CONFORMAL EINSTEIN EQUATIONS

(A NEW APPROACH)

Based on:

gr-qc/0408072 (AHP, '05)

and gr-qc/0412020 with P. Chruściel (CMP, '05)

Alternate approach to construction of global solutions of (vacuum) Einstein equations via conformal compactifications

Penrose / H. Friedrich

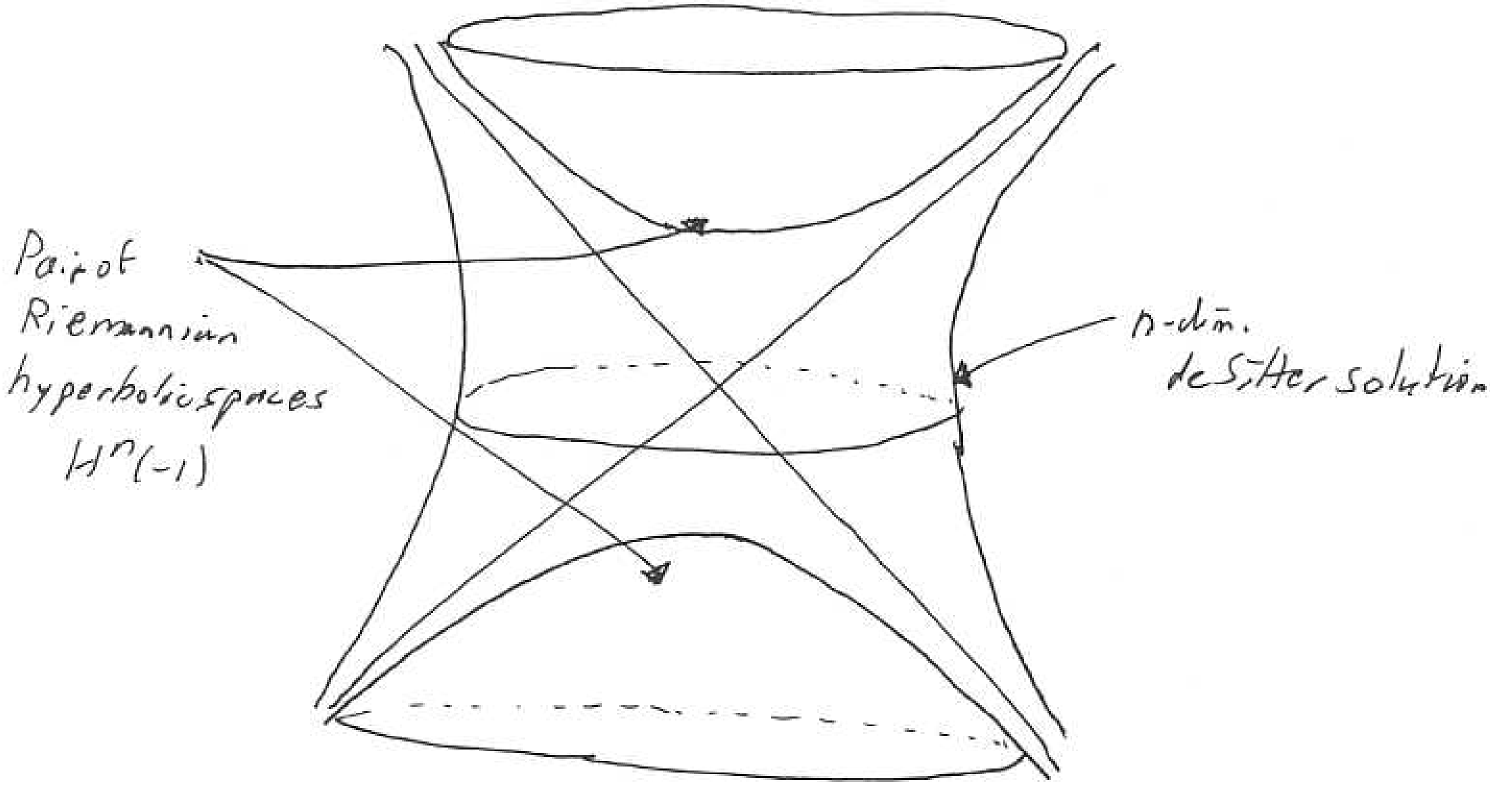
Basic Points:

- Simplicity / Naturality
- Construction holds for all <u>even</u> dimensions
- Method (probably) fails in odd dimensions

Hollands, Ishibashi, Wald

Derivation of Conformal Einstein Equations

$(\mathbb{R}^{n+1}, \eta)$ = $(n+1)$-dimensional Minkowski space-time



Self-similar structure

$$\mathcal{L}_X \eta = 2\eta \quad , \quad X = \text{"position vector field"}$$

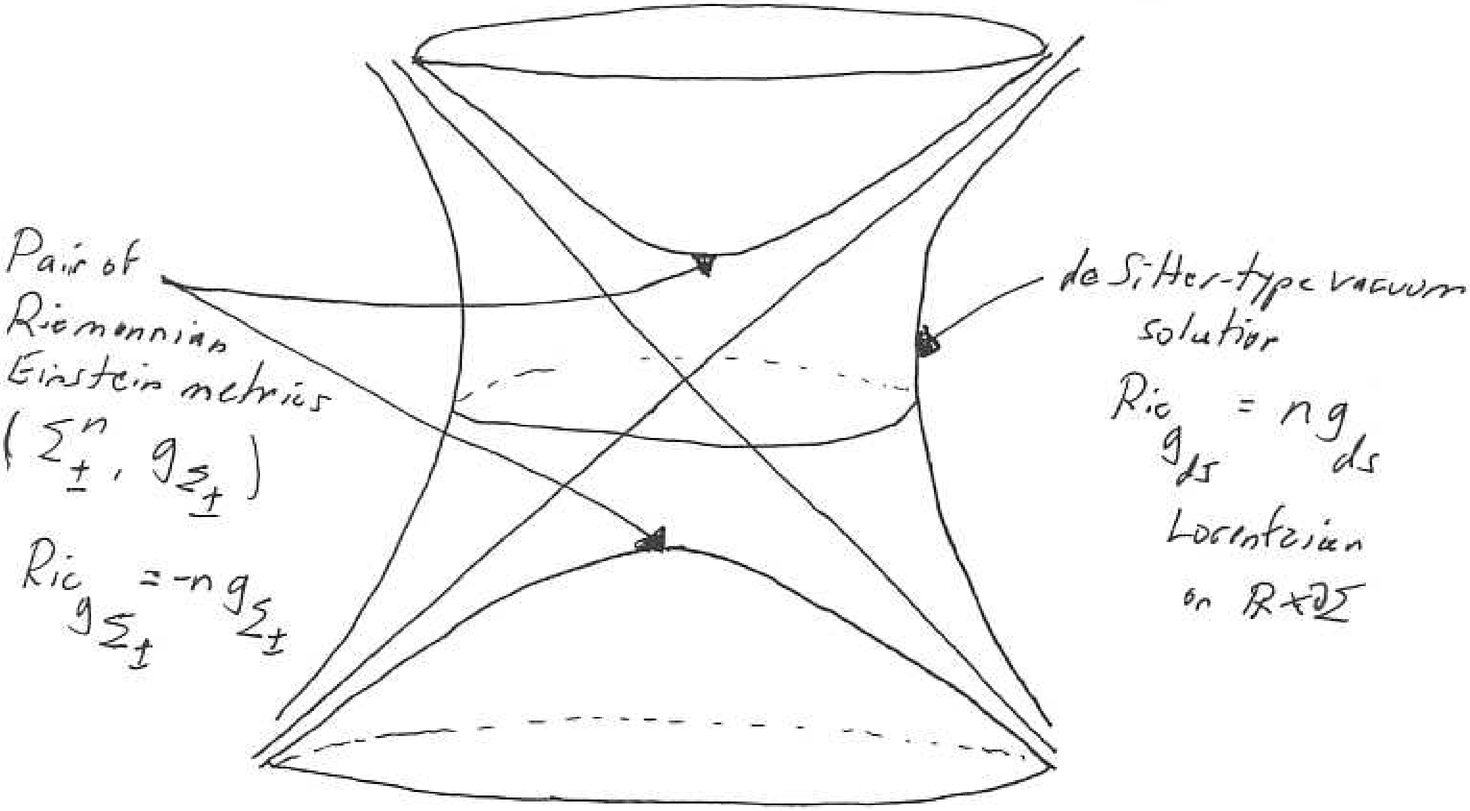
In $(3+1)$ dimensions, this is only self-similar vacuum solution

-rigid-

Not so in higher dimensions

General self-similar "vacuum" solution

$(M^{n+1}, \mathfrak{g})$ $\quad \mathcal{L}_X \mathfrak{g} = 2\mathfrak{g}$



"Evolution" inside/outside light cones = trivial rescalings

Now suppose $(M^{n+1}, \mathfrak{g})$ conformally compact

- $\mathfrak{g}$ has conformal extension (smooth) to $\mathcal{I}^+$ (or $\mathcal{I}^-$)

$\Longleftrightarrow$ (Σ_+, g_{Σ_+}) conformally compact

so $\quad \bar{\Sigma} = \Sigma \cup \partial\Sigma$: manifold with boundary

$$\bar{g}_\Sigma = \rho^2 g_\Sigma \quad , \rho = 0 \text{ on } \partial\Sigma$$
$$d\rho \neq 0 \text{ on } \partial\Sigma$$

Take formal expansion of metric $\tilde{g}$ at $\mathcal{I}^+$

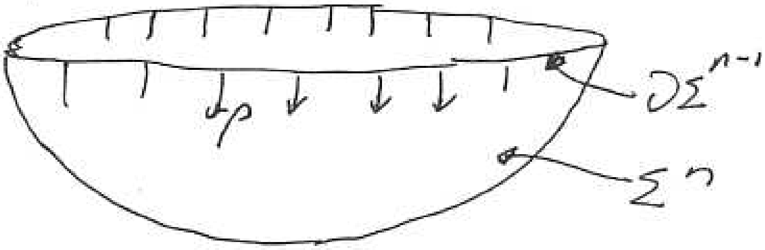
- Bondi, van der Berg, Metzner type expansion in $(n+1)$ dimensions

Rescaling evolution $\Rightarrow$

suffices to restrict $\tilde{g}$ to

(Σ, g_Σ)

and expand g_Σ near $\partial\Sigma = \{\rho = 0\}$



Write

$$\bar{g}_\Sigma = \rho^2 g_\Sigma$$

$$\underbrace{\bar{g}_\Sigma = d\rho^2 + g_\rho}_{\text{Gaussian coords}}, \quad g_\rho = \text{curve of metrics on } \partial\Sigma$$

so $g_\Sigma = \rho^{-2}(d\rho^2 + g_\rho)$

Einstein equations for g_Σ $\Rightarrow$ degenerate equations for $\bar{g}_\Sigma$

$\Rightarrow$ degenerate equations for curve g_ρ at $\rho = 0$.

Formal expansion for g_ρ = Fefferman–Graham expansion ('86)

- Works for <u>any</u> signature

$$\dim \Sigma = n, \quad \dim \partial\Sigma = n-1$$

<u>n even</u>

$$g_\rho \sim \underbrace{g_{(0)} + \rho^2 g_{(2)} + \cdots \rho^{n-2} g_{(n-2)}}_{\text{determined by}} + \rho^{n-1} g_{(n-1)} + \cdots$$

<u>determined</u> by
Einstein eqns + ∂-metric $g_{(0)}$
$g_{(k)} \sim \partial^k g_{(0)}$

$g_{(n-1)}$: <u>undetermined</u>, except for constraints
$\mathrm{tr}\, g_{(n-1)} = \delta g_{(n-1)} = 0$
"freely specifiable"

All coefficients determined by $(g_{(0)}, g_{(n-1)})$

<u>n odd</u>

$$g_\rho \sim \underbrace{g_{(0)} + \rho^2 g_{(2)} + \cdots + \rho^{n-3} g_{(n-3)}}_{\text{determined}} + \rho^{n-1} g_{(n-1)} + \rho^{n-1}\log\rho\, \mathcal{H} + \cdots$$

$g_{(n-1)}$: free

$\cdots$: power series in ρ, $\log\rho$, "polyhomogeneous"

$\mathcal{H}$ is <u>determined</u>, $\mathcal{H} \sim \partial^{n-1} g_{(0)}$

$\mathcal{H}$ = Fefferman–Graham (obstruction) tensor

Properties of $\mathcal{H}$

(1) $\mathcal{H}$ is conformal covariant

$$\mathcal{H}(\phi^2 g) = \phi^{-(n-3)} \mathcal{H}(g)$$

(freedom of choice in conformal factor ρ)

(2) Form of $\mathcal{H}$:

$$\mathcal{H}(g) = \Delta_g^{\frac{n-1}{2}-2}\left[\Delta_g P + D^2(\mathrm{tr}\, P)\right] + \mathcal{F}^{n-3}(g)$$

$\Delta_g = D^* D$ - Laplace operator : wave operator if g is Lorentzian

$$P = Ric_g - \frac{R}{2(n-2)} g$$

(3) If g is conformal to Einstein metric

$Ric_g = \lambda g$, any λ, any signature

then $\underline{\underline{\mathcal{H}(g) = 0}}$

(4) $\mathrm{tr}\, \mathcal{H} = \delta \mathcal{H} = 0$

$\underline{\mathcal{H}(g) = 0}$ = conformal Einstein equations

On $(\partial \Sigma^{n-1}, g)$: system of PDE's, order $(n-1)$ on g

Interpretation (AdS/CFT)

$\mathcal{H}$ comes from Lagrangian

$$\mathcal{H}(g) = \delta_g L$$

L = conformal anomaly (Henningson-Skenderis '98)

= logarithmic divergent term in expansion of Einstein-Hilbert action (on shell)

$$S_{EH} = \int_{\Sigma^n} R - 2\Lambda = \lim_{\varepsilon \to 0} \int_{\Sigma^n_\varepsilon} R - 2\Lambda \qquad \Sigma_\varepsilon = \{\rho \geq \varepsilon\}$$

n odd

$$= V_{(n-1)} \cancel{\rho}_\varepsilon^{-(n-1)} + V_{(n-3)} \cancel{\rho}_\varepsilon^{-(n-3)} + \cdots + V_{(2)} \cancel{\rho}_\varepsilon^{-2} + \underline{\underline{L}} \log \cancel{\rho}_{\varepsilon^{-1}} + V + o(1).$$

Conformal Einstein Equations

$$\mathcal{H} = 0$$

- For any metric signature, <u>even</u> dimensions only

Use to study global properties of solutions to Einstein equations

$$(\partial\Sigma^{n-1}, \hat{g}) \longrightarrow (M^{n+1}, g)$$

$$\mathcal{H}(g) = \Box_g^{\frac{n+1}{2}-2}(\Box_g P + D^2 tr P) + \mathcal{F}^{n-1}(g) = 0$$

$$P = Ric_g - \frac{R}{2n} g$$

<u>$n=3$</u> $\mathcal{H} = 0$ = Bach equations

$$\Box Ric + \frac{1}{3} D^2 R + \frac{1}{6} \Box R \cdot g = 2 Riem(Ric) - \frac{2}{3} R \cdot Ric - \frac{1}{2}\left[|Ric|^2 - \frac{R^2}{3}\right] g$$

or

$$B_{ij} = P_{ij,k}{}^{k} - P_{ik,j}{}^{k} - P^{kl} W_{kijl} = 0$$

<u>$n=5$</u>

$$\mathcal{H}_{ij} = B_{ij,k}{}^{k} + l.o.t.$$

System $\mathcal{H} = 0$ has

- diffeomorphism invariance $\longleftrightarrow$ Ga[illegible] wave (F[illegible]
- Conformal invariance $\longleftrightarrow$ $R = $ co[illegible]

In these gauges, CEE have simp[illegible]

$$(*) \qquad \Box_g^{\frac{n+1}{2}} g + \mathcal{F}^{n-1}(g) = 0$$

Prop. The system (*) can be reduced to a [illegible] symmetric hyperbolic system.

Cauchy Problem

(Σ, γ) = space-like hypersurface

Gaussian coords

$$g = -dt^2 + \gamma_t$$

$$k^{(i)} = \tfrac{1}{2} \mathcal{L}_T^{(i)} g \big|_{t=0} = \tfrac{1}{2} \partial_t^{(i)} \gamma_t \big|_{t=0}$$

Cauchy data : $(\Sigma, \gamma, k^{(1)}, \ldots, k^{(n)})$

Freely specifiable modulo Constraint Eq[illegible]

$$\mathcal{H}(T, \cdot) = 0 \qquad (n+1) \text{ E[illegible]}$$

$\delta \mathcal{H} = 0 \Rightarrow$ constraints preserved under e[illegible]

- Any solution of vacuum Einstein constraints = $\mathcal{H}$ - constraint eqns.

Constraints are conformally invariant

$$g \text{ a soln of CEE} \longleftrightarrow \tilde{g} = \Omega^2 g \text{ also solution}$$

Write $\omega = \Omega|_\Sigma$, $\omega^{(j)} = \underbrace{T \cdots T}_{j}(\Omega)|_\Sigma$: freely specifiable, $\omega > 0$

$$\tilde{\gamma} = \omega^2 \gamma$$

$$\tilde{k}^{(1)} = \omega k^{(1)} + \omega^{(1)} \gamma$$

etc : conformal transformations of $(\gamma, k^{(1)}, \dots, k^{(n)})$

<u>Theorem</u> Let $(\Sigma, [\gamma, k^{(1)}, \dots, k^{(n)}])$ be any class satisfying constraint equations, with

$$(\gamma, k^{(1)}, \dots, k^{(n)}) \in H^s_{loc}(\Sigma) \times H^{s-1}_{loc}(\Sigma) \times \cdots \times H^{s-n}_{loc}(\Sigma)$$

$$s > \frac{n}{2} + n + 1$$

Then, up to isometry, there exists a unique maximal, globally hyperbolic conformal space-time $(\mathcal{M}, [g])$ satisfying

$$\mathcal{H}(g) = 0$$

with given initial data. Cauchy problem is well-posed.

If initial data satisfy conformal <u>Einstein</u> constraint eqns, then solution is conformally Einstein.

<u>Idea</u> Solve $(*)$, show gauge choices preserved via

$$\delta \mathcal{H} = \operatorname{tr} \mathcal{H} = 0$$

Choose $\omega^{(2)}, \dots \omega^{(n)}$ s.t. $R_g = \text{const}$ to order $(n-2)$ on Σ

Applications

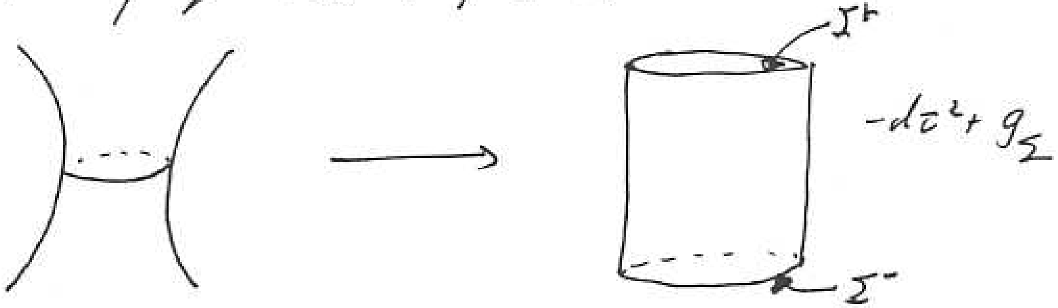
① Global Stability of de Sitter spaces

(generalized) de Sitter

$$(M^{n+1}, g_{dS}),\quad g_{dS} = -dt^2 + \cosh^2 t\, g_\Sigma$$

$$M = \mathbb{R} \times \Sigma \qquad\qquad Ric_{g_\Sigma} = n\, g_\Sigma$$

Conformally equivalent to product



<u>Einstein</u> Cauchy data on $\mathcal{I}^+$ (or $\mathcal{I}^-$)

$(\gamma, g_{(n)})$, $g_{(n)}$ transverse-traceless (as in F-G expansion)

dS^+ = space of glob. hyperbolic vacuum Einstein solns, $\Lambda > 0$, conf. compact on $\mathcal{I}^+$

dS^- = "

$dS^\pm = dS^+ \cap dS^-$

<u>Cor</u> Given $(\Sigma^n, \gamma, g_{(n)})$, <u>n odd</u>, Cauchy data $(\gamma, g_{(n)}) \in H^s \times H^{s-n}$

$s > \frac{n}{2} + n + 1$

Then $\exists!$ solution in dS^+, with given data on $\mathcal{I}^+$

Cauchy problem well posed at $\mathcal{I}^+$

The space $dS^\pm$ is <u>open</u> (global stability)

Let $\overline{dS^{\pm}}$ = closure of $dS^{\pm}$ w.r.t. $H^{s} \times H^{s-n}$ topology on $\mathcal{I}^{+} \cup \mathcal{I}^{-}$

$\partial dS^{\pm} = \overline{dS^{\pm}} \setminus dS^{\pm}$

= limits of spaces in $dS^{\pm}$

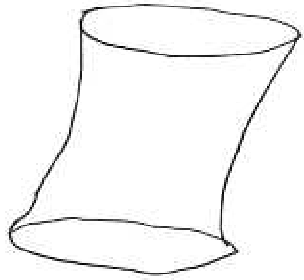
Structure of Limits

In even dimensions, spaces in $\partial dS^{\pm}$ are of one of the following types:

(i) Pair of complete solutions

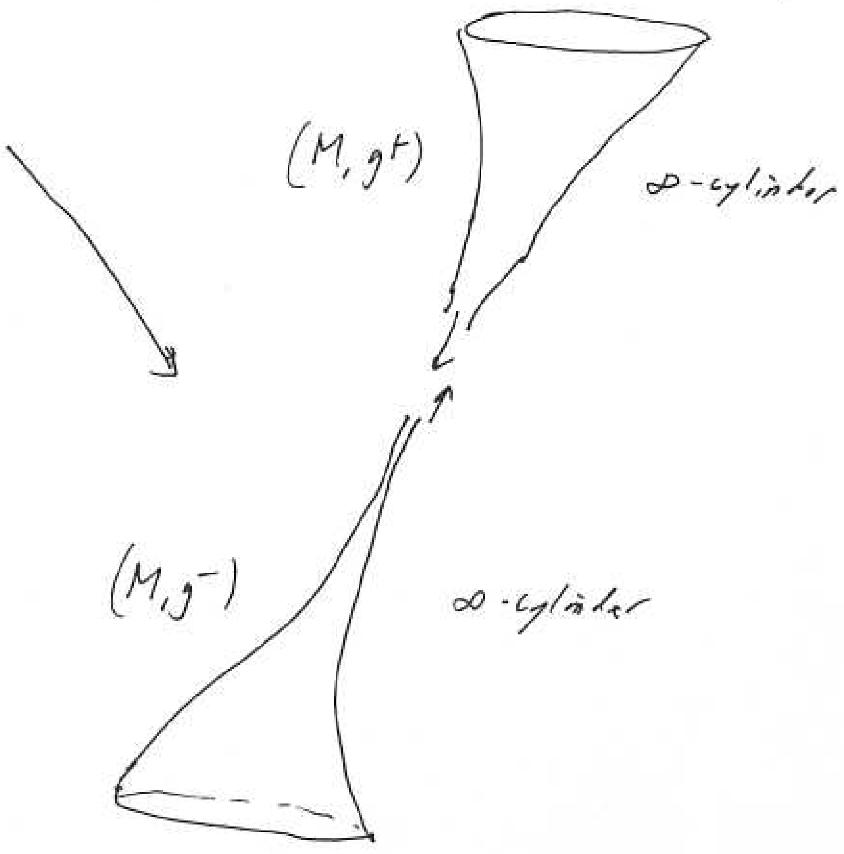
$(M, g^{+}) \in dS^{+}$, $(M, g^{-}) \in dS^{-}$ both geodesically complete, glob. hyp. "∞ far apart"

$\mathcal{I}^{-} = \emptyset$ $\qquad$ $\mathcal{I}^{+} = \emptyset$



$(M, g) \in dS^{\pm}$

glob. conf. compact or finite cylinder



Examples

dS Taub-NUT solutions on $\mathbb{R} \times S^{3}$

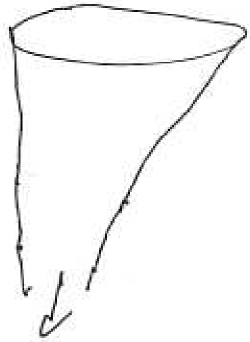
(ii) Single complete solution

$(M, g) \in dS^+$ geodesically complete

globally hyperbolic

$\mathcal{I}^- = \emptyset$

or $\mathcal{I}^-$ open (not conf. compact on $\mathcal{I}^-$)



No examples

(iii) Single complete solution

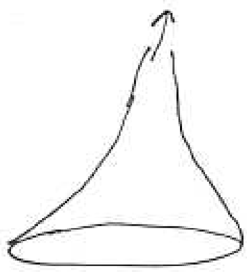
$(M, g) \in dS^-$

geod. complete

glob. hyp.

$\mathcal{I}^+ = \emptyset$

or $\mathcal{I}^+$ open



No examples

Main Point

Singularities do not form on

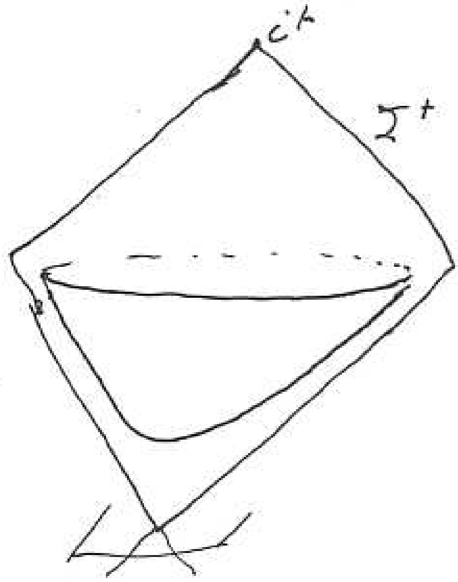
$\partial dS^\pm$

Outside the boundary $\partial dS^\pm$, expect

singularities / black holes

to form from smooth initial data

② Future hyperboloidal stability



Standard hyperboloidal initial data:

(B^n, γ_0, k_0)

γ_0 = hyperbolic (Poincaré) metric

$k_0 = \gamma_0$

H^s conformally compact, $\forall s$, ie

$[\gamma_0, k_0^{(1)}, \dots, k_0^{(n)}]$ has H^s extension to ∂B^n.

Cor. Suppose (B^n, γ, k) = initial data set for vacuum Einstein eqns

n odd, H^s conformally compact, $s > \frac{n}{2} + n + 1$

Then $\exists\, \varepsilon = \varepsilon(n) > 0$ s.t. if

$[\gamma, k^{(1)}, \dots k^{(n)}]$ is ε-close to $[\gamma_0, k_0^{(1)}, \dots k_0^{(n)}]$

in $H^s \times \dots \times H^{s-n}$,

then maximal glob. hyp. vacuum development of (B^n, γ, k)

is future complete, H^s conformally compact, containing

regular future infinity i^+.

Using

Cutler-Wald / Corvino-Schoen / Chruściel-Delay

technique

<u>Cor</u> There exists an infinite dimensional space of geodesically complete, asymptotically simple, globally hyperbolic solutions of vacuum Einstein equations in all even dimensions.