Initial Data Engineering

INI
25 Aug '05

- over 50 years of studying

Initial Data (γ, K) satisfying Einstein Constraint Eqs

$$R - K^2 + (\mathrm{tr}K)^2 = \rho$$
$$\nabla \cdot K - \nabla(\mathrm{tr}K) = J$$

mostly via Conformal Method

$$\gamma_{ab} = \phi^4 \lambda_{ab}$$
$$K_{cd} = \phi^{-2}(\sigma_{cd} + LW_{cd}) + \phi^4 \lambda_{cd} \tau$$

$$(\nabla \cdot L) W = \phi^6 \nabla \tau$$
$$\Delta \phi = R\phi - (\sigma + LW)^2 \phi^{-7} + \tau^2 \phi^5$$

- Much Understood

- Existence of Solutions
  - on any closed 3 manifold
  - smooth or rough
- Space of CMC Solutions & Near-CMC Solutions
  - Parametrization
  - Manifold structure

- Much Not Understood

- Non CMC Solutions
- Initial data for Colliding Black Holes

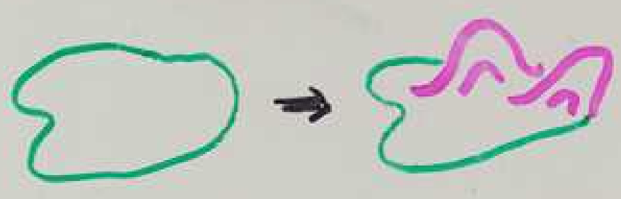
⇒ Data Engineering Problem:

Build sets of data to accomplish certain tasks

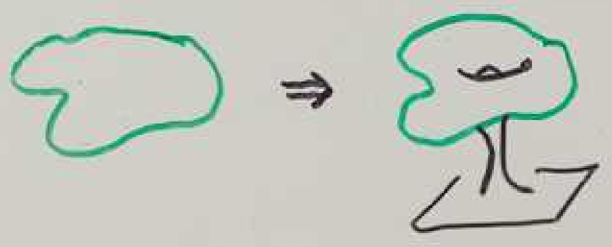
Possibly using spare (known) sets of data

Some Others:

- Add wormholes to given spacetime



- Add Black Holes to given spacetime



- Embed given spacetime region in Schwarzschild extension

- Produce "Reflexive" Data

$\gamma \to \gamma$

$k \to -k$

—New Engineering Tool:

last 5 years

Gluing

- Handles some of these tasks (not all)

- The Talk:
  - Idea of Gluing
  - Main Gluing Theorem
  - Applications of Gluing
  - Proof (sketch)
  - To Do

Idea of Gluing

Given

pair of solutions of constraints

$(\Sigma_1, \gamma_1, k_1)$ $\qquad$ $(\Sigma_2, \gamma_2, k_2)$

Obtain

new solution

$(\hat{\Sigma}, \hat{\gamma}, \hat{k})$

with

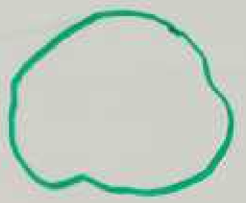
- $\hat{\Sigma} = \Sigma_1 \# \Sigma_2$
- solution unchanged away from bridge

¿Can we always glue?

→ No

Pick

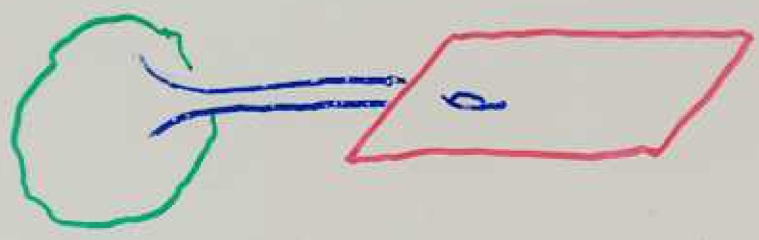
(S^3, γ, k) $(\mathbb{R}^3, \text{flat}, k=0)$



If could glue

violate Positive Energy Theorem

since



has Energy $=0$

but not Minkowski

Main Gluing Theorem

Chrusciel
Pollack
I
Mazzeo
Delay

Thm

Let $(\Sigma_1, \gamma_1, k_1)$ and $(\Sigma_2, \gamma_2, k_2)$ be sol'ns of the constraints.

Let $p_1 \in \Sigma_1$ and $p_2 \in \Sigma_2$ each satisfy no KIDs condition

Then gluing across p_1 and p_2 works

Note:

No restriction on
- topology
- asymptotics

$\Longrightarrow$ Can be
closed
asymptotically Euclidean
asymptotically hyperbolic

¿What does No KIDs Mean?

Version 1:

A constraint solution (Σ, γ, K) has

no KIDs at $p \in \Sigma$

if $\nexists$ any open set $U \ni p$

such that

$D(U)$ contains Killing field

Version 2:

Let $\mathcal{L}_{(\Sigma,\gamma,K)} = DC_{(\gamma,K)}$

be the linearization of the constraints C

If $\nexists$ $U \ni p$ such that

$\mathcal{L}_{(\Sigma,\gamma,K)}$ has nontrivial kernel

then (Σ, γ, K) has no KIDs at $p \in \Sigma$

¿How Restrictive is No KIDs?

Thm (Beig, Chruściel, Schoen)

For generic point in generic soln

no KIDs.

Applications of Gluing

1) No Topological Obstructions

Thm

For any closed 3 manifold Σ

- $\exists$ solution on Σ
- $\exists$ Asympt Euclidean soln on $\Sigma \setminus \{p\}$
- $\exists$ Asympt Hyperbolic soln on $\Sigma \setminus \{p\}$

Idea of Proof

c – Use

  Lemma:

   For any closed Σ

    $\exists$ Riemannian metric g with

$$R(g) = -6$$

      No KIDS

– Then

  (Σ, g, g) is soln on Σ

Æ AH – Use Beig-Chrus-Schoen to verify

  * $\exists$ Æ sol'n $(\mathbb{R}^3, \gamma_E, k_E)$

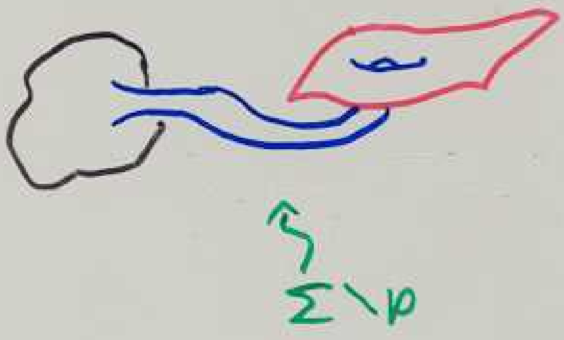
   with no KIDs

  * $\exists$ AH soln $(\mathbb{R}^3, \gamma_H, k_H)$

   with no KIDS

- Glue

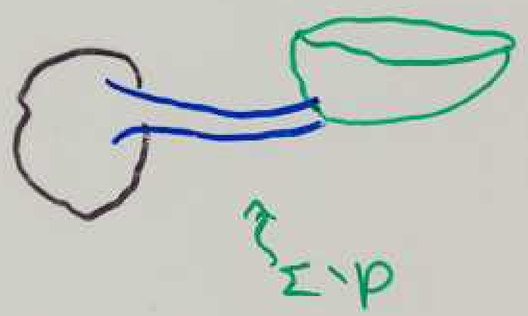
(Σ, g, g) to $(\mathbb{R}^3, \hat{\gamma}_E, k_E)$



$\Rightarrow$ Æ soln on $\Sigma \setminus p$

- Glue

(Σ, g, g) to $(\mathbb{R}^3, \gamma_H, k_H)$



$\Rightarrow$ AH soln on $\Sigma \setminus p$

2) Adding Wormholes to Your Universe

Claim

Let (Σ, γ, k) be soln of constraints

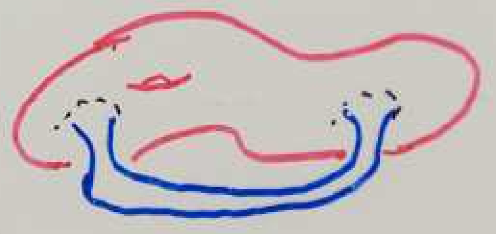
If $p, q \in \Sigma$

have no KIDs

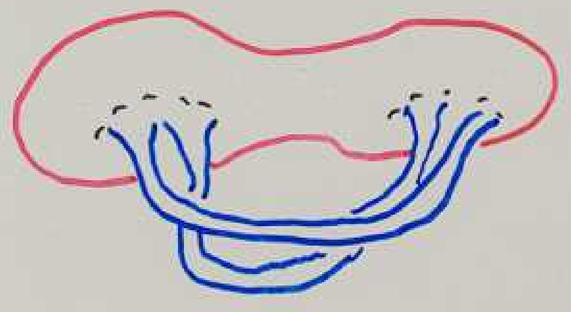
then $\exists$ soln

$(\Sigma \# \text{wormhole}, \hat{\gamma}, \hat{k})$

- with wormhole
- unchanged away from nbhd of p, q.



keep adding more...

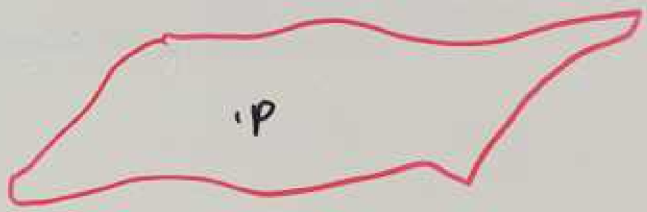


¿How long do they last?

3) Constructing Multi Black Hole Spacetimes

Thm

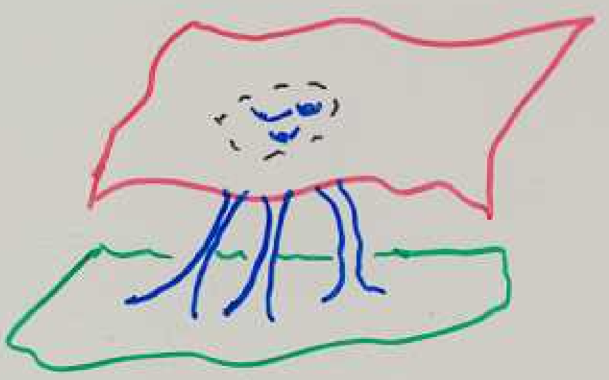
Let (Σ, γ, K) be an Asympt Euclidean sol'n of the constraints



If p has no KIDs
then $\exists$ soln

$$(\Sigma \setminus \{p_1, \dots p_n\}, \hat{\gamma}, \hat{K})$$

- with N black holes in $D(\Sigma \setminus \{p_1 \dots p_n\}, \hat{\gamma}, \hat{K})$
- unchanged away from nbhd of p



Idea of Proof

- Glue generic Æ $(\tilde{\Sigma}, \tilde{\gamma}, \tilde{K})$ to (Σ, γ, K) at $p_1, \dots p_N \in$ nbhd of p
- Argue that spacetime development has N disjoint horizons

Chruściel Delay

4) Asymptotically Flat Spacetimes with No Maximal Slice

Thm

$\exists$ spacetimes which
- satisfy vac Einstein
- are globally hyperbolic (max)
- are asymptotically flat

and admit no maximal Cauchy surface

$\mathrm{tr}\,k=0$

Idea of Proof

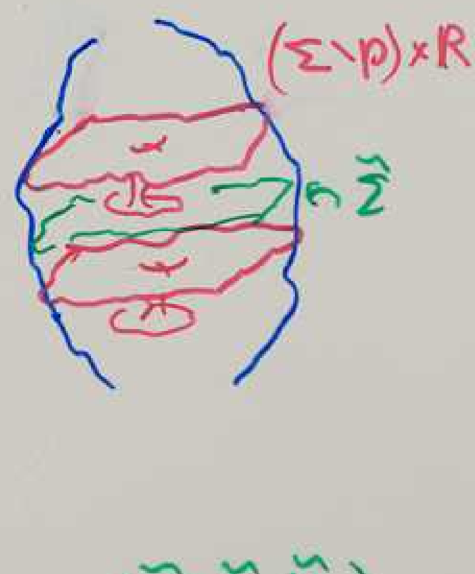
- Pick closed Σ which admits no metrics with $R \geq 0$

- Apply "No Topolog Obstruction Thm"

$\Rightarrow \exists$ asympt Euclidean soln $(\Sigma \setminus p, \gamma, k)$ of constraints

- Consider development

$D(\Sigma \setminus p, \gamma, k)$



Let $\check{\Sigma}$ be maximal Cauchy surface in $D(\Sigma \setminus p, \gamma, k)$

$\Rightarrow$
- $\check{\Sigma} \approx \Sigma \setminus p$
- $R(\check{\gamma}) \geq 0$ for data $(\check{\Sigma}, \check{\gamma}, \check{k})$
- $(\check{\Sigma}, \check{\gamma})$ is AE.

- Use

Lemma:

If $(\tilde{\Sigma}, \tilde{\gamma})$ is Æ and has $R(\tilde{\gamma}) \geq 0$

then $\tilde{\Sigma}$ has compactification which admits $\tilde{\tilde{\gamma}}$ with $R(\tilde{\tilde{\gamma}}) > 0$.

- Since the compactification of $\tilde{\Sigma}$ is Σ

$\Rightarrow$ Contradiction

$\Rightarrow$ No maximal slice

one more application:

(the one that motivated the gluing program)

5) Spatially Compact Spacetimes with No CMC Slice

Thm

$\exists$ spacetimes which
- satisfy vac Einstein
- are globally hyperbolic (max)
- are spatially compact

and admit no CMC Cauchy surface

$\boxed{tr\,K = const}$

Idea of Proof

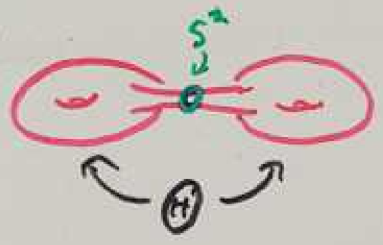
- Key idea:

Reflexive Initial Data

- $\Sigma = \hat{\Sigma} \# \hat{\Sigma}$

- $\exists$ "reflexion map"

$\Theta : \Sigma \to \Sigma$



with

$\Theta^* \gamma = \gamma$

$\Theta^* K = -K$

– How to Use It:

Prop (Bartnik; Eardley-Witt)

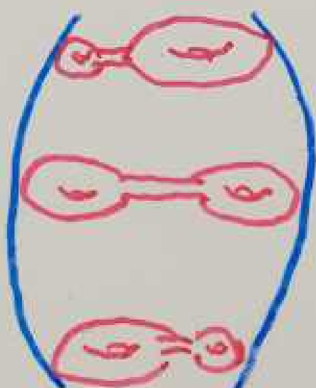
Let Σ admit no metric with $R \geq 0$
Let (Σ, γ, k) be reflexive data
The maximal development $D(\Sigma, \gamma, k)$
admits no CMC Cauchy surface

Proof of Prop

- Say $D(\Sigma, \gamma, k)$ admits CMC with $\tau = \mathrm{tr}\, k$
- $\tau \neq 0$
  since $\tau = 0 \rightarrow R = k^2 \geq 0$
- Note

  Claim

  Let $D(\Sigma, \gamma, k)$ be the max developmt of reflexive data
  If $(\hat{\Sigma}, \hat{\gamma}, \hat{k})$ is Cauchy data in $D(\Sigma, \gamma, k)$
  then $(\check{\Sigma}, \hat{\gamma}, -\hat{k})$ is as well.



$(\hat{\Sigma}, \hat{\gamma}, \hat{k})$

(Σ, γ, k)

$(\check{\Sigma}, \hat{\gamma}, -\hat{k})$

$\Rightarrow$ If $\exists$ data with CMC τ
then $\exists$ data with CMC $-\tau$

- Use Barrier Thm:

  $\Rightarrow$ $\exists$ data with CMC $= 0$

  $\Rightarrow$ Contradiction ▲

– How to Construct It:
(T³ # T³ version)

- Use conformal method & Bartnik deformation to get soln

  (T^3, γ, k)

  no KIDs

- Note

  $(T^3, \gamma, -k)$ is also soln with kids

- Glue

  $(T^3, \gamma, -k)$ to (T^3, γ, k)

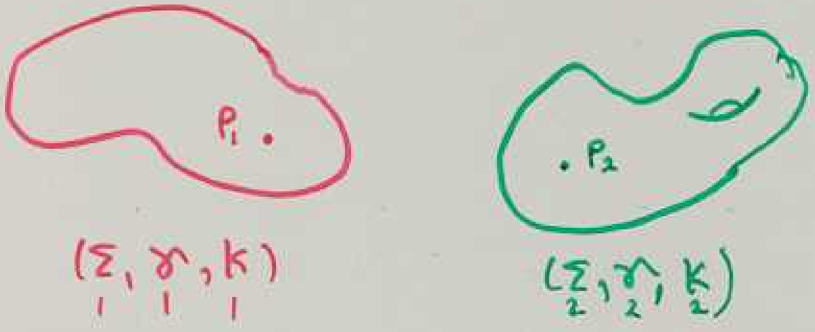
  on equivalent points.

- Argue that it's reflexive ■

¿ How many of these?

- Lots more with reflexive data
- ¿ Others??

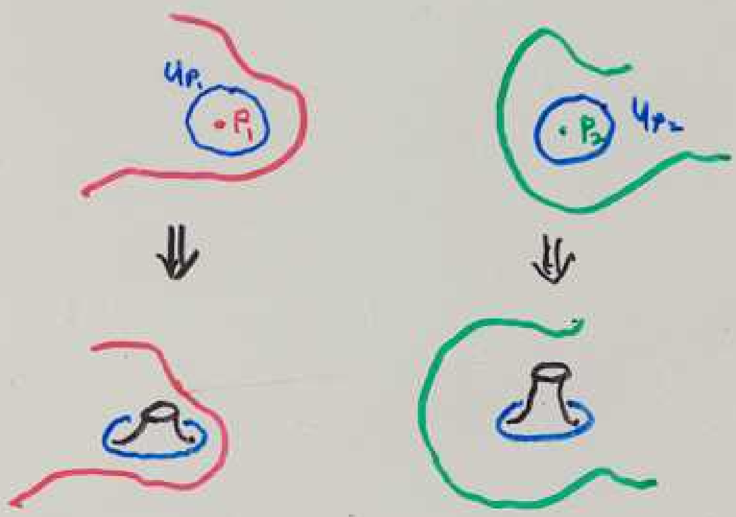
On Proving the Main Gluing Theorem



Key Steps:

- Preparing the Neighborhoods of p_1 and p_2

Use Bartnik Deformation
to get local CMC regions



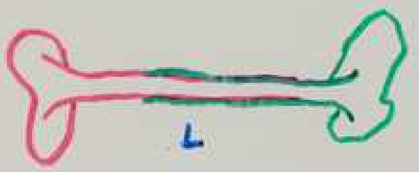
→ Same CMC value!

– Local Gluing of Local CMC Regions

V_{p_1} $\qquad$ V_{p_2}

- Conformal Blow Up at p_1 and p_2

- Connect sum V_{p_1} and V_{p_2} and patch the fields



$\Rightarrow$ Family of approximate solutions

- Solve the constraints on V_L

$\Rightarrow$ family of solutions

$$(V_L, \gamma_L, K_L)$$

with $\xrightarrow[\text{large } L]{}$ unchanged away from neck.

- Reassembling Pieces

- Original solutions on

$\Sigma_1 \setminus U_{p_1}$ and $\Sigma_2 \setminus U_{p_2}$

- New glued solutions on

V_ϵ

- Transition pieces

$U_{p_1} \setminus V_{p_1}$ $\quad$ $U_{p_2} \setminus V_{p_2}$

Smooth the seams using

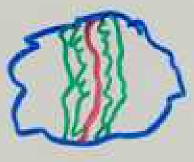
Chrusciel-Delay Thm

If (γ, K) on open region Λ

- is soln everywhere except codim Φ
- is arbitrarily close to soln on Φ
- has no KIDs

Then $\exists$ deformation $(\hat{\gamma}, \hat{K})$ s.t.

- soln everywhere on Λ
- $(\hat{\gamma}, \hat{K}) = (\gamma, K)$ on $\Lambda \setminus$ nbhd of Φ



$\Longrightarrow$ Gluing is done

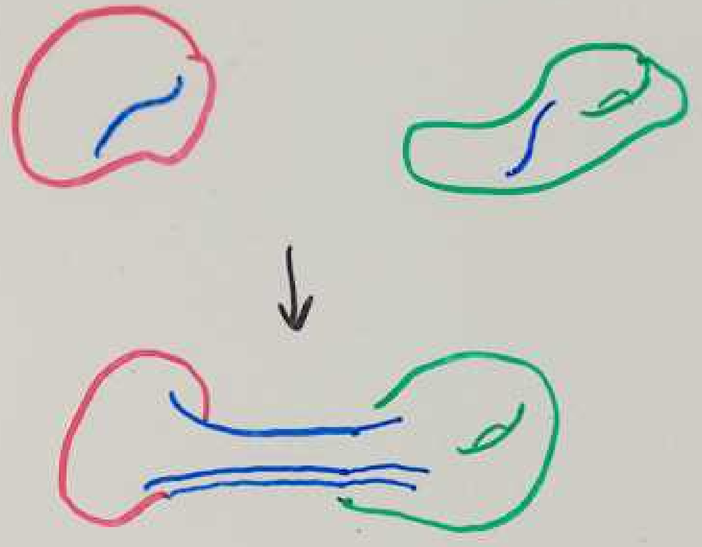
Things to Do

- Remove "No KIDs"
  ⇒ No
  Likely a sharp condition

- Include matter
  ⇒ Mostly done
  Maxwell

- Higher Dimensions
  ⇒ Mostly done

- Glue on Ridges



⇒ Thinking about it.

- Evolving Glued Data
  ⇒ Thinking about it