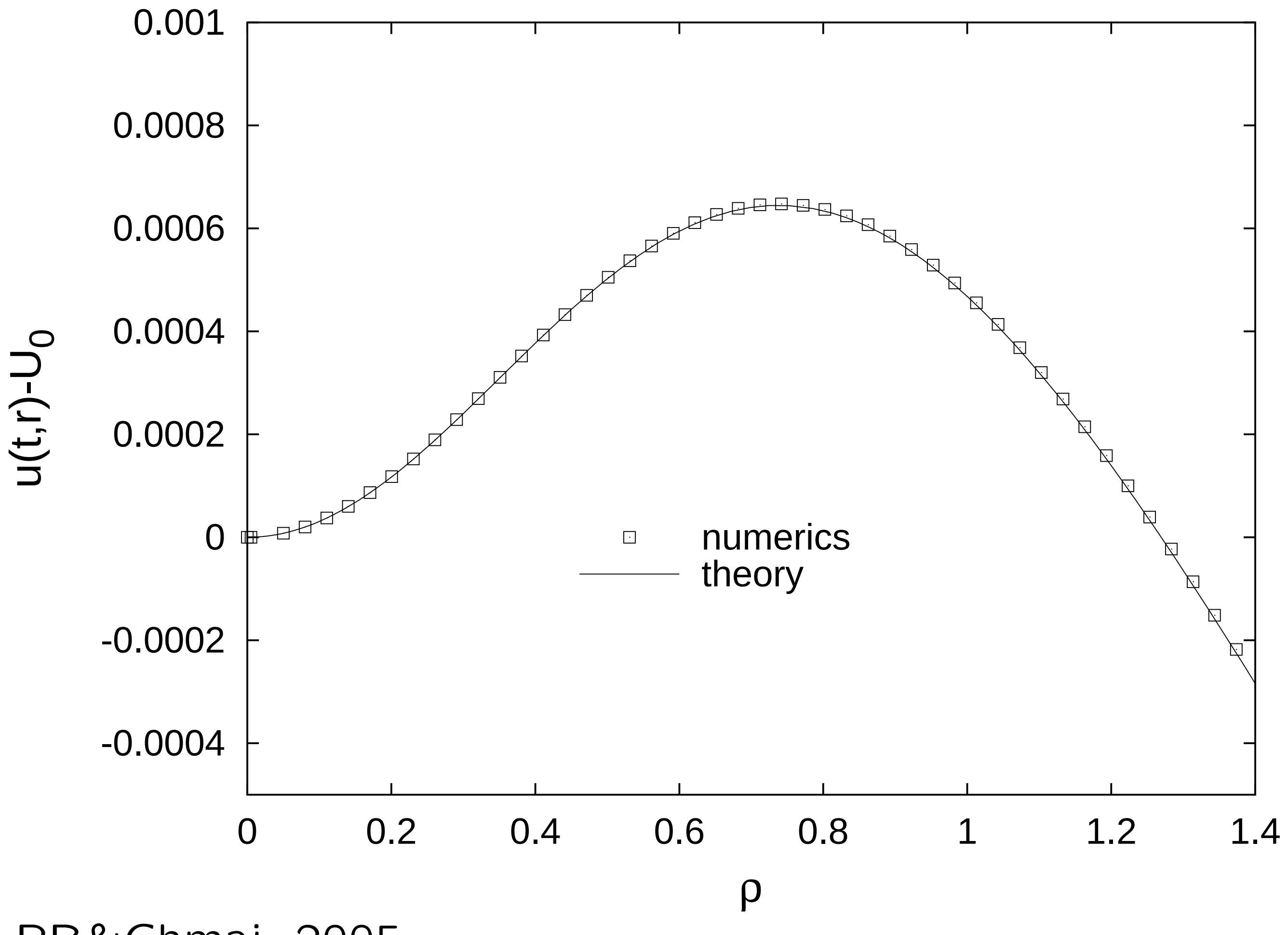
$$V(U_0) = \begin{cases} \frac{2(1-6\rho^2+\rho^4)}{(1+\rho^2)^2} & \text{WM} \\ \\ \frac{6(25-90\rho^2+33\rho^4)}{(5+3\rho^2)^2} & \text{YM} \end{cases}$$

Eigenvalues

n	λ_n (Yang-Mills)	λ_n (wave maps)
0	1	1
1	-0.588904	-0.543359
2	-2.181597	-2
3	-3.570756	-3.398472
4	-5.043294	-4.765254
5	-6.486835	-6.102056
6	-7.912777	-7.296764
	...	...

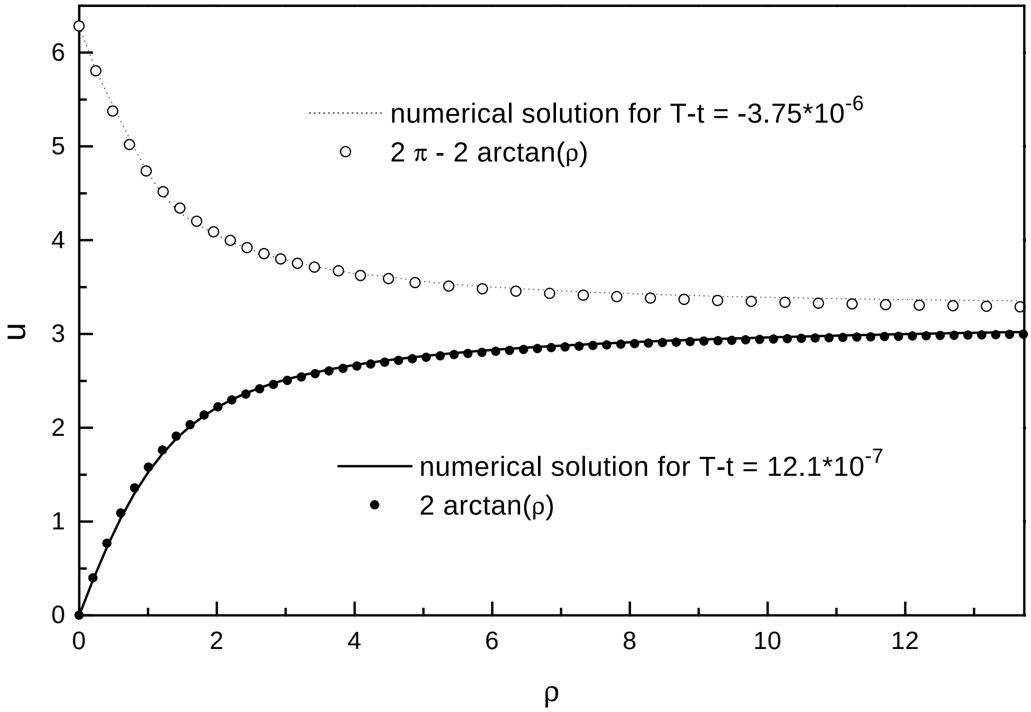
Convergence to the self-similar solution for $t \nearrow T$

$$u(t,r) = U_0(\rho) + \sum_{k=1} c_k e^{\lambda_k \tau} v_k(\rho) \sim U_0(\rho) + c_1 e^{\lambda_1 \tau} v_1(\rho)$$



PB&Chmaj, 2005

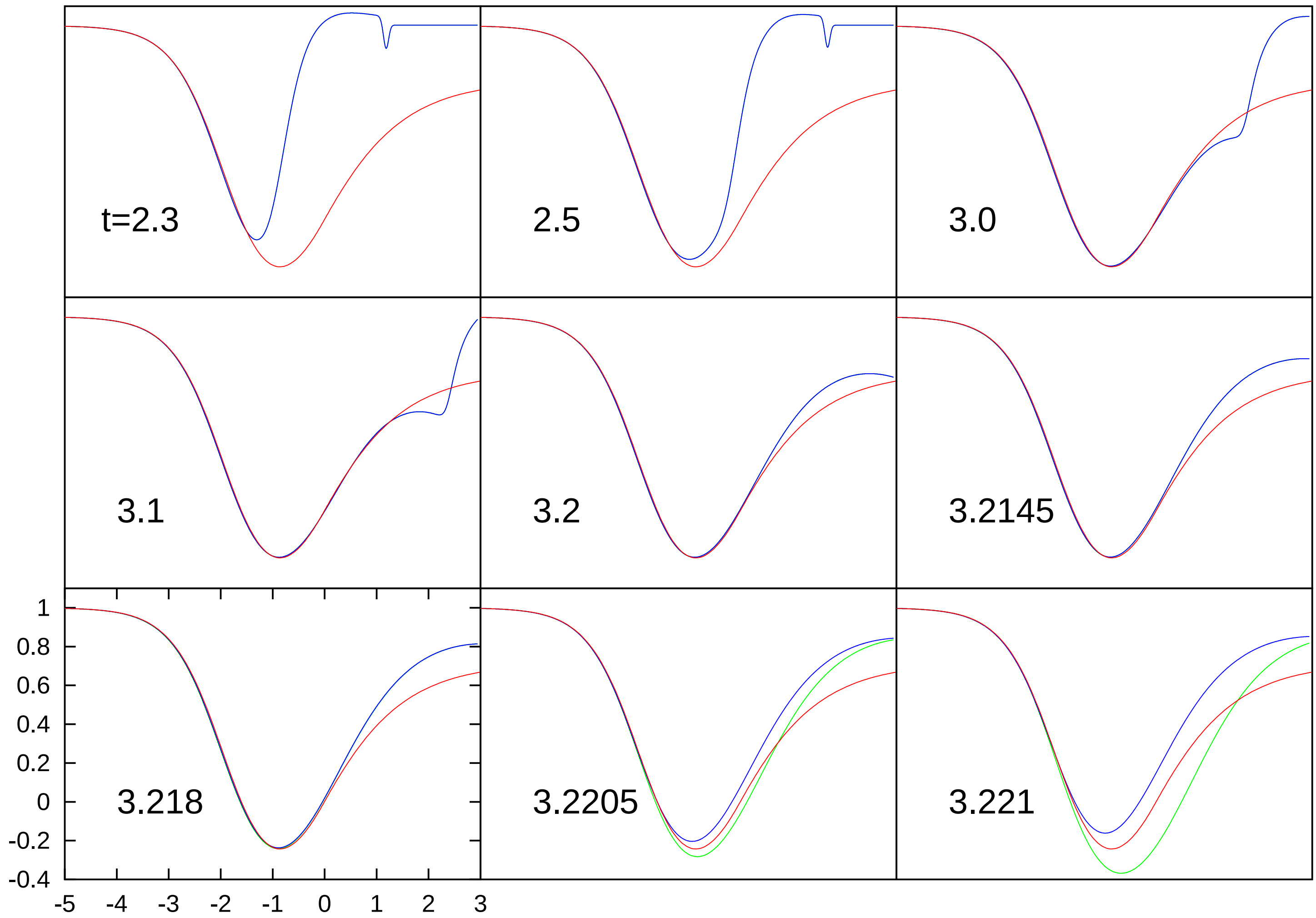
Continuation beyond blowup



- The self-similar solution U_0 can be continued analytically into the future light cone of the singularity.

- Speculation: After the blowup the solution immediately recovers smoothness and remains smooth until the next blowup occurs. After a finite number of blowups (at times $T_1 < T_2 < \cdots < T_n$) *any* solution eventually disperses.

Threshold for blowup (for $5+1$ Yang-Mills)



$A = A^* - 10^{-15}$

$A = A^* + 10^{-15}$

U_1 - critical solution

PB&Tabor, 2002
(similar picture for 3+1 wave maps, PB,Chmaj&Tabor, 2000; Hirschmann,Isenberg&Liebling, 2000)

Critical phenomena near the threshold

Intermediate asymptotics

$$u(t,r) \sim U_1(\rho) + c_1(A)(T-t)^{-\lambda_1} v_1(\rho) + \text{decaying modes}$$

v_1 - the unstable mode with positive eigenvalue λ_1

Scaling law for marginally subcritical data:

$$\max\{\text{energy density}(t,0)\} \sim (A^* - A)^{-4/\lambda_1}$$

- Lesson: Critical phenomena at the threshold for singularity formation (universality, self-similarity, scaling), originally observed for Einstein's equations, seem to be robust features of supercritical evolution equations.

- Advantage: Better analytic insight - proof of existence of the critical solution and its single-mode instability.

Critical case $N = 2$

- Rigorous results: For all "reasonable" targets the wave map equation is globally well-posed for small energy initial data (Tao 2001, Tataru 2003). A similar result is believed to hold (but not proved yet) for $4 + 1$ Yang-Mills eqs.

- What happens for large energy data?

- Small energy global regularity would imply large energy global regularity if the energy could not concentrate. Can energy concentrate?
  - For equivariant $2 + 1$ wave maps the energy cannot concentrate for convex (Shatah&Tahvildar-Zadeh 1994) and certain non-convex targets (Grillakis)
  - It seems that the possibility of energy concentration is intimately related to the existence of static solutions.

Static solutions

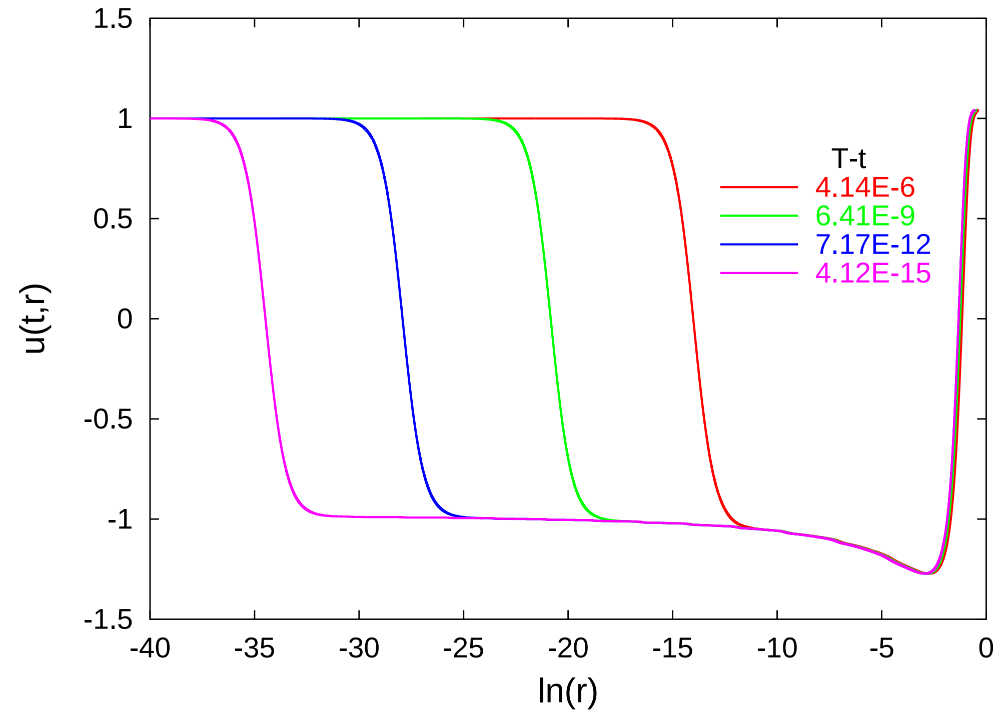
In critical models scaling invariance does not exclude static solutions.

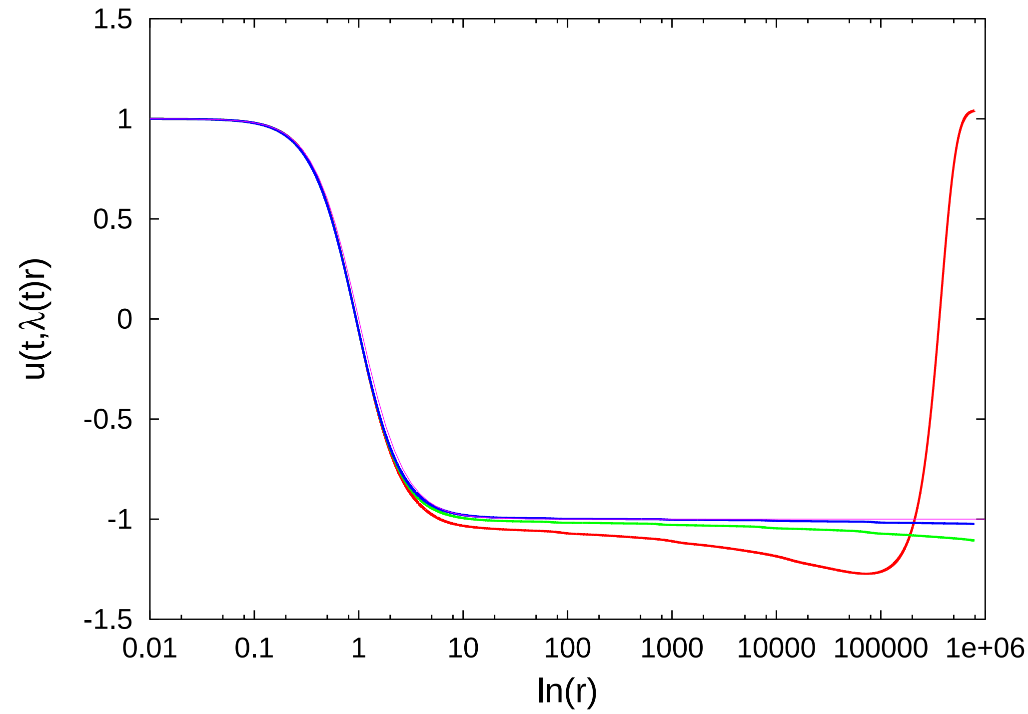
$$u_S(r) = \begin{cases} 2\arctan(r) & \text{harmonic map } R^2 \to S^2 \\ \frac{1-r^2}{1+r^2} & \text{Yang-Mills instanton in } R^4 \end{cases}$$

- By scaling invariance, the static solution has no scale: $u_S^\lambda(r) = u_S(r/\lambda)$ is also a solution with the same energy.

- These static solutions are marginally stable.
  Proof: The zero mode
  $$v_0 = \frac{d}{d\lambda} u_S(r/\lambda)\Big|_{\lambda=1} = -r u_S'(r)$$
  has no nodes.

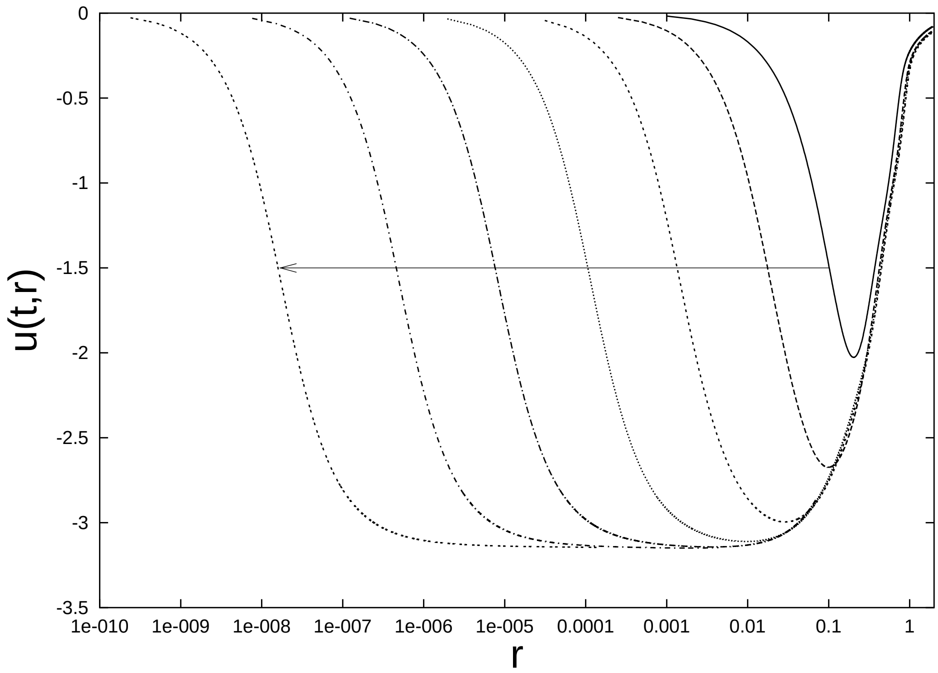
Blowup for Yang-Mills in $4+1$ dimensions

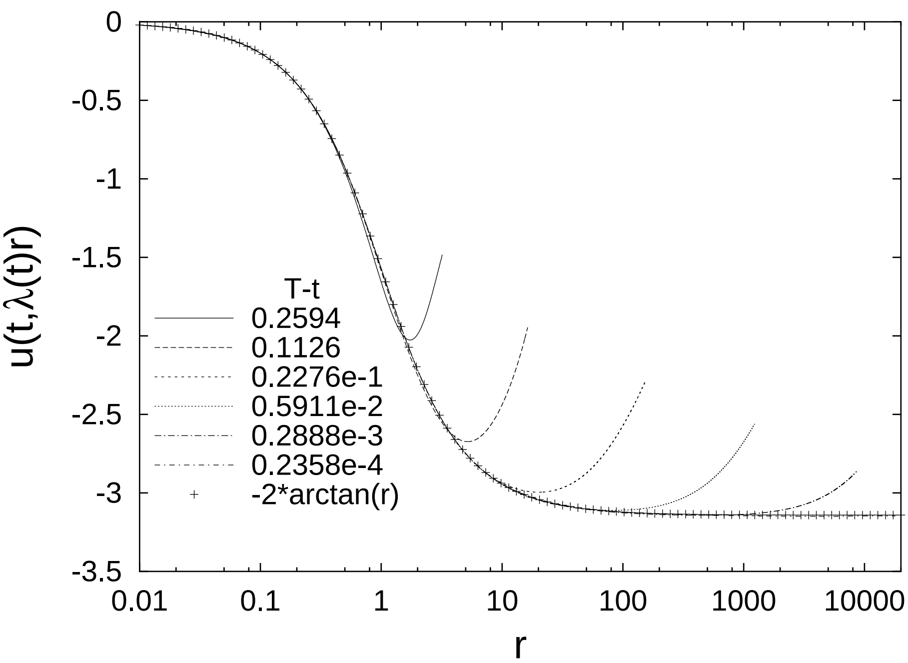




PB&Tabor 2002

Blowup for $2+1$ wave maps into S^2





PB,Chmaj&Tabor 2000; Isenberg&Liebling 2000

Conjecture 2 (On critical blowup). *Large energy solutions do blow up in finite time. The asymptotic profile of blowup is given by the static solution. More precisely, there exists a positive function* $\lambda(t) \to 0$ *for* $t \to T$ *such that*

$$\lim_{t\to T} u(t, \lambda(t)r) = u_S(r).$$

Moreover, $\dot{\lambda}(t) \to 0$ *as* $t \to T$.

- How can we understand this?

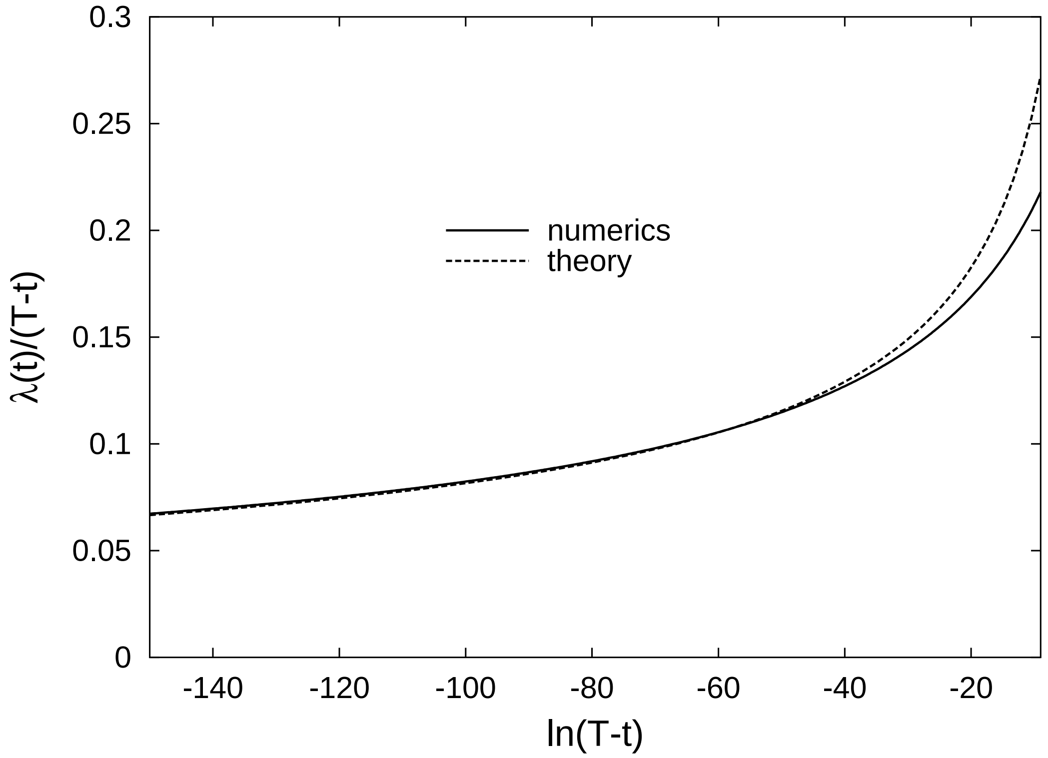
- What is the rate of blowup?

Modulation equation for Yang-Mills
(PB, Sigal, Ovchinnikov 2003)

$$\lambda \ddot{\lambda} = \frac{3}{4} \dot{\lambda}^4$$

The asymptotic solution for $t \to T$

$$\lambda(t) \sim \sqrt{\frac{2}{3}} \frac{T-t}{\sqrt{-\ln(T-t)}}$$



An analogous calculation for wave maps is harder
(Sigal et al., in preparation)

Energy concentration

For solutions which blowup we define the kinetic and the potential energy at time $t < T$ inside the past light cone of the singularity

$$E_K(t) = \int_0^{T-t} u_t^2 r dr, \quad E_P(t) = \int_0^{T-t} \left(u_r^2 + \frac{(1-u^2)^2}{r^2} \right) r dr.$$

Substituting $u = u_S(r/\lambda(t))$ into this we get

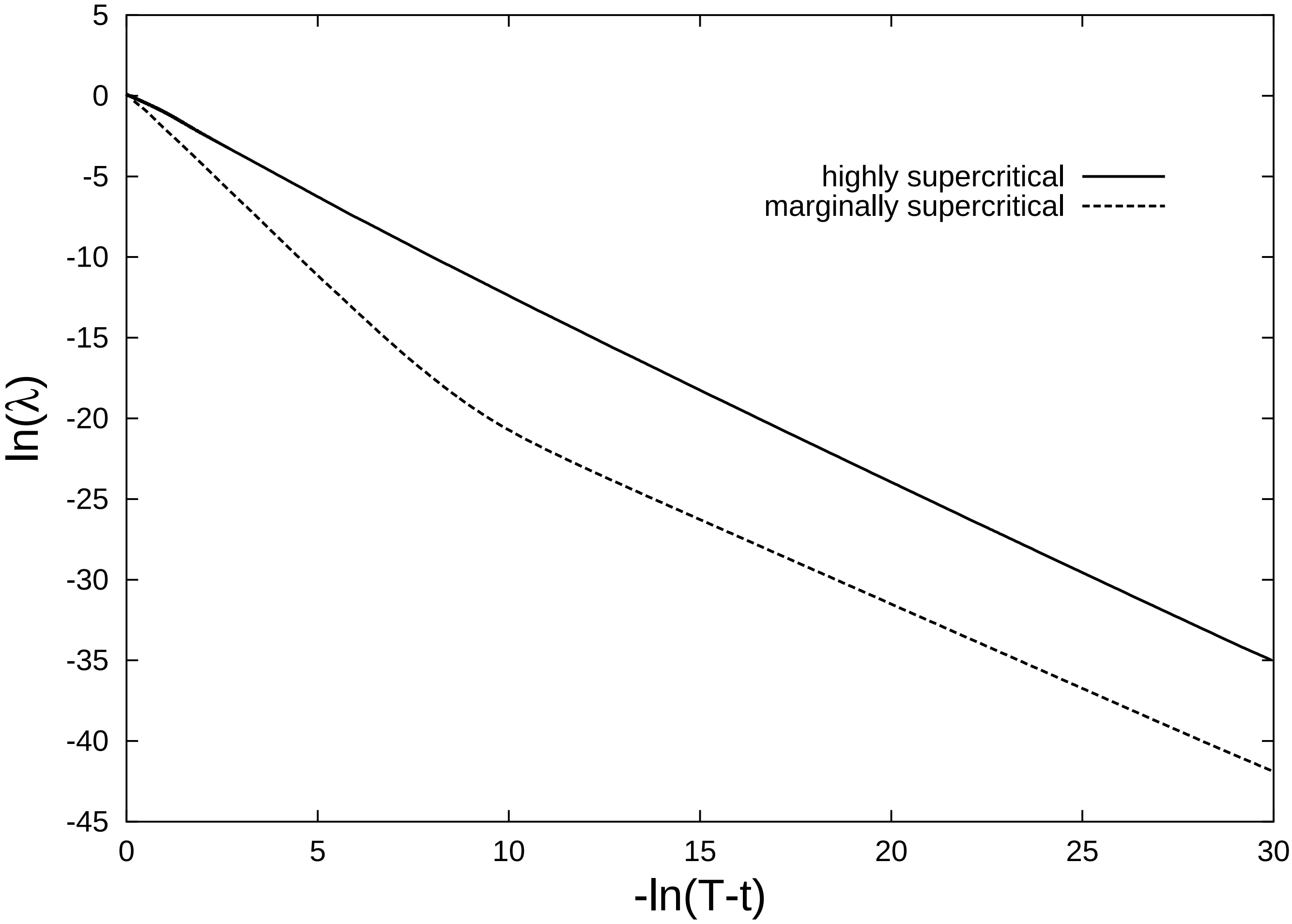
$$E_K(t) = \dot{\lambda}^2 \int_0^{\frac{T-t}{\lambda(t)}} {u_S'}^2 \, r^3 dr,$$

$$E_P(t) = \int_0^{\frac{T-t}{\lambda(t)}} \left({u_S'}^2 + \frac{(1-u_S^2)^2}{2r^2} \right) r dr.$$

Thus (assuming that $\lambda(t)/(T-t) \to 0$ for $t \to T$)

$$\lim_{t \nearrow T} E_K(t) = 0, \quad \lim_{t \nearrow T} E_P(t) = E(u_S).$$

Threshold for blowup in $N = 2$



There is no critical solution
in the critical dimension!

Remarks:

- Impact on mathematical studies
  - Helps to put research on the right track:
    Example: Until recently it was believed that Yang-Mills equations in $4+1$ dimensions are globally regular
  - Provides insight:
    Example: Theorem (Struwe 2003)
    If a solution of the $2+1$ equivariant wave map equation blows up, then it does so in a manner described in Conjecture 2.
    Corollary: Global regularity for targets which do not admit nonconstant static solutions (harmonic maps).

- Reliability of numerics