One-dimensional Spatiotemporal Chaos

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Kuramoto-Sivashinsky equation

Kuramoto-Sivashinsky (KS) equation:

\[ u_t + u_{xxxx} + u_{xx} + uu_x = 0 \]

\( x \in [0, L] \), periodic BCs
Overview

- Kuramoto-Sivashinsky (KS) equation:
  - introduction
  - wavelet analysis and scale-by-scale structure
- Large-scale contributions to dynamics
  - effective large-scale stochasticity
- (De)stabilized KS equation
  - dissipativity and analyticity
  - viscous shock solution
  - scaling of bounds on the attractor
- Nikolaevskii equation
  - amplitude equations
  - scaling regimes
Recent Developments: Absorbing Ball for KS

Bronski & Gambill (2005, in preparation):
- Lyapunov-type argument with improved construction of gauge function $\phi$ to prove

$$\limsup_{t \to \infty} ||u||^2 \leq K L^3$$

- Demonstrate optimality of exponent by this approach

Giacomelli & Otto (2005):
- Improved bound for KS:

$$\limsup_{t \to \infty} ||u||^2 \leq o(L^3)$$

- Introduce new approach: on large spatial scales, KS solution behaves like entropy solution of Burgers equation.
Nikolaevskii equation

\[ u_t = -\partial_x^2 [r - (1 + \partial_x^2)^2]u - uu_x \]

- Model for short-wave instabilities with continuous symmetry:
  - Shift symmetries \( t \mapsto t + c, \ x \mapsto x + c \)
  - Reflection \( x \mapsto -x, \ u \mapsto -u \)
  - Galilean invariance \( u \mapsto u + V, \ x \mapsto x - Vt \)
- Dispersion relation: \( \gamma(k) = k^2 [r - (1 - k^2)^2] \)

(damped) KS

Nikolaevskii

- All solutions decay for \( r \leq 0 \): write \( r = \varepsilon^2 \)
- \( L^2 \) absorbing ball and analyticity (following method of Collet et al.; non-optimal in \( \varepsilon L \)):

\[
\limsup_{t \to \infty} \|u\|^2 \leq K \varepsilon^{38/7} L^{24/7} = K \varepsilon^2 (\varepsilon L)^{24/7}
\]
Roll solutions

- Band of unstable modes width $O(\varepsilon)$: $|k - 1| < \frac{\varepsilon}{2} (1 + \text{h.o.t.})$
- Maximum Fourier mode growth rate $\omega = \varepsilon^2$ at $k = 1$
- Expect slow length scale $X = \varepsilon x$, time scale $T = \varepsilon^2 t$.

- No. of unstable modes $\frac{\varepsilon L}{2\pi}$ — need $\gg 1$ to observe large-system limit

Rolls:
- Steady $O(\varepsilon)$ roll solutions via weakly nonlinear analysis: for $|q| < 0.5$

$$u_r(x) = \varepsilon u_1(x) + \varepsilon^2 u_2(x) + \ldots$$

$$= 6\varepsilon \sqrt{1 - 4q^2} e^{i(1+\varepsilon q)x} + \text{c.c.} + O(\varepsilon^2)$$

- All roll solutions unstable (Tribelsky & Velarde 1996, Matthews & Cox 2000)
- Direct transition to STC as $r = \varepsilon^2$ increases through 0 (Tribelsky & Tsuboi 1996)
- “soft-mode turbulence” / “Nikolaevskii chaos” - a new type of spatiotemporal chaos? (Tanaka 2005)
KS-like behaviour for “large” $\varepsilon^2$

\[ \varepsilon^2 = 1.0 \quad \varepsilon^2 = 0.2 \]

Spatial cross-section and evolution

Power spectra

Nikolaevskii
$\varepsilon^2 = 0.5$

Kuramoto-Sivashinsky

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Separation of scales for small $\varepsilon^2$

Nikolaevskii equation:

\[ L = 1600, \quad \varepsilon^2 = 0.04 \]

Power spectra

Spatial cross-section and evolution
Amplitude equations I

- Couple amplitude of small-scale pattern ($k = 1$ mode) with Goldstone ($k = 0$) mode.
  - Slow scales $X = \varepsilon x$, $T = \varepsilon^2 t$
  - Ansatz: $u(x, t) = \varepsilon^\alpha A(X, T) e^{ix} + \text{c.c.} + \varepsilon^\beta f(X, T) + \text{h.o.t.}$

- Ginzburg-Landau scaling: $\alpha = 1$, $\beta = 1$:
  
  $$u(x, t) = \varepsilon A(X, T) e^{ix} + \text{c.c.} + \varepsilon f(X, T) + \text{h.o.t.}$$

  $$A_T = A + 4A_{XX} - \frac{|A|^2 A}{36} - i f A/\varepsilon - (f A)_X$$  

  $$(s\text{GL})$$

  $$f_T = f_{XX} - f f_X - |A|^2_X$$

- $1/\varepsilon$ term: inconsistent scaling
Amplitude equations I

- Couple amplitude of small-scale pattern ($k = 1$ mode) with Goldstone ($k = 0$) mode.
  - Slow scales $X = \varepsilon x$, $T = \varepsilon^2 t$
  - Ansatz: $u(x, t) = \varepsilon^\alpha A(X, T) e^{ix} + \text{c.c.} + \varepsilon^\beta f(X, T) + \text{h.o.t.}$

- Ginzburg-Landau (GL) scaling: $\alpha = 1$, $\beta = 1$:
  $$u(x, t) = \varepsilon B(X, T) e^{ix} + \text{c.c.} + \varepsilon g(X, T) + \text{h.o.t.}$$

$$B_T = B + 4B_{XX} - \frac{|B|^2 B}{36} - igB/\varepsilon - (gB)_X \quad \text{(sGL)}$$
$$g_T = g_{XX} - gg_X - |B|^2$$

- $1/\varepsilon$ term: inconsistent scaling
- reduce notational confusion: write $B, g$ for amplitudes in GL scaling
Amplitude equations II

- Couple amplitude of small-scale pattern ($k = 1$ mode) with Goldstone ($k = 0$) mode.
  - Slow scales $X = \varepsilon x$, $T = \varepsilon^2 t$
  - Ansatz: $u(x, t) = \varepsilon^\alpha A(X, T)e^{ix} + \text{c.c.} + \varepsilon^\beta f(X, T) + \text{h.o.t.}$
    \[
    U_1(X, T) \quad \text{and} \quad U_0(X, T)
    \]

- Matthews-Cox (MC) scaling (Matthews & Cox, 2000): $\alpha = \frac{3}{2}$, $\beta = 2$:
  \[
  u(x, t) = \varepsilon^{3/2} A(X, T)e^{ix} + \text{c.c.} + \varepsilon^2 f(X, T) + \text{h.o.t.}
  \]

\[
A_T = A + 4AXX - ifA \quad \text{(MC}\_1)
\]

\[
f_T = fXX - |A|^2_X
\]

- Consistent scaling: $\varepsilon$ scaled out
Amplitude equations II

- Couple amplitude of small-scale pattern \((k = 1\) mode) with Goldstone \((k = 0\) mode).
  - Slow scales \(X = \varepsilon x, T = \varepsilon^2 t\)
  - Ansatz: \(u(x, t) = \varepsilon^\alpha A(X, T) e^{ix} + \text{c.c.} + \varepsilon^\beta f(X, T) + \text{h.o.t.}\)
    \[
    \begin{align*}
    U_1(X,T) \quad & U_0(X,T)
    \end{align*}
    \]

- Matthews-Cox (MC) scaling (Matthews & Cox, 2000): \(\alpha = 3/2, \beta = 2\):
  \(\varepsilon^3 A(X, T)e^{ix} + \text{c.c.} + \varepsilon^2 f(X, T) + \text{h.o.t.}\)

\[
A_T = A + 4AXX - ifA
\]
\[
f_T = fXX - |A|^2_X
\]

(MC\(_1\))

- Consistent scaling: \(\varepsilon\) scaled out
- Exponentially growing spatially homogeneous solutions:
  for \(f = |A|_X = 0, A(X, T) = A(T) = A_0 e^T\)
  (unstable: spatial perturbations grow more rapidly)
- Incomplete description of dynamics: MC equations do not contain \(O(\varepsilon)\) roll solution
Scaling exponents: numerical investigation

Computations in progress (RW, K.F. Poon, D.J. Muraki)

- rms scaling of $u$: $\text{rms}(u) \sim \varepsilon^\gamma$ with $\gamma \approx 1.48$
- Extract $U_1 = \varepsilon^{\alpha} A$, $U_0 = \varepsilon^{\beta} f$ from $u$ using Fourier filters:
  - obtain $U_0$ from $\{\hat{u} | |k| < 0.5\}$; $U_1$ from $\{\hat{u} | |k - 1| < 0.5\}$

![Graphs showing scaling of $U_1$ and $U_0$.](image)

- scaling of $U_1$: $\alpha \approx 1.4$
- scaling of $U_0$: $\beta \approx 1.8$

- Fully developed chaotic regime appears consistent with MC scaling for $A$, less conclusively for $f$: corrections to scaling?
Higher-order amplitude equations

Ginzburg-Landau scaling: \( u = \varepsilon B(X, T)e^{ix} + \text{c.c.} + \varepsilon g(X, T) + \text{h.o.t.} \)

\[
B_T = B + 4B_{XX} - \frac{|B|^2B}{36} - igB/\varepsilon - (gB)_X
\]

\[
g_T = gxX - ggX - |B|^2_X
\]

Roll solutions \( B = 6\sqrt{1 - 4q^2}e^{iqX} \) have \( u = \mathcal{O}(\varepsilon) \): GL scaling
Higher-order amplitude equations

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B_T = B + 4B_{XX} - \frac{|B|^2 B}{36} - igB/\varepsilon - (gB)_X \\
g_T = g_{XX} - gg - |B|^2_X 
\]

roll solutions \( B = 6\sqrt{1 - 4q^2e^{iqX}} \) have \( u = \mathcal{O}(\varepsilon) \): GL scaling

via \( B = \varepsilon^{1/2}A, g = \varepsilon f \), equivalent to Matthews-Cox equations with nonlinear \( \mathcal{O}(\varepsilon) \) correction terms: \( u = \varepsilon^{3/2}A(X, T)e^{ix} + \text{c.c.} + \varepsilon^2 f(X, T) + \text{h.o.t.} \)

\[
A_T = A + 4A_{XX} - ifA - \varepsilon \left[ \frac{|A|^2 A}{36} + (fA)_X \right] \\
f_T = f_{XX} - |A|^2_X - \varepsilon ff_X 
\]
Higher-order amplitude equations

Ginzburg-Landau scaling: \( u = \epsilon B(X, T)e^{ix} + \text{c.c.} + \epsilon g(X, T) + \text{h.o.t.} \)

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g_T = g_{XX} - ggX - |B|_X^2
\]  
(sGL)

roll solutions \( B = 6\sqrt{1 - 4q^2e^{iqX}} \) have \( u = \mathcal{O}(\epsilon) \): GL scaling

via \( B = \epsilon^{1/2}A, g = \epsilon f \), equivalent to Matthews-Cox equations with nonlinear \( \mathcal{O}(\epsilon) \) correction terms: \( u = \epsilon^{3/2}A(X, T)e^{ix} + \text{c.c.} + \epsilon^2 f(X, T) + \text{h.o.t.} \)

full multiple-scale expansion to \( \mathcal{O}(\epsilon) \) yields additional linear corrections to MC equations

\[
A_T = A + 4A_{XX} - ifA - \epsilon \left[ \frac{|A|^2 A}{36} + (fA)_X \right] - 2i\epsilon[A_X + 6A_{XX}XX] \\
f_T = f_{XX} - |A|_X^2 - \epsilon ff_X
\]  
(MC\(\epsilon\))

stabilizing cubic term enters at \( \mathcal{O}(\epsilon) \); rolls have size \( \mathcal{O}(\epsilon^{-1/2}) \) in MC scaling, captured by MC equations with \( \mathcal{O}(\epsilon) \) corrections

uniform description of dynamics via higher-order amplitude equations?
Pointwise validity of amplitude equations

- Compare full Nikolaevskii evolution with \( u \) reconstructed from amplitude equations

![Solution of full Nikolaevskii equation: \( L = 2000, \varepsilon^2 = 0.001 \)](image)

\[
\text{Nikolaevskii} \\
L = 2000 \\
\varepsilon^2 = 0.001
\]

- Tracking via amplitude equations:

  - \( \mathcal{O}(1) \) Matthews-Cox model

  ![Solution of amplitude equations: Matthews-Cox scaling, \( \mathcal{O}(1) \)](image)

  \[
  \mathcal{O}(\varepsilon) \text{ corrections} \\
  \text{sGL & MC}_\varepsilon
  \]

  ![Solution of amplitude equations: Ginzburg-Landau scaling](image)
  ![Solution of amplitude equations: Matthews-Cox scaling, \( \mathcal{O}(\varepsilon) \)](image)

- In progress: pointwise comparison - compute residuals \( A_T - [A + 4A_{XX} - ifA], \)
  \( f_T - [f_{XX} - |A|^2_X] \). Are significant \( \mathcal{O}(\varepsilon) \) contributions spatially localized?
Transition between scaling regimes

Scaling regimes:

- **GL scaling** \( u = \mathcal{O}(\varepsilon) \):
  \[ u = \varepsilon B(X, T) e^{ix} + \text{c.c.} + \varepsilon g(X, T) + \text{h.o.t.} \text{ with } B, g = \mathcal{O}(1) \]

- **MC scaling** \( u = \mathcal{O}(\varepsilon^{3/2}) \):
  \[ u = \varepsilon^{3/2} A(X, T) e^{ix} + \text{c.c.} + \varepsilon^2 f(X, T) + \text{h.o.t.} \text{ with } A, f = \mathcal{O}(1) \]

Both regimes are relevant:

- Expect global attractor has diameter \( \mathcal{O}(\varepsilon) \)
- Roll solutions, \( \mathcal{O}(\varepsilon) \) scaling, lie on unstable submanifold
- Transient dynamics near rolls; instability and collapse to \( \mathcal{O}(\varepsilon^{3/2}) \) fully chaotic regime: “Nikolaevskii chaos”

Growth and collapse of roll solutions:

\[ L = 3200 \]
\[ \varepsilon^2 = 0.004 \]
Heuristic mechanism of roll destabilization

Work in GL scaling:

\[ u = \varepsilon B(X, T)e^{ix} + \text{c.c.} + \varepsilon g(X, T) + \text{h.o.t.} \]

\[ B_T = B + 4B x x - \frac{|B|^2 B}{36} - igB/\varepsilon - (gB)_x \]

\[ g_T = g x x - gg x - |B|^2 x \]
Heuristic mechanism of roll destabilization

GL scaling—separate amplitude, phase:

\[ u = \varepsilon r(X, T) e^{i\theta(X, T)} e^{ix} + \text{c.c.} + \varepsilon g(X, T) + \text{h.o.t.} \]

\[ r_T = (1 - 4\theta_X^2) r + 4r_x x - \frac{r^3}{36} - (gr)_x \]

\[ \theta_T = -g/\varepsilon + 4\theta_x x + 8\frac{r_x}{r} \theta_x - g\theta_x \]

\[ g_T = g xx - gg x - (r^2)_x \]
Heuristic mechanism of roll destabilization

GL scaling—separate amplitude, phase:

\[ u = \varepsilon r(X, T)e^{i\theta(X, T)}e^{ix} + \text{c.c.} + \varepsilon g(X, T) + \text{h.o.t.} \]

\[ r_T = (1 - 4\phi^2)r + 4r_X X - \frac{r^3}{36} - (gr)_X \]

\[ \phi_T = -g_X/\varepsilon + 4\phi_X X + 8 \left( \frac{r_X}{r} \phi \right)_X - (g\phi)_X \]

\[ g_T = g_X X - gg_X - (r^2)_X \]

only phase gradients dynamically relevant: write i.t.o. \( \phi = \theta_X \)
Heuristic mechanism of roll destabilization

GL scaling—separate amplitude, phase:

\[ u = \varepsilon r(X, T)e^{i\theta(X, T)}e^{ix} + \text{c.c.} + \varepsilon g(X, T) + \text{h.o.t.} \]

\[ r_T = (1 - 4\phi^2)r + 4r_x x - \frac{r^3}{36} - (gr)_x \]

\[ \phi_T = -g_x / \varepsilon + 4\phi_x x + 8 \left( \frac{r_x}{r} \phi \right)_x - (g\phi)_x \]

\[ g_T = g_x x - gg_x - (r^2)_x \]

- only phase gradients dynamically relevant: write i.t.o. \( \phi = \theta_x \)

- Roll solutions (\( f = r_x = \theta_x x = 0 \)):

\[ \phi = \theta_x = q, \quad r = 6\sqrt{1 - 4q^2} \quad \text{for } |q| < 1/2 \]

- Heuristic description of instability: growth of gradients of \( g \) to \( g = \mathcal{O}(\varepsilon) \) leads to amplification of phase gradients to \( \phi = \theta_x = \mathcal{O}(1) \). Once \( 1 - 4\phi^2 < 0 \), the roll amplitude \( r \) collapses to \( r = \mathcal{O}(\varepsilon^{1/2}) \), and we enter the MC scaling regime

- Space and time scales of instability not yet fully understood (eg. \( \varepsilon^4 = 0.04 \))
Large-amplitude initial data

Large-scale perturbations of rolls

- Burgers-type behaviour: viscous shock solutions
- gradients in large-scale mode induce phase gradients which suppress rolls
- relation to GL scaling regime: for $r = 0$, $g$ satisfies (viscous) Burgers equation
- special exact solutions?

Long-lived transient states displaying localized shock fronts and roll suppression
- Persistent dynamics retain Burgers-like solutions at large scales: GL scaling?
- Observation: anomalous scaling, particularly for $U_0$, due to spatially localized GL scaling regimes?
Summary

- Potentially anomalous (non-MC) scaling exponents: spatial intermittency?
- Investigate hybrid behaviour - coexistence of scaling regimes
- Connection of linear instability of rolls with uniform amplitude equation formalism?
- Study special solutions of amplitude equations

Further thoughts

- Compare \( u \) from Nikolaevskii equation with \( A, f \) from amplitude equations via probability distributions
- Lyapunov exponents: time scale for chaotic divergence of trajectories
- Effective stochastic dynamics at large scales?
  - time scale separation between \( f \) and \( A \)

*Time evolution of Fourier modes*: \( L = 3200, \varepsilon^2 = 0.004 \)

![Fourier modes plot](image1)

\( k \approx 0 \)

\( k \approx 1 \)

- Gaussian PDFs at large scales?
- Nikolaevskii equation in KPZ universality class?