Higher Dimensional Black Holes

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There has been much progress recently in understanding the properties of black holes in n spacetime dimensions. In this talk I will attempt to to give an overview emphasising mathematical aspects and open mathematical problems. However, it is important to realise that the work is driven by physical considerations .

Motivation

- Understanding black holes in all dimensions should help understand 4-dimensional black holes, and how they are special.

- Higher dimensions arise in most attempts to unify gravity with the standard model (Kaluza-Klein theory, Supergravity, String theory, M-theory)

- If the extra dimensions are large ($\leq .1mm$) black holes and Hawking radiation could be produced at LHC starting in 2007.

- AdS/CFT: black holes in asymptotically AdS_5 backgrounds are dual to $SU(N)$ Yang-Mills Theory at non-zero temperature on the conformal boundary.

- Black holes give insight into the properties of very heavy BPS string states.

- Construction of positive definite Einstein metrics by Wick rotating and taking limits which generalise the Page metric on the non-trivial S^2 bundle over S^2. These have applications to string theory, particularly if they admit Killings spinors. for example Einstein-Sasaski metrics.

- Higher dimensional black holes may be used to study Black Hole entropy and state counting

Field equations

We shall mainly restrict our ourselves to the vacuum equations with cosmological term

$$R_{\mu\nu} = \pm \frac{1}{l^2}(n-1)g_{\mu\nu} \,. \quad (1)$$

The upper sign gives a positive cosmological constant. The lower sign applies to $AdS_n \equiv SO(n-1,2)/SO(n-1,1)$ or its universal covering spacetime $CAdS_n$ whose conformal boundary is the Einstein Static Universe $ESU_n = S^{n-1} \times R$ should be thought of as a timelike ultra-static cylinder at infinity.

There is a great known about charged black holes (with multiple $U(1)$'s) , with scalars and and in AdS backgrounds but it will largely be ignored in what follows.

Basic mathematical questions

- Existence
- Uniqueness
- Stability

Static Black Holes

The basic static examples are the **Schwarzschild-Kottler-Tangherlini Solutions**

$$ds^2 = -\Delta dt^2 + \frac{dr^2}{\Delta} + r^2 h_{mn}(y) dy^m dy^n \,, \tag{2}$$

$$\Delta = k - \frac{2m}{r^{n-3}} \mp \frac{r^2}{l^2}, \tag{3}$$

where $k = 0, \pm 1$ $m, n = 1, 2, \ldots, n-2$, and $h_{mn}(y)$ is any positive definite Einstein metric scaled such that

$$R_{mn} = k(n-3) h_{mn} \tag{4}$$

Birkhoff's Theorem applies: these solutions cannot be made time-dependent by allowing g_{tt} and g_{rr} to depend non-trivially on time keeping $h_{mn}(y)$ fixed.

Einstein Metrics

Two and three-dimensional Einstein metrics must be of constant curvature but in four and higher dimensions this is not so. If the scalar curvature is positive then Einstein metrics seem to be very plentiful in five and higher dimensions. Obviously we take S^n/Γ. More strikingly for $5 \leq d \leq 9$ Böhm showed, that there is an entire integer's worth of Einstein metrics on S^d ! These are cohomogeneity one, having isometry group $SO(p) \times S0(q)$ with orbits $S^p \times S^q$ $p, q > 1$ and $p+q+1=d$. They were obtained by reducing the equations to ODE's.

$$ds^2 = dt^2 + a^2(t)d\Omega_p^2 + b^2(t)d\Omega_q^2 \tag{5}$$

On $S^2 \times S^3$ Cvetic, Lü Pope, Page (hep-th/0505223) constructed a triple integer's worth of explicit co-homogeneity two Einstein-Sasaski metrics,obtained as Page style limits of explicit rotating de-Sitter black hole metrics and ∃ many more non-explicit examples.

These Einstein metrics give, for $k = 1, \frac{1}{l^2} = 0$, solutions which are regular outside a regular event horizon, even if the horizon is topologically spherical. Thus any attempt to prove uniqueness must make non-trivial global assumptions about asymptotic flatness.

Horizon Topology

Cai and Galloway (hep-th/0102149) have shown that for vanishing cosmological constant an apparent horizon (i.e. a stable mininal $(n-2)$ -surface in an $n-1$ dimensional inital value surface with non-negative scalar curvature) must be able to carry a metric of positive scalar curvature. That is have positive Yamabe constant.

This condition is trivially satisfied it the horizon carries an Einstein metric with positive scalar curvature.

If $n = 5$ the horizon may be a finite connected sum of homotopy spheres, possibly with identifications, and $k \geq 0$ copies of $S^2 \times S^1$.

Uniqueness of Minkowski Spacetime ?

If no black holes are present, and $\frac{1}{l^2} = 0$, it is simple to generalise Lichnerowicz's theorem to show that the globally static Minkowski ground state is unique subject to a strong AE assumption. One simple argument applies the maximum principle to $\sqrt{-g_{tt}}$ to obtain ultra-staticity

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j \,, \tag{6}$$

with g_{ij} a complete non-singular Ricci flat Riemannian manifold.

If we assume AE then, by Witten or Schoen Yau, $g_{ij} = \delta_{ij}$.

Another simple argument uses the positive mass theorem.

Failure of Lichnerowizc

The uniqueness fails badly if we drop AE condition.

For example take $n = 4$, and chose any gravitational instanton. Among these are ALE Hyperkähler 4-metrics which tend to E^4/Γ outside a compact set. One case is Eguchi-Hanson for which $\Gamma = Z_2$ or multi-centre metrics with $\Gamma = Z_k$,

However the metrics need not be ALE, they could be AF (e.g. Self-dual Taub-NUT or or even Atiyah-Hitchin) They need not be Hyper-Kähler (e.g. Taub-BOLT) In higher dimensions the possibilities are even more complicated.

Uniqueness of AdS

The ground state uniqueness result was been extended to the globally static AdS_n ground state by Boucher, Gibbons, Horowitz (n=4) and to higher dimensions by Wang (math.dg 0210165) and Qing (math.dg/0310281) The positive mass theorem plays an essential role,

There is also a proof of Anderson, as a special case of a stronger result*.

However, as with Minkowski spacetime we must make a strong AAdS assumption.

*see later

Uniqueness of Static Tangherlini Black Holes

If $\frac{1}{l^2} = 0$ Gibbons, Ida and Shiromizu (gr-qc/0203004 , hep-th/0206049) generalised the Bunting and Masood-ul-Alam proof of Israel's Theorem which makes use of the positive mass theorem to all dimensions. Hwang had independently done this earlier in 1998 in an unnoticed paper in *Geometriae Dedicata*. The use of the positive mass theorem excluded Tangherlini- Böhm black holes since they are not asymptotically flat.

The extension to the uniqueness of the Kottler solution in four dimensions has been reviewed by Chrusciel and Simon (gr-qc/0004032). There is a suggestive argument based on Cosmic Censorship/Isoperimetic bounds of the mass, which might generalise to higher dimensions *.

*see later

Anderson's approach

(see hep-th/0403087) Takes a static Lorentzian solution and Wick rotates to give a Riemannian solution with $SO(2)$ isometry group. This requires all connected components of the horizon have the same surface gravity.

He then shows that, for negative cosmological constant , any isometry of the conformal boundary extends to an isometry of the interior. If the boundary is $S^1 \times S^{n-2}$ with isometry group $SO(2) \times SO(n-1)$ one may now reduce to ODE's.

Thus Schwarzschild-AdS is unique in all dimensions.

Linear Stability of Static Black holes

Gibbons and Hartnoll (hep-th/0206202) pointed out that if $d > 2$, there are non-trivial tensor harmonics on S^d and hence an entirely new type of gravitational wave fluctuation around a Tangherlini (or Kottler-Tangherlini) black hole. These are tensor modes in addition to the usual scalar and vector modes. The tensor modes are also present for the general Schwarzschild-Kottler-Tangherlini solution with arbitrary Einstein metric on the 'base manifold'$\{B, h_{mn}\}$. The problem may be reduced to the spectrum a radial equation of Schrödinger form whose potential depends upon the spectrum of the Lichnerowicz operator(i.e. linearised Einstein tensor in de Donder gauge) on $\{B, h_{mn}\}$. A stability criterion was obtained:

$$\lambda_{\text{min}} \leq 4 - \frac{(7-n)^2}{4} \qquad \Longleftrightarrow \text{instability} \tag{7}$$

According to this criterion if $\{B, h_{mn}\}$ is the standard round metric on S^{n-2} the solution is stable against tensor modes. Ishibashi and Kodama (hep-th/0305147) shown stability against scalar and vector modes .

Gibbons, Hartnoll and Pope (hep-th/0208031) showed that Böhm metrics on S^5 and $S^2 \times S^3$ give rise to unstable black holes but Einstein-Sasaki metrics (in fact which admit Killing spinors *) are stable.

These instabilities are no longer present for a sufficiently negative cosmological constant.

*An Einstein-Sasaski manifold is such that the cone over it it is Ricci-flat Kähler (i.e. Calabi-Yau)

Non-linear stability

In a recent paper Bizon, Chmaj and Schmidt(gr-qc/0506074)) have studied numerically a time-dependent co-homogeneity two ($U(2)$-invariant) ansatz which corresponds to a fully non-linear $l = 1$ excitation of the five-dimensional Tangherlini-metric.

$$ds^2 = -Ae^{-2\delta}dt^2 + A^{-1}dr^2 + \frac{r^2}{4}(e^{2B}(\sigma_1^2 + \sigma_2^2) + e^{-4B}\sigma_3^2) \,. \tag{8}$$

This is a very rich ansatz and incorporates as special cases many of the static metrics mentioned above as well as Kaluza-Klein Black holes. This is also at least one exact time-dependent solution (see Gibbons, Lu and Pope (hep-th/ hep-th/0501117)) in which dynamical decompactification takes place

Rotating Black holes

The analogue of the Kerr solutions were found by Myers and Perry in (1986). A body in $(n-1)$ spatial dimensions may rotate in $[\frac{n-1}{2}]$ orthogonal two-planes and hence have $[\frac{n-1}{2}]$ angular velocities Ω_i and $[\frac{n-1}{2}]$ conjugate angular momenta J_i , The metric contains as parameters a 'mass'm and $[\frac{n-2}{2}]$ 'rotation parameters a_i. The isometry group is, in general, reduced from $R \times SO(n-1)$ to $R \times SO(2)^{[\frac{n-1}{2}]}$, the latter being the maximal torus of $SO(n-1)$. In this case, the metric is of cohomogeneity one.

If some of the rotation parameters are equal there can be enhanced symmetry. Thus if $a_1 = a_2 = \ldots = a_{[\frac{n-1}{2}]}$ and n is odd $U([\frac{n-1}{2}])$ acts. In this case, the metric is of cohomogeneity one and in principle the Einstein equations reduce to ODE's.

However, in general, the metrics are of cohomogeneity $\frac{n-1}{2}$ or $\frac{n}{2}$ as n is odd or even respectively. If $n \geq 5$ the cohomogeneity exceeds 2 and standard solution generating techniques (generalised Ernst equations etc)for obtaining such metrics are inapplicable. *

Thus some other technique must be used. This is ....

*If $n = 5$, however, such techniques do apply and may be used to study may used,see later

The Kerr-Schild Ansatz

The key to Myers and Perry's success is that if

$$
\begin{aligned}
g_{\mu\nu} &= \bar{g}_{\mu\nu} + h_{\mu\nu}\,, && (9)\\
h_{\mu\nu} &= \frac{2m}{U} k_\mu k_\nu\,, && (10)
\end{aligned}
$$

the Einstein equations linearise! provided

(i) $\bar{g}_{\mu\nu}$ is Einstein,

(ii) $g^{\mu\nu} k_\mu k_\nu = 0\,,$

(iii) k^μ is tangent to a null geodesic congruence.

This is because

$$\bar{g}^{\mu\nu} k_\mu k_\nu = 0 \,, \tag{11}$$
$$\bar{g}^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu} \,, \tag{12}$$
$$k^\mu = g^{\mu\nu} k_\mu \,, \tag{13}$$
$$= \bar{g}^{\mu\nu} k_\nu \,, \tag{14}$$

and if k^μ is tangent to a null geodesic congruence

$$R^\mu_\nu = \bar{R}^\mu_\nu - h^\mu_\rho \bar{R}^\rho_\nu + \frac{1}{2} \bar{\nabla}^\rho \bar{\nabla}_\nu h^{\mu\rho} + \frac{1}{2} \bar{\nabla}^\rho \nabla_\mu h_{\nu\rho} - \frac{1}{2} \bar{\nabla}^\rho \bar{\nabla}_\rho h^\mu_\nu \tag{15}$$

Kerr solution in n even spacetime dimensions: the background is Minkowski spacetime; the null vector is

$$k = k_\mu dx^\mu = dt + \sum_{i=1}^{[\frac{n-1}{2}]} \frac{r(x_i dx_i + y_i dy_i) + a_i(x_i dy_i - y_i dx_i)}{r^2 + a_i^2} + \frac{z dz}{r} \,, \quad (16)$$

$$U = \frac{1}{r}(1 - \sum_{i=1}^{[\frac{n-1}{2}]} \frac{a_i^2 (x_i^2 + y_i^2)}{(r^2 + a_i^2)^2}) \prod_{j=1}^{[\frac{n-1}{2}]} (r^2 + a_j^2) \,, \quad (17)$$

$$\sum_{i=1}^{[\frac{n-1}{2}]} \frac{x_i^2 + y_i^2}{r^2 + a_i^2} + \frac{z^2}{r^2} = 1 \,. \quad (18)$$

Note that these metrics are most naturally presented in Boyer-Lindquist coordinates which are spheroidal rather than spherical at infinity.

Non-Uniqueness and stability

One might have anticipated that the usual uniqueness and stability results would hold in higher dimensions. Indeed, in five(and no higher than five) dimensions where the metric is cohomogeneity two Morisawa and Ida (gr-qc/0401100) have shown, following work of Maison, that the Einstein equations are of generalised Ernst non-linear sigma model form with target $SL(3,R)/SO(3)$ and obtained Robinson-Mazur type identities. Indeed Morisawa and Ida construct a proof of uniqueness assuming that the isometry group is $R \times SO(2) \times SO(2)$ * and that the horizon has spherical topology

However...

*Hawking's argument, if applicable would seem to give just $R \times SO(2)$

Black Rings

Emparan and Reall(hep-th/0110260) have constructed asymptotically flat five dimensional ‘black ring ’solutions whose horizon topology is $S^1 \times S^2$. The solution is obtained by Wick rotating a Kaluza-Klein C-metric.

The isometry group is $R \times SO(2) \times SO(2)$ and the metric is of chomogeneity two. However it has just one angular momentum non=zero, and tends to the metric of a Myers-Perry black hole with one rotation parameter non-zero. For certain values of J and M one may have three solutions, one Myer-Perry black hole and two ring solutions.

$$
\begin{aligned}
ds^2 &= -\frac{F(y)}{F(x)}(dt + C(\nu)R\frac{1+y}{F(y)}d\psi)^2 && (19)\\
&+ \frac{R^2}{(x-y)2}(-\frac{G(y)}{F(y)}d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2) && (20)\\
C(\nu) &= \frac{\nu}{1+\nu^2}(\frac{2(1+\nu)^3}{1-\nu})^{\frac{1}{2}}\,, && (21)\\
F(u) &= 1 + \frac{2\nu}{1+\nu^2}u\,, && (22)\\
G(u) &= (1-u^2)(1+\nu u)\,. && (23)
\end{aligned}
$$

ϕ and ψ have period $\frac{2\pi}{\sqrt{1+\nu^2}}$.

$$
E = \frac{3\pi}{2}R^2\frac{\nu}{(1-\nu)(1+\nu^2)}\,, \qquad J = \frac{\pi}{2}R^2\frac{C(\nu)}{(1-\nu)\sqrt{1+\nu^2}}\,. \qquad (24)
$$

Ergo-regions and Penrose Process

Both the Myers-Perry and the Emparan-Reall rotating black holes have ergo-regions and the Penrose process has been investigated by Nozawa and Maeda (hep-th/0502166).

Geodesisc can be found in the equatorial plane, but in general the Hamilton-Jacobi equation does not seem to separate.

Ist Law of Thermodynamics

This includes a dipole term. (see Copsey and Horowitz (hep-th/0505278)

Higher Dimensions

No other asymptotically flat solutions are known in higher dimensions, but if $n >$ the angular momenta of Meyers-Perry black holes at fixed mass can be arbitrarily large and the horizon rather oblate. Emparan and Myers (hep-th/0308056) have conjectured that they become unstable by analogy with the known Gregory-Laflamme instability of the 'black string', Schwarzschild$_4$ $\times R$.

Kerr AdS Black Holes

If $n = 4$, this is due to Carter in 1968 It may be cast into Kerr-Schild form and the background is AdS_4 in spheroidal Boyer-Lindquist coordinates. The generalisation to five dimensions was given by Hawking Hunter and Taylor-Robinson (hep-th/9811056) who also gave the case of a single rotation parameter in all dimensions (which has enhanced symmetry) essentially using a guessing method. All are of Kerr-Schild form which suggested to Gibbons,Lu, Page and Pope (hep-th/0404008, hep-th/0409155) a more systematic procedure, reducing the problem to a linear one. This allowed the construction of an ansatz in all dimensions with the full number of rotation parameters which was checked using computer algebra in all dimensions up to eleven.

The idea was to write down the AdS_n metric in Boyer-Lindquist spheroidal coordinates and then make a guess for k_μ and U . The metric comes out automatically in a frame which is rotating at infinity with angular velocity

$$\Omega_i^\infty = -\frac{a_i}{l^2} \tag{25}$$

This must be taken into account when evaluating the energies and angular momenta.

Boyer-Lindquist for AdS

If n is odd and $\lambda = -\frac{1}{l^2}$,

$$d\bar{s}^2 = -(1-\lambda y^2)dt^2 + \frac{dy^2}{1-\lambda y^2} + y^2 \sum_{k=1}^{\frac{n-1}{2}} (d\hat{\mu}_k^2 + \hat{\mu}_k^2\, d\phi_k^2)\,, \tag{26}$$

$$\sum_{i=1}^{\frac{n-1}{2}} \hat{\mu}_i^2 = 1\,. \tag{27}$$

We now define new "spheroidal coordinates" (r, μ_i) according to

$$(1+\lambda a_i^2)\, y^2\, \hat{\mu}_i^2 = (r^2 + a_i^2)\, \mu_i^2\,, \tag{28}$$

$$\sum_{i=1}^{\frac{n-1}{2}} \mu_i^2 = 1\,. \tag{29}$$

Note that (28) and (29) imply that

$$y^2 = \sum_{i=1}^{\frac{n-1}{2}} \frac{(r^2 + a_i^2)\mu_i^2}{1+\lambda a_i^2}\,. \tag{30}$$

In terms of the new coordinates (r, μ_i), the de Sitter metric (26) becomes

$$\begin{aligned} d\bar{s}^2 \;&=\; -W\,(1-\lambda\,r^2)\,dt^2 + F\,dr^2 + \sum_{i=1}^{\frac{n-1}{2}} \frac{r^2+a_i^2}{1+\lambda\,a_i^2}\,(d\mu_i^2 + \mu_i^2\,d\phi_i^2) \\ &+ \frac{\lambda}{W\,(1-\lambda\,r^2)}\,\Big(\sum_{i=1}^{\frac{n-1}{2}} \frac{(r^2+a_i^2)\,\mu_i\,d\mu_i}{1+\lambda\,a_i^2}\Big)^2\,, \end{aligned} \tag{31}$$

where

$$W \equiv \sum_{i=1}^{\frac{n-1}{2}} \frac{\mu_i^2}{1 + \lambda\, a_i^2}\,, \qquad\qquad F \equiv \frac{r^2}{1 - \lambda\, r^2} \sum_{i=1}^{\frac{n-1}{2}} \frac{\mu_i^2}{r^2 + a_i^2}\,. \tag{32}$$

One gets Kerr AdS by setting

$$k_\mu \, dx^\mu = W \, dt + F \, dr - \sum_{i=1}^{\frac{n-1}{2}} \frac{a_i \, \mu_i^2}{1 + \lambda \, a_i^2} \, d\phi_i \,, \tag{33}$$

$$U = \sum_{i=1}^{\frac{n-1}{2}} \frac{\mu_i^2}{r^2 + a_i^2} \prod_{j=1}^{\frac{n-1}{2}} (r^2 + a_j^2) \,. \tag{34}$$

There are similar expressions for even n

Conserved Charges

Possession of the explicit metrics allowed Gibbons, Perry and Pope (hep-th/0408217) to calculate the surface gravities κ, event horizon areas A, angular velocities Ω_i relative to a non-rotating frame at infinity and energies E and angular momenta J_i . This is non-trivial and requires not only passing to a frame which is non-rotating at infinity but to asymptotically spherical coordinates.

Subsequent work by Deruelle and Katz (gr-qc/0410135), Barnich and Compere gr-qc/0412029) Deser,Kanik and Tekim(gr-qc 0506057) has confirmed these results.

Thermodynamics

One may also check that the charges satisfy the 1st Law of Thermodynamics

$$dE = TdS + \Omega_i dJ_i \,, \qquad T = \frac{\kappa}{\pi} \,, \qquad S = \frac{1}{4} A \,, \tag{35}$$

and the Quantum Statistical Relation, that is

$$\frac{1}{T}(E - TS - \Omega_i J_i) = \text{Euclidean Action} \tag{36}$$

and also the (supersymmetric) positivity bounds

$$\boxed{E \geq \frac{1}{l} |J_i|} \tag{37}$$

Explicitly, if

If $\Xi_i = 1 - \frac{a_i^2}{l^2}$,

and $\mathcal{A}_{n-2} = \mathrm{vol}(S^{n-2})$

$$\boxed{J_i = \frac{m\, a_i\, \mathcal{A}_{n-2}}{4\pi\, \Xi_i\, (\prod_j \Xi_j)}} \tag{38}$$

n odd
$$\boxed{E = \frac{m\, \mathcal{A}_{n-2}}{4\pi\, (\prod_j \Xi_j)} \Big(\sum_{i=1}^{\frac{n-1}{2}} \frac{1}{\Xi_i} - \frac{1}{2} \Big)} \tag{39}$$

n even
$$\boxed{E = \frac{m\, \mathcal{A}_{n-2}}{4\pi\, (\prod_j \Xi_j)} \sum_{i=1}^{\frac{n-2}{2}} \frac{1}{\Xi_i}} \tag{40}$$

Instability of Kerr AdS ?

Work by Cardoso (hep-th/0405006) on a massless scalar field points to an instability of small four-dimensional Kerr-AdS black holes due to super-radiant scattering which is analogous to the Black Hole Bomb. Because gravitational waves reflect off the conformal boundary of AdS, amplification should take place. It is highly likely that that also occurs for higher dimensional Kerr AdS black holes.

This could be checked because....

Separability of Hamilton Jacobi and Wave equations

If $n = 4$, despite the fact that the metric is cohomogeneity, the Hamilton-Jacobi equation is separable and hence the geodesic flow of the Kerr-AdS metric is Liouville integrable. This is due to the presence of an extra constant of the motion, arising from a Killing-Staeckel Tensor of rank two. Similarly, the Wave equation separates.

According to Kunduri and Lucietti(hep-th/ hep-th/0502124) This remains true if $n = 5$. There are similar results in > 5 dimensions in the case that the rotation parameters take only two distinct values. (see Vasudevan Stevens (gr-qc/0507096)) .

$$K = K^{\mu\nu} p_\mu p_\nu \qquad H = g^{\mu\nu} p_\mu p_\nu \tag{41}$$

$$\{H, K\} = 0 \Leftrightarrow \nabla^{(\mu} K^{\nu\sigma)} = 0\,. \tag{42}$$

Despite separability of the Dirac equation if $n = 5$ and $J_1 = J_2$ (Davis, Kunduri,and Lucietti, hep-th/0508169), at unlike the case $n = 4$, there seems to be no Killing-Yano two form.

$$K_{\mu\nu} = g^{\alpha\beta} Y_{\mu\alpha} Y_{\nu\beta} \tag{43}$$

$$\nabla_{(\mu} Y_{\nu)\sigma} = 0\,. \tag{44}$$

Hopefully, these results indicate that separating the gravitational perturbation equations may be feasible.

Cosmic Censorship Inequalities and AdS Bekenstein Bounds

Verlinde (hep-th/0008141) has suggested that a CFT in $(n-1)$ dimensions should satisfy, at least at large energies, a so called Cardy-Verlinde-de formula.

$$S = \frac{2\pi l}{n-2}\sqrt{E_c(2E - E_c)}\,, \tag{45}$$

where E_c is given by

$$\boxed{E_c := (n-2)[(1 + \frac{1}{n-2})E - TS - \Omega J - \Phi Q]\,.} \tag{46}$$

and is, since $PV = \frac{E}{n-2}$, a measure of non-extensivity.

Inverting,

$$E = \frac{1}{2}E_c + \frac{1}{2E_c}[\frac{(n-2)S}{2\pi l}]^2 . \qquad (47)$$

Minimisation with respect to E_c leads to the lower bound

$$\boxed{2\pi l E \geq (n-2)S\,.} \qquad (48)$$

which we shall refer to as the *Anti-de Sitter Bekenstein Bound*, regardless of whether it arises from a Cardy-Verlinde formula. Clearly the anti-de Sitter Bekenstein bound is a necessary, *but not sufficient* condition for the existence of a Cardy-Verlinde formula.

Remarkably, although CFT's do not satisfy the Bekenstein bound and hence cannot satisfy the Cardy-Verlinde formula, Kottler black holes satisfy both. Moreover equality is occurs at the Hawking-Page phase transition.

Gibbons, Perry and Pope (hep-th/0506233) recently examined all known AdS black holes, charged and rotating in four and higher dimensions and found that while the Cardy-Verlinde formula does not hold in general * the Bekenstein bound always does.

This appears to be a consequence of a Cosmic Censorship Area inequality, which holds in all cases we have checked..

*There is a subtlety with Kerr-AdS

$$E \geq \frac{(n-2)A}{16\pi l}\,[l\,(\frac{A}{\mathcal{A}_{n-2}})^{-\frac{1}{n-2}} + \frac{1}{l}\,(\frac{A}{\mathcal{A}_{n-2}})^{\frac{1}{n-2}}]\,. \qquad (49)$$

Some consequences of this bound, which is a more global extension of Hawking's variational principle for black holes are:

(1) If $l \to \infty$, then (49) reduces to a bound first proposed in $n = 4$ dimensions by Penrose, who observed that it is a necessary condition for the cosmic censorship hypothesis.

(2) The proposed bound (49) implies a generalisation of the AdS$_n$ Bekenstein bound, to the non-stationary case. Noting that the quantity in square brackets in (49) must be greater than or equal to 2, we have a generalisation of Bekenstein's bound to the time-dependent case when one may no longer equate entropy with 1/4 of the area of an apparent horizon:

$$E \geq \frac{(n-2)A}{8\pi l}. \tag{50}$$

(3) The cosmic censorship bound (49) is attained for the case of a Schwarzschild-anti-de Sitter black hole, and one conjectures that this is the only case for which it is saturated.

In four spacetime dimensions, the Geroch-Wald-Jang-Huisken-Ilmanen proof of the original Penrose Inequality formally extends to the AdS version. In higher dimensions, even the Penrose Inequality appears has not yet been settled.

It is striking however that this inequality also figured in the work of Chrusciel and Simon on the uniqueness of the Schwarzschild AdS mentioned earlier.

The AdS Bekenstein bound does not contain Newton's constant, and might therefore be expected to hold for a (free) QFT in AdS_n.

Recent work by Gibbons, Perry and Pope shows that it does not

Kerr dS Black Holes

Our rotating black hole solutions are also valid if the cosmological constant is taken to be positive, just send $l^2 \to -l^2$. In this case, an additional, cosmological, horizon is present. We verified that all known solutions, including many with matter, support the General Conjecture that the area A_C of the cosmological horizon satisfies

$$\boxed{A_C \leq \mathcal{A}_{n-2}\, l^2\,,} \tag{51}$$

and the black hole horizon satisfies

$$\boxed{A_H \leq \mathcal{A}_{n-2}\, l^{n-2}\, (\frac{n-3}{n-1})^{\frac{n-2}{2}}\,.} \tag{52}$$

The should extend to marginally inner or outer trapped surfaces. Can these be proved using the inverse mean curvature flow?