A BREAK-DOWN CRITERION FOR THE EINSTEIN EQUATIONS IN VACUUM

Globally hyperbolic space-time foliated by Σ_t,

$$\mathbf{g} = -n^2 dt^2 + g_{ij} dx^i dx^j$$

1. Σ_t maximal, A.F.

2. Σ_t compact , Yamabe type -1, CMC.

$$\text{tr} k = t, \qquad t < 0.$$

THEOREM. First time t_* of a breakdown is characterized by

$$\limsup_{t \to t_*^-} \left(\|k(t)\|_{L^\infty} + \|\nabla \log n(t)\|_{L^\infty} \right) = \infty.$$

RELATED RESULTS

1. **M. Anderson** (2001): Breakdown cannot occure unless

$$\limsup_{t \to t_*^-} \|\mathbf{R}(t)\|_{L^\infty} = \infty.$$

2. **Beale-Kato -Majda** (1982): Breakdown, for Euler equations, cannot occure unless

$$\int_{t_0}^{t_*} \|(\nabla \times v)(t)\|_{L^\infty} dt = \infty.$$

3. **Eardley-Moncrief** (1982): No break-down can occcure for the Yang-Mills equations in $\mathbb{R}^{3+1}$.

MINKOWSKI SPACE: $(\mathbb{R}^{n+1}, \mathbf{m})$

- Standard coordinates $t = x^0, x = (x^1, \ldots, x^n)$.

- $\mathcal{J}^+(p), \mathcal{J}^-(p)$ past and future sets $p = (\underline{t}, \underline{x})$.

- $\mathcal{N}^\pm(p)$ null hypersurface, $u_p^\pm = 0$.

$$u_p^+ = t - \underline{t} - |x - \underline{x}|, \quad u_p^- = \underline{t} - t - |x - \underline{x}|$$

- Optical function $\mathbf{m}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$

- Null gradient $\mathbf{L} = -(\mathbf{m}^{\alpha\beta}\partial_\beta u)\partial_\alpha$.

$$< L, L >= 0, \qquad D_{\mathbf{L}}\mathbf{L} = 0$$

- Affine parameter $\mathbf{L}(s) = 1, \quad \mathbf{L} = \frac{d}{ds}$

WAVE PROPAGATION IN $\mathbb{R}^{3+1}$

WAVE EQUATION

$$\Box\phi = f; \quad \phi(0) = \partial_t\phi(0) = 0.$$

- **Solution** (Kirchoff)

$$\phi(p) = 4\pi^{-1}\int_{t\geq 0} r^{-1}\,\delta(\,u_p^-\,)\,f$$

$$\begin{aligned}\Box\Big(w\delta(u)\Big) &= \Box w\,\delta(u) \qquad (w = r^{-1} = |x - \underline{x}|^{-1})\\ &+ (-2L(w) + w\Box u)\delta'(u)\\ &+ \Big(\mathbf{m}^{a\beta}\partial_\alpha u\partial_\beta u\Big)\delta''(u)\\ &= 4\pi\delta(p),\end{aligned}$$

- **Eikonal equation** $\mathbf{m}^{a\beta}\partial_\alpha u\partial_\beta u = 0$

- **Transport equation** $-2L(w) + w\,\Box u = 0$

CONSERVATION LAWS

THEOREM. *En- momentum tensor Q, Div $Q = 0$. For any v-field X, with* $^{(X)}\pi = \mathcal{L}_X \mathbf{g}$,

$$\text{En}(t_2) + \text{Flux}_p(t_1, t_2) = \text{En}(t_1) + \int_{t_1}^{t_2} \int_{\Sigma_s} \mathbf{Q} \cdot {}^{(X)}\pi$$

- $\text{En}(t) = \int_{\Sigma_t} Q(X, \partial_t)$.

- $\text{Flux}_p(t_1, t_2) = \int_{\mathcal{N}^-_{t_1,t_2}(p)} Q(X, L)$.

Proof. Integrate, on $\mathcal{J}^-(p)$ between $t = t_1, t = t_2$,

$$D^\alpha(Q_{\alpha\beta}X^\beta) = \frac{1}{2}Q^{\alpha\beta}(D_\alpha X_\beta + D_\beta X_\alpha)$$

EXAMPLE: $X = \mathbf{T} = \partial_t, \ \ \pi = {}^{(\mathbf{T})}\pi$,

$$\boxed{\text{En}(t_2) + \text{Flux}_p(t_1, t_2) = \text{En}(t_1) + \int_{t_1}^{t_2} \int_{\Sigma_s} \mathbf{Q} \cdot \pi}$$

EARDLEY-MONCRIEF THEOREM

Yang-Mills: $F = dA + [A, A]$,

$$D^{\beta}_{(A)} F_{\alpha\beta} := \partial^{\beta} F_{\alpha\beta} + [A^{\beta}, F_{\alpha\beta}] = 0.$$

Bianchi $\Rightarrow$ $\square_{(A)} F + F * F = 0$

$$\boxed{\text{Energy}(t) + \text{Flux}_p(0, t) = \text{Energy}(0)}$$

THEOREM: *Global Regularity for all smooth data.*

$$\begin{aligned} F(p) &= (4\pi)^{-1} \int_{\mathcal{N}^-(p)} r^{-1} F \star F + F^{(0)}(p) \\ &- (4\pi)^{-1} \int_{\mathcal{N}^-(p)} r^{-1} (\square_{(A)} - \square) F. \end{aligned}$$

$$\left| \int_{\mathcal{N}^-(p)} r^{-1} F \star F \right| \le t^{\frac{1}{2}} \cdot \text{Flux} \cdot \|F\|_{L^\infty}$$

Neglecting the third term(<u>need a good gauge</u>),

$$\|F\|_{L^\infty} \le (t^{\frac{1}{2}} \cdot \text{Flux}) \cdot \|F\|_{L^\infty} + \mathcal{I}_0 \lesssim \mathcal{I}_0$$

BEL- ROBINSON ENERGY

$$\mathbf{Q}(\mathbf{R})_{\alpha\beta\gamma\delta} = \frac{1}{2}(\mathbf{R}_{\alpha\ \beta}^{\ \mu\ \nu}\mathbf{R}_{\gamma\mu\delta\nu} + {}^*\mathbf{R}_{\alpha\ \beta}^{\ \mu\ \nu}\ {}^*\mathbf{R}_{\gamma\mu\delta\nu}).$$

PROPOSITION. $\mathbf{T}$ *unit normal to* Σ_t, $\pi = \mathcal{L}_{\mathbf{T}}\mathbf{g}$.

$$\boxed{\text{En}(t_2) + \text{Flux}_p(t_1, t_2) = \text{En}(t_1) + \text{Err}}$$

$$\begin{aligned}
\text{En}(t) &= \int_{\Sigma_t} Q(\mathbf{T}, \mathbf{T}, \mathbf{T}, \mathbf{T}) dv_g \\
\text{Flux}_p(t_1, t_2) &= \int_{\mathcal{N}^-(p)} \mathbf{Q}(\mathbf{L}, \mathbf{T}, \mathbf{T}, \mathbf{T}) \\
\text{Err} &= \frac{3}{2}\int_{t_0}^{t}\int_{\Sigma_{t'}} \mathbf{Q}_{\alpha\beta\gamma\delta}\, \pi^{\alpha\beta}\mathbf{T}^\gamma\mathbf{T}^\delta\ n dv_g
\end{aligned}$$

THEOREM. Assume,

$$\int_{t_0}^{t_*} \|\pi(t)\|_{L^\infty} dt < \infty.$$

Then,

$$\|\mathbf{R}(t)\|_{L^2} \lesssim \|\mathbf{R}(t_0)\|_{L^2}.$$

HIGHER ENERGY ESTIMATES

Proposition. *Assume, <u>in addition</u>,*

$$\sup_{t_0 \le t < t_*} \|\mathbf{R}(t)\|_{L^\infty} < \infty \qquad \Rightarrow$$

Solutions stay smooth for all $t < t_*$.

Proof.

$$\text{Div } \mathbf{Q}(\mathcal{L}_{\mathbf{T}}\mathbf{R}) \;=\; \pi \cdot \mathbf{DR} \cdot \mathbf{DR} + \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{DR}$$

By integration,

$$\begin{aligned} \frac{d}{dt}\|\mathcal{L}_{\mathbf{T}}\mathbf{R}(t)\|_{L^2}^2 \;&\lesssim\; \|\pi(t)\|_{L^\infty} \cdot \|\mathbf{DR}(t)\|_{L^2}^2 \\ &+\; \|\mathbf{R}(t)\|_{L^4}^2 \cdot \|\mathbf{DR}(t)\|_{L^2} \end{aligned}$$

By L^2- elliptic estimates,

$$\|\mathbf{DR}(t)\|_{L^2} \lesssim \|\mathcal{L}_T\mathbf{R}(t)\|_{L^2}.$$

Conclude that, if $\int_{t_0}^{t_*} \|\mathbf{R}(t)\|_{L^4}^2 < \infty$,

$$\|\mathbf{DR}(t)\|_{L^2} \lesssim \|\mathbf{DR}(0)\|_{L^2}$$

LORENTZIAN CAUSALITY

- Causal sets $\mathcal{J}^{\pm}(p)$. Null boundaries $\mathcal{N}^{\pm}(p)$.
- Null geodesic generating vectorfield L
- Null 2^{nd} fund. form $\quad \chi(X,Y) = < D_X L, Y >$
- Affine parameter $L(s) = 1$.
- Jacobi equation $\frac{d}{ds}\chi + \chi^2 = \text{Riem}(\cdot\,, L, \cdot, L)$
- Raychadhouri equation

$$\frac{d}{ds}(\text{tr}\,\chi) + \frac{1}{2}(\text{tr}\,\chi)^2 + |\hat{\chi}|^2 = \text{Ric}(L, L) = 0$$

- Expansion $\quad \frac{d}{ds}\,\text{Area}(S_s) = \int_{S_s} \text{tr}\,\chi$
- Radius of injectivity. Radius of conjugacy,

WAVE PROPAGATION, HADAMARD PARAMETRIX

$$\Box_{\mathbf{g}}\phi = f, \qquad \phi\Big|_{\Sigma_0} = \partial_t\phi\Big|_{\Sigma_0} = 0 \qquad (1)$$

- **Eikonal equation**

$$\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0, \quad u|_{\mathcal{N}^-(p)} = 0.$$

- **Transport Equation**, along $\mathcal{N}^-(p)$

$$-2\frac{d}{ds}w - w\,\mathrm{tr}\,\chi = 0, \quad sw(p) = 1.$$

- **Main Calculation**

$$\Box\Big(w\delta(u)\Big) = 4\pi\delta(p) - N(\nabla_S^2\,\mathrm{tr}\,\chi\,,\ldots)\,\delta(u).$$

THEOREM. *For any solution of equation* (1)*,*

$$4\pi\phi(p) \;=\; \int_{\mathcal{N}^-(p)} wf + \int_{\mathcal{N}^-(p)} (\nabla_S^2\mathrm{tr}\,\chi + \ldots)\,\cdot\,\phi$$

REMARK. In general $\nabla_S^2\mathrm{tr}\,\chi$ depends on two derivatives of Riem.

UNIFORM CURVATURE BOUNDS.

- Curvature $\mathbf{R}$ verifies $\square_{\mathbf{g}}\mathbf{R} = \mathbf{R} \star \mathbf{R}$
- Restricted Hadamard (Kirchoff) parametrix $\Rightarrow$

$$\mathbf{R}(p) = -4\pi^{-1}\int_{\mathcal{N}^-(p)} w \cdot (\mathbf{R}\star\mathbf{R}) + \mathbf{R}_0(p) + \int_{\mathcal{N}^-(p)} N(\nabla^2 \mathrm{tr}\,\chi + \ldots)\cdot\mathbf{R}.$$

- As for Yang-Mills,

$$\Big|\int_{\mathcal{N}^-(p)} w \cdot (\mathbf{R}\star\mathbf{R})\Big| \le \|R\|_{L^\infty}\cdot \mathrm{Flux}_p \cdot \Big(\int_{\mathcal{N}^-(p)} w^2\Big)^{\frac{1}{2}}$$

$$\Big|\int_{\mathcal{N}^-(p)} N\cdot\mathbf{R}\Big| \le \|R\|_{L^\infty}\cdot\int_{\mathcal{N}^-(p)}|N|$$

- Need to control, in terms of the flux, the norms

$\|\mathrm{tr}\,\chi - \overline{\mathrm{tr}\,\chi}\|_{L^\infty(\mathcal{N})}, \|\nabla \mathrm{tr}\,\chi\|_{L^2(\mathcal{N})}, \|\nabla^2\mathrm{tr}\,\chi\|_{L^1(\mathcal{N})}$

- Lower bounds for the radius of injectivity of the congruence of past null geodesics from p.

THEOREM [Kl-Rodn(2003)] *Let* $(\mathbf{M}, \mathbf{g})$ *be an Einstein-vacuum manifold and* $\mathcal{N}^+$ *an outg. null hypersurface foliated by* S_s*.*

- $\mathcal{I}_0$ *be an initial data norm, measuring the deviation of* $S_0 \subset \Sigma$ *from the standard sphere.*

- $\mathcal{R}$ *geodesic curvature flux*

If $\mathcal{R}$ *and* $\mathcal{I}_0$ *are sufficiently small* $\Rightarrow$ *lower bound for the radius of conjugacy* c_**. In particular tr*χ *remains finite for* $0 \le s \le c_*$

Comparison to Riemannian geometry: Radius of conjugacy. Radius of injectivity?

Main Ideas

1. From

$$\frac{d}{ds}(\text{tr}\chi) + \frac{1}{2}(\text{tr}\chi)^2 = -|\widehat{\chi}|^2$$

we deduce that $\int_0^1 |\widehat{\chi}|^2 ds$ has to be controlled. Equation

$$\nabla_L \widehat{\chi} + \frac{1}{2}\text{tr}\chi \cdot \widehat{\chi} = -\alpha$$

does not seem helpful since we don't control $\int_0^1 |\alpha|^2 ds$.

2. Codazzi as a Hodge elliptic system on S_s,

$$\not{d}\text{iv}\, \widehat{\chi} - \frac{1}{2}\nabla \text{tr}\, \chi = -\beta + \frac{1}{2}\text{tr}\chi\zeta - \zeta \cdot \widehat{\chi}.$$

$$\widehat{\chi} = \mathcal{D}^{-1}\beta + \frac{1}{2}\mathcal{D}^{-1}(\nabla \text{tr}\chi) + \text{l.o.t.}$$

Thus, $\int_0^1 |\widehat{\chi}|^2 = I_1 + I_2 + \text{l.o.t.}$

$$I_1 = \int_0^1 |\mathcal{D}^{-1}\beta|^2, \quad I_2 = \int_0^1 |\mathcal{D}^{-1}(\nabla \text{tr}\chi)|^2$$

3. Would like to use the (false !) trace estimate

$$\int_0^1 |\mathcal{D}^{-1}\beta|^2 \lesssim \int_{\mathcal{H}} |\nabla\mathcal{D}^{-1}(\beta)|^2 \lesssim \int_{\mathcal{H}} |\beta|^2.$$

4. There exists a correct sharp trace theorem, in flat space,

$$\int_0^1 |\nabla_L \pi|^2 \lesssim \int_{\mathcal{H}} |\overline{\nabla}^2 \pi|^2$$

Can we express $\mathcal{D}^{-1}\beta = \nabla_L \pi +$ l.o.t. with

$$\int_{\mathcal{H}} |\overline{\nabla}^2 \pi|^2 \lesssim \mathcal{R}$$

5. There is such a miracle due to the null Bianchi identities:

$$L(\rho) = \not{\text{d}}\text{iv}\,\beta + \text{l.o.t.}, \quad L(\sigma) = -\text{curl}\,\beta + \text{l.o.t.}$$

Thus, $\beta = \mathcal{D}^{-1}L(\rho,\sigma) +$ l.o.t. or,

$$\begin{aligned} \mathcal{D}^{-1}\beta &= \mathcal{D}^{-2}L(\rho,\sigma) + \text{l.o.t.} \\ &= \nabla_L \mathcal{D}^{-2}(\rho,\sigma) + \text{l.o.t.} \end{aligned}$$

6. We have an additional problem with term $I_2 = \int_0^1 |\mathcal{D}^{-1}(\nabla \text{tr}\chi)|^2$ which needs to be controlled in $L^\infty(S_0)$. Unfortunately operators of order zero not bounded in L^∞.

7. We need to replace $L^\infty(S_0)$ with a norm which is stable with respect to pseudodifferential operators of order zero, such as Besov space $B^1_{2,1}(S_0)$.

8. To implement 7. we need a stronger version of the sharp trace theorem,

$$\| \int_0^1 \nabla_L \pi_1 \cdot \pi_2 \|_{B^0_{2,1}} \lesssim \|\overline{\nabla}\pi_1\|_{L^2(\mathcal{H})} \|\overline{\nabla}\pi_2\|_{L^2(\mathcal{H})}$$

9. This requires a geometric definition of Besov spaces on manifolds, i.e. geom. Littlewood-Paley calculus.

$$F = \sum_k P_k F$$

10. Additional difficulties to estimate ζ and $\underline{\chi}$. Recall that,

$$\nabla_L \zeta = -\beta + \text{l.o.t.}.$$

We also had,

$$\text{curl}\, \zeta = -\sigma + \frac{1}{2}\widehat{\chi} \cdot \widehat{\underline{\chi}}\,.$$

Introduce *mass aspect function*.

$$\mu = -\text{d}\!\!/\text{iv}\,\zeta + \frac{1}{2}\widehat{\chi} \cdot \widehat{\underline{\chi}} - \rho$$

Derive a transport equation for μ:

$$\nabla_L \mu = \widehat{\chi}\cdot(\nabla\cdot\zeta) + 2\zeta\cdot\beta + \text{tr}\,\chi\rho + 2\zeta\cdot\nabla\text{tr}\,\chi + \text{l.o.t.}$$

Solve it together with the Hodge elliptic system:

$$\begin{aligned} \text{d}\!\!/\text{iv}\,\zeta &= -\mu + \rho + \frac{1}{2}\widehat{\chi} \cdot \widehat{\underline{\chi}} \\ \text{curl}\,\zeta &= -\sigma + \frac{1}{2}\widehat{\chi} \cdot \widehat{\underline{\chi}} \end{aligned}$$

11. Need to control commutators terms such as,

$$[\nabla_L, \mathcal{D}^{-1}],\ [\nabla_L, \nabla \cdot \mathcal{D}^{-2}],\ [\nabla_L, P_k]$$

12. Define LP projections on S_s,

$$P_k F = \int_0^\infty m_k(\tau) U(\tau) F d\tau$$

$m_k(\tau) = 2^{2k} m(2^{2k}\tau)$, $m(\tau) \in \mathcal{S}\Big([0,\infty)\Big)$ with a finite number of vanishing moments.

$$\partial_\tau U(\tau)F - \Delta U(\tau)F = 0, \quad U(0)F = F.$$