Asymptotic expansions for cosmological solutions of the Einstein equations

Introduction

This talk is concerned with solutions of the Einstein equations, coupled to matter, which can be considered as cosmological models.

More specifically, it is concerned with two important asymptotic regimes: the approach to an initial singularity and a phase of unending expansion.

In the first case there are past-incomplete causal geodesics; in the second future-directed causal geodesics are complete.

Consider a globally hyperbolic spacetime with a global time function t taking values in the interval (t_-, t_+). We want to understand what happens in the limit $t \to t_\pm$.

This problem will be called tractable if the geometry and matter variables have asymptotic expansions of a certain kind. For a generic function f the defining property is

$$\|f(t,x) - \sum_{i=0}^{N} f_i(x) g_i(t)\| \leq C \|g_{N+1}(t)\| \quad (1)$$

Furthermore the relations obtained by differentiating (1) term by term with respect to the time or space variables should also hold.

The existence of an expansion of this kind represents a true restriction on the function f. Supposing that f_0 is non-zero it follows that $\partial_x f/f = \partial_x f_0/f_0 + o(1)$.

If f satisfies $f_{tt} = f_{xx}$ with periodic boundary conditions an expansion of the type (1) cannot hold (persistent waves).

Although the two asymptotic regimes we are considering are very different there are mathematical similarities and insights gained in one can be transferred to the other.

Some tractable problems

Generic solutions of the vacuum Einstein equations with $\Lambda = 0$ are not tractable in the sense introduced above in either time direction due to BKL oscillations in the past and persistent waves in the future.

A case where it has been proved that the asymptotic problem is tractable is that of the future behaviour of solutions of the vacuum Einstein equations with $\Lambda > 0$ (H. Friedrich, 1986).

A small data restriction is needed, e.g. data close to those for the de Sitter solution $-dt^2 + e^{2Ht}(dx^2 + dy^2 + dz^2)$, $H = \sqrt{\Lambda/3}$.

The metric is written in Gauss coordinates $-dt^2 + g_{ij}(t,x)dx^i dx^j$ and the asymptotic expansion is

$$g_{ij}(t,x) = g^0_{ij}(x)e^{2Ht} + g^2_{ij}(x) + g^3_{ij}(x)e^{-Ht} + \ldots, t \to \infty \qquad (2)$$

Any data close enough to de Sitter data gives rise to a solution which exists globally in the future and has asymptotic behaviour of this kind. The proof uses conformal methods to transform the global problem to a local one.

The proof is special to four dimensions. A new approach allowing it to be extended to higher even dimensions has been developed by M. Anderson (2004).

Solutions of the Einstein equations with $\Lambda > 0$ exhibit accelerated expansion and this can be also be obtained using certain scalar field models.

Consider the Einstein equations with $\Lambda = 0$ minimally coupled to a nonlinear scalar field with potential $V(\phi)$.

When the potential is exponential, $V(\phi) = V_0 e^{-k\phi}$ with $k > 0$ a sufficiently small constant accelerated expansion is known to occur in the homogeneous case (power-law inflation).

By taking the above results of Anderson and applying Kaluza-Klein reduction M. Heinzle and A. Rendall (2005) showed that results qualitatively similar to (2) for the above potential and a discrete set of values of k.

For the Einstein vacuum equations with $\Lambda = 0$ L. Andersson and V. Moncrief have obtained information on late-time asymptotic behaviour for data which are small perturbations of those for the Milne model $-dt^2 + t^2(dx^2 + dy^2 + dz^2)$.

However this problem probably does not belong to the tractable class. A one-term asymptotic expansion of the form (1) is obtained but can presumably not be extended to higher order due to persistent waves.

In the case of approach to the singularity there is not a single case where a problem has been proved to be tractable under the sole restriction of small initial data.

In the absence of direct results determining the asymptotics of an open set of initial data an alternative is to prove the existence of very general solutions with prescribed asymptotics.

Prescribed asymptotics

Results on prescribed asymptotics can be obtained using the theory of Fuchsian systems. These comprise a class of singular PDE.

Fuchsian techniques were applied by L. Andersson and A. Rendall (2001) to the Einstein equations coupled to a linear scalar field.

According to the general heuristic picture of spacetime singularities due to Belinskii, Khalatnikov and Lifshitz (BKL) a scalar field suppresses the oscillations present for most types of matter fields.

It has been proved that there exist very general solutions which have an expansion of the form (1) near the initial singularity.

Here the leading term is a homogeneous Bianchi type I solution which varies from one spatial point to the next.

These results confirm some features of the general BKL picture rigorously in this particular case.

Near the singularity the evolution at different spatial points decouples and can be modelled by a homogeneous solution.

The homogeneous solution depends on the spatial coordinates as parameters.

Let λ_i be the eigenvalues of the second fundamental form of the hypersurfaces $t =$const. and let $p_i = \lambda_i / \sum_j \lambda_j$.

The generalized Kasner exponents (GKE) $p_i(t,x)$ are functions on spacetime which converge uniformly to a function $p_i^\infty(x)$ as the singularity is approached.

In the case of the expansion (2) the GKE all tend to 1/3, which corresponds to isotropization. This indicates a way in which the expanding direction is better than the contracting one when determining asymptotics.

In space dimensions ≥ 10 a generalization of the BKL procedure indicates that the approach to the singularity should be tractable for vacuum solutions.

This has been confirmed rigorously by T. Damour, M. Henneaux, A. Rendall and M. Weaver (2002) who also allowed a variety of matter fields.

Results for a wider class of problems can be obtained if asymptotic expansions of genuine solutions are replaced by formal asymptotic expansions.

With this replacement there are results on the expanding phase for the Einstein equations with $\Lambda > 0$ and a perfect fluid with linear equation of state $p = (\gamma - 1)\rho$.

The expansions were first written down by A. Starobinsky (1983) and put into a clear mathematical framework by A. Rendall (2004).

It is probably possible to get existence theorems for corresponding solutions using Fuchsian techniques but this has not been done yet.

Notice that this is the first result we have mentioned where a model of ordinary (phenomenological) matter such as a fluid or kinetic matter is included.

It is the latter which is needed for applications to astrophysics, where this matter is needed to describe galaxies.

Note that when inhomogeneous models are considered in astrophysical cosmology it is almost always in the context of a linearized approximation.

A careful analysis on the level of formal asymptotic expansions has been carried out by R. Bieli (2005) for certain nonlinear curvature-coupled scalar fields.

The BKL picture is a source of ideas about asymptotic expansions near the initial singularity but they are almost always accompanied by oscillations.

Up to now there are no rigorous results for any class of inhomogeneous models which exhibits oscillations.

Another complication concerns localized structure. Here the blow-up of $\partial_x f/f$ prevents applicability of the tractable case,

A variety of different heuristic and numerical approaches to the problem of the structure of initial singularities exist but on the level of rigorous mathemtics we are still waiting for the breakthrough.

There is one other class of rigorous results on the structure of singularities where no symmetry is assumed, namely those on isotropic singularities.

This work was motivated by Penrose's Weyl curvature hypothesis and has been completed for matter described by the Euler equation (linear equation of state) or the Vlasov equation.

The proofs, due to K. Anguige, C. Claudel, R. Newman and P. Tod, (1993-2000) are related to Fuchsian methods.

The case of matter described by the Boltzmann equation has been studied by P. Tod on the level of formal power series (2003).

Conclusions

1. There are various results on the asymptotics of cosmological solutions of the Einstein equations without any symmetries, most of them in the case described here as tractable.

2. Fuchsian techniques have been very successful and they still have many more potential applications to the Einstein equations.

3. The problem of getting rigorous resulys on oscillatory behaviour in the asymptotics of inhomogeneous solutions is still wide open.