On the topology of black holes in higher dimensions

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based on joint work with
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Introduction

- There has been increased interest in higher dimensional black holes in recent years. It is of interest to determine which properties of black holes in 4-dim extend to higher dimensions.
- Here we consider the topology of higher dimensional black holes. Hawking's original theorem on black hole topology states roughly:

  In a 4-dimensional spacetime obeying suitable energy conditions the surface of a steady state black hole is spherical (topologically S^2)

  This result also extends to *apparent horizons*.
- Aim is to present a generalization of Hawking's theorem to higher dimensions. The natural conclusion in higher dimensions is that the surface of a steady state black hole is of positive Yamabe type, i.e., admits a metric of positive scalar curvature.

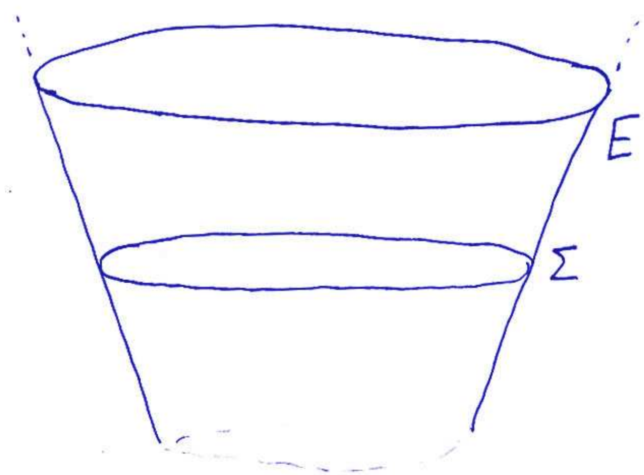
Hawking´s theorem

Theorem Suppose (M, g) is a 4-dimensional AF stationary black hole spacetime obeying the dominant energy condition (DEC). Then cross sections of the event horizon are spherical.

AF = admits a regular scri (conformal infinity) $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}-$

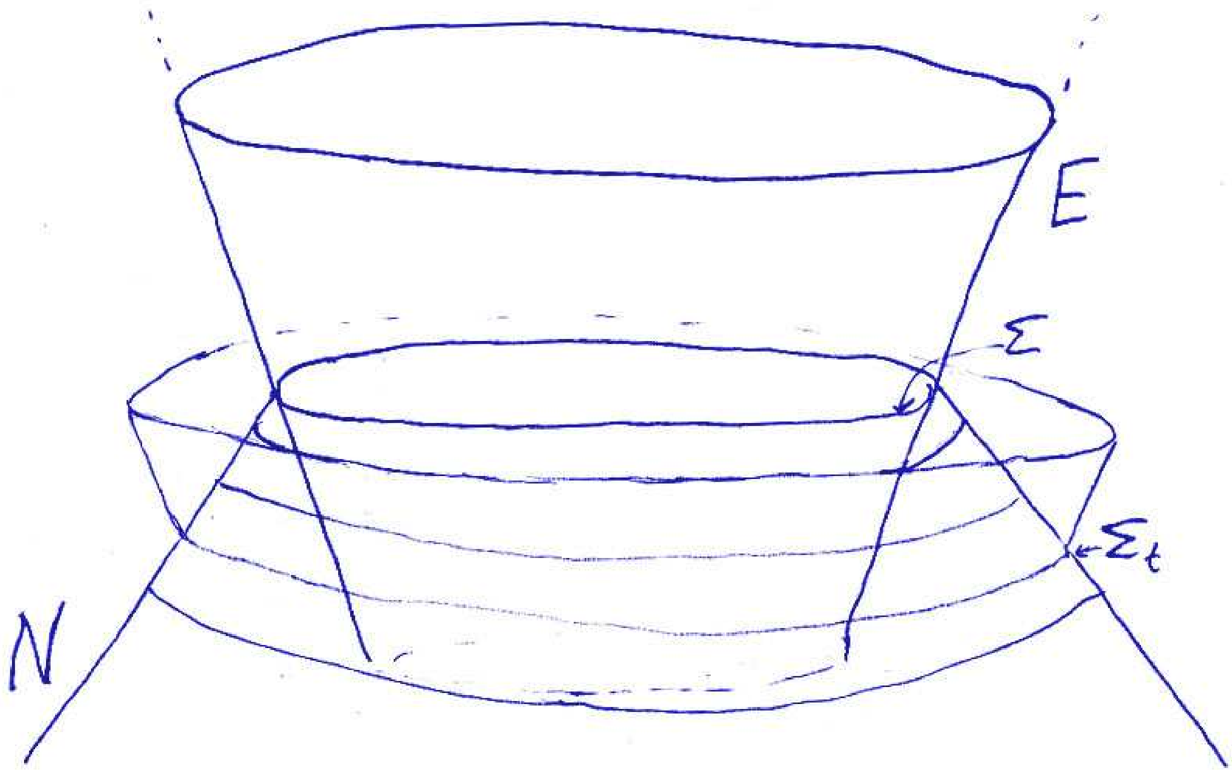
E = event horizon = $\partial I^-(\mathscr{I}^+)$

cross section = closed 2-surface Σ obtained as intersection of E with a spacelike hypersurface.



Hawking's theorem

Idea of proof: If $\Sigma \not\approx S^2$, i.e., if $g \geq 1$ then can deform Σ to an outer trapped surface outside the black hole region, which is forbidden by standard results.



Hawking's theorem

Remark: Actually the torus T^2 is borderline of this argument.

$$g = 1 \quad \Rightarrow \quad \left.\frac{\partial \theta}{\partial t}\right|_{t=0} \leq 0$$

But can have $\Sigma \approx T^2$ only under special circumstances:

- Σ must be flat
- null expansion and shear vanish on Σ
- A certain energy-momentum term vanishes along Σ.

So, by Hawkings' argument, generically, $\Sigma \approx S^2$.

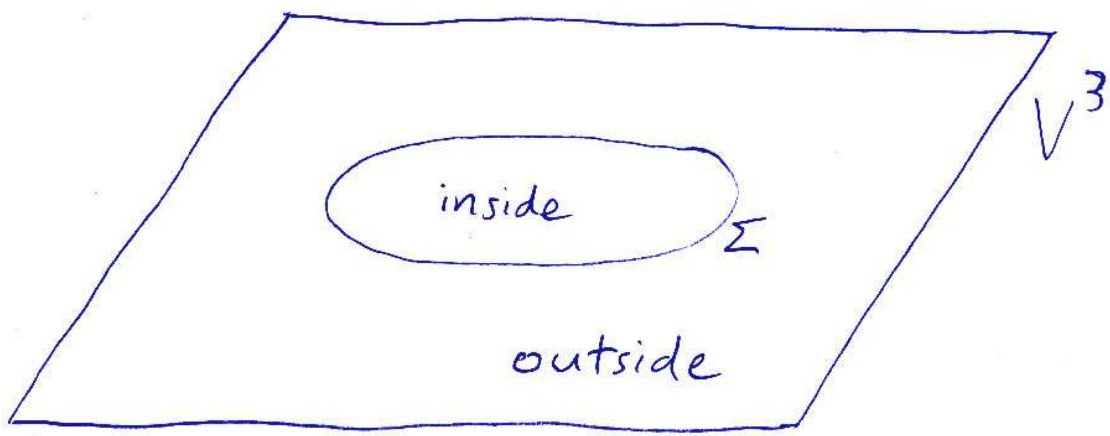
Hawking's theorem for apparent horizons

The conclusion of Hawking's theorem holds for apparent horizons in spacetimes that needn't be stationary.

Consider,

$$
\begin{aligned}
M^4 &= \text{4-dim spacetime} \\
V^3 &= \text{spacelike hypersurface in } M^4 \\
\Sigma &= \text{closed 2-surface in } V^3
\end{aligned}
$$

Suppose Σ separates V^3 into an "inside" and an "outside":



Hawking´s theorem for apparent horizons

We say Σ is an outer apparent horizon provided:

- Σ is marginally outer trapped,

$$\theta = 0 \quad \text{wrt outward null normal}$$

- there are no outer trapped surfaces outside of Σ.

Heuristically, Σ is the "outer limit" of outer trapped surfaces.

Theorem Let M^4 be a spacetime satisfying the DEC. If Σ is an outer apparent horizon in V^3 then $\Sigma \approx S^2$ (generically).

Remark: This theorem includes the previous theorem on the topology of stationary black holes as a special case.

Our aim is to obtain a higher dimensional version of this theorem.

Topological censorship

A completely different approach to studying black hole topology arose in the 90's based on the notion of topological censorship.

Friedman, Schleich, Witt, Woolgar, G., ..

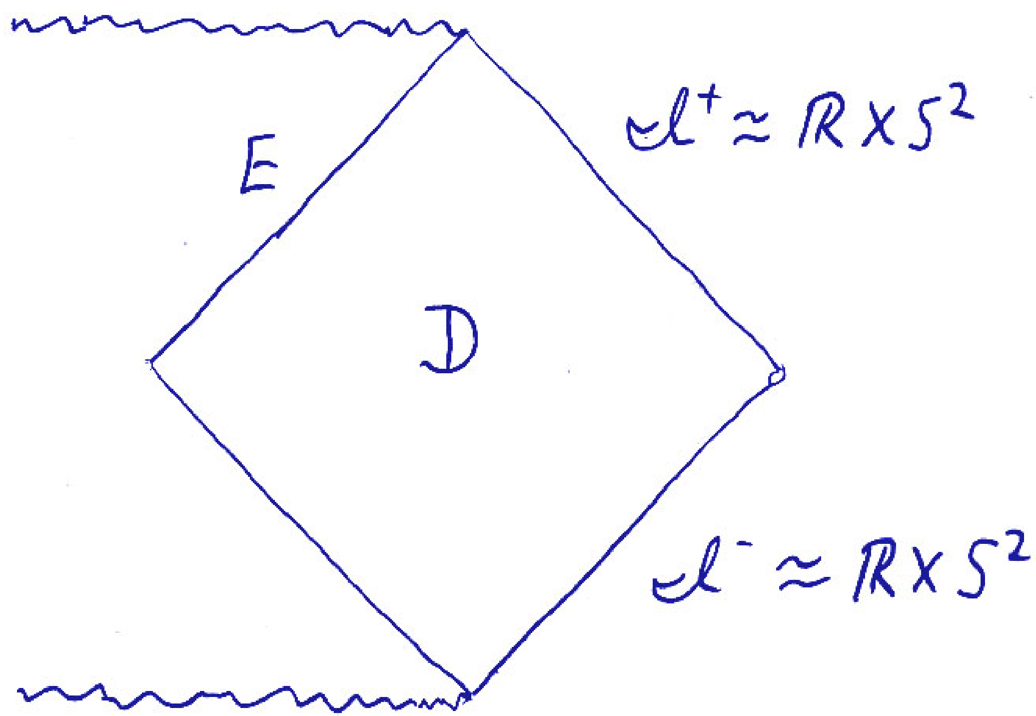
From the point of view of topological censorship the domain of outer communications - the region outside of all black holes and white holes - should have simple topology.

Top cen and the topology of black holes

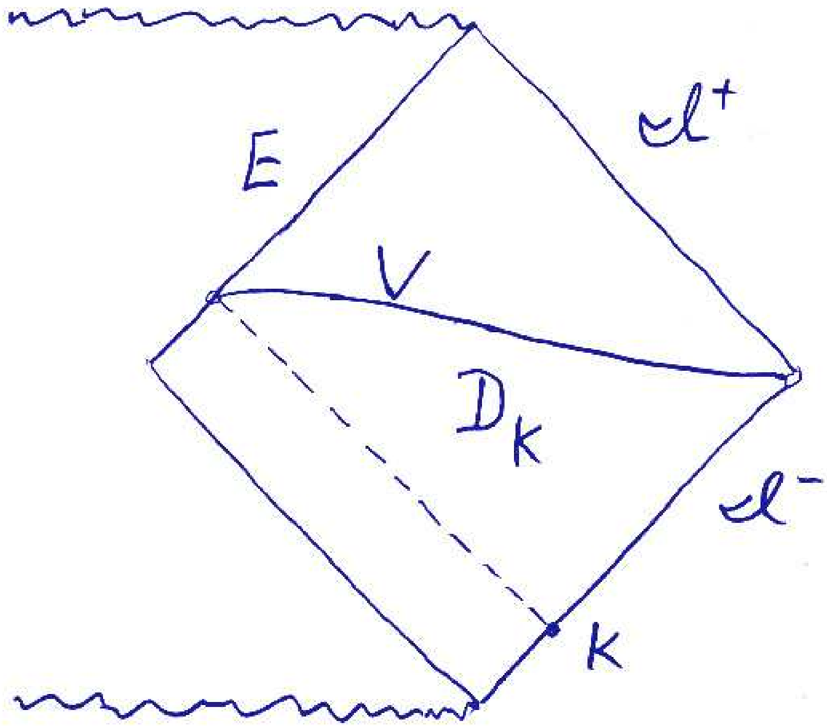
Theorem (G.) Let M be a 4-dim AF spacetime obeying the NEC. Suppose that the DOC

$$D = I^{-}(\mathscr{I}^{+}) \cap I^{+}(\mathscr{I}-)$$

is globally hyperbolic. Then D is simply connected.



Top cen and the topology of black holes



<u>Fact:</u> D_K is globally hyperbolic and simply connected.

$$\overline{V} = \text{closure of } V \text{ in } M$$
$$= V \cup \Sigma$$

$\overline{V}$ simply connected $\Rightarrow \Sigma \approx S^2$

Top cen and the topology of black holes

Remark: Topological censorship holds in arbitrary dimension. As long as scri is simply connected the DOC will be simply connected. However, this fact cannot be used to determine the topology of black holes in higher dimensions. Only in spatial dimension 3 does the simple connectivity of a manifold-with-boundary determine the topology of its boundary.

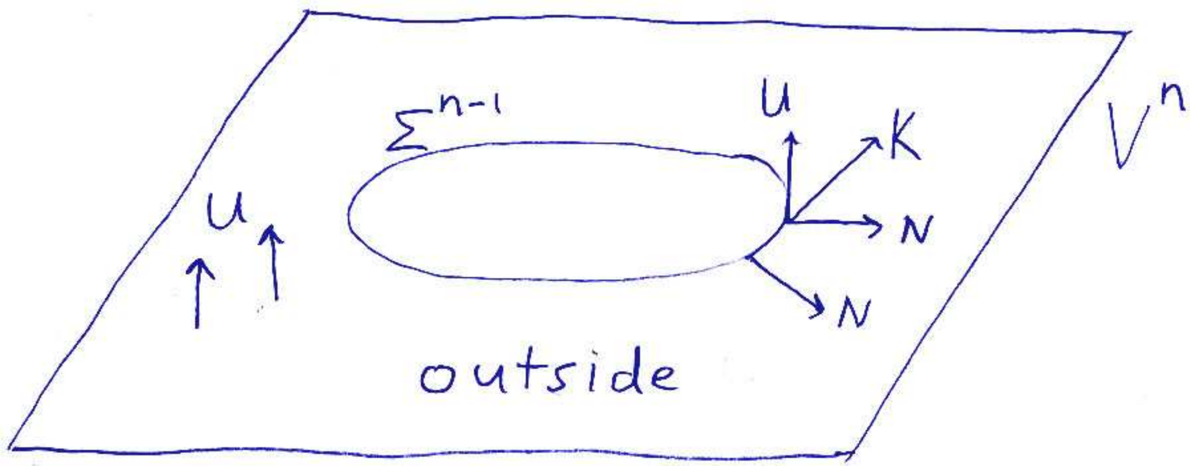
See, however, the recent paper of Oz and collaborators, hep-th/0509013.

Generalization of Hawking's Theorem

Let

$$M^{n+1} = n+1\text{-dim spacetime}, \ n \geq 3$$
$$V^n = \text{spacelike hypersurface in } M^{n+1}$$
$$\Sigma^{n-1} = \text{closed } n-1\text{-surface in } V^n$$

Suppose Σ separates V^n into an "inside" and an "outside":



$K = U + N =$ outward null normal to Σ

Generalization of Hawking´s Theorem

Null expansion of Σ (wrt to K):

$$\chi : T_p\Sigma \times T_p\Sigma \to \mathbb{R}\,, \qquad \chi(X,Y) = \langle \nabla_X K, Y\rangle$$

$$\begin{aligned} \theta &= \operatorname{tr}\chi = h^{ab}\chi_{ab} \\ &= \operatorname{div}_\Sigma K \end{aligned}$$

Def. We say Σ^{n-1} is an outer apparent horizon provided:

- Σ is marginally outer trapped,

  $\theta = 0$ wrt outward null normal

- there are no outer trapped surfaces outside of Σ.

Generalization of Hawking's Theorem

Theorem (G., Schoen) Let (M^{n+1}, g), $n \geq 3$, be a spacetime satisfying the DEC. If Σ^{n-1} is an outer apparent horizon in V^n then Σ^{n-1} is of positive Yamabe type, i.e., admits a metric of positive scalar curvature, unless,

- Σ is Ricci flat (flat if $n = 3, 4$) in the induced metric
- $\chi \equiv 0$ on Σ
- $\mathcal{T}(U, K) = T_{ab}U^a V^b \equiv 0$ on Σ

Topological restrictions

There are many known topological obstructions to the existence of positive scalar curvature metrics in higher dimensions, beginning with the famous result of Lichnerowicz on the vanishing of the $\hat{A}$-genus (cf. also results of Hitchin).

A key advance was made in the late 70′s/early 80′s by Schoen-Yau and Gromov-Lawson.

Focus attention on dim $\Sigma = 3$ (dim $M = 5$) case.

Fact: If Σ is a closed orientable 3-manifold of positive Yamabe type then Σ must be a connected sum of spherical manifolds and $S^2 \times S^1$′s.

Thus, the basic horizon topologies in dim $\Sigma = 3$ case are S^3 and $S^2 \times S^1$.

Proof of the theorem

Proof: We consider normal variations of Σ in V, i.e., variations $t \to \Sigma_t$ of $\Sigma = \Sigma_0$ with variation vector field

$$V = \frac{\partial}{\partial t}|_{t=0} = \phi N, \quad \phi \in C^\infty(\Sigma)\,.$$

Let

$$\theta(t) = \text{ the null expansion of } \Sigma_t,$$

where $K = U + N_t$ and N_t is the unit normal field to Σ_t in V.

Since there are no outer trapped surfaces outside of Σ, one can show that there exists a normal variation of Σ with

$$\phi > 0 \quad \text{and} \quad \left.\frac{\partial \theta}{\partial t}\right|_{t=0} \geq 0$$

A computation shows,

$$\left.\frac{\partial\theta}{\partial t}\right|_{t=0} = -\triangle\phi + (Q + \operatorname{div} X)\,\phi + \phi|\nabla\ln\phi|^2 - \phi|X - \nabla\ln\phi|^2$$

where

$$Q = \frac{1}{2}S - \mathcal{T}(U,K) - \frac{1}{2}|\chi|^2 \quad \text{and} \quad X = \tan(\nabla_N U)$$

Setting $u = \ln\phi$ we obtain,

$$\frac{\partial\theta}{\partial t} = e^u\left(-\triangle u + Q + \operatorname{div} X - |X - \nabla u|^2\right)$$

Since $\left.\frac{\partial\theta}{\partial t}\right|_{t=0} \geq 0$, we obtain,

$$\boxed{-\triangle u + Q + \operatorname{div} X - |X - \nabla u|^2 \geq 0}$$

Absorbing laplacian term into divergence term,

$$Q + \operatorname{div}(X - \nabla u) - |X - \nabla u|^2 \geq 0$$

Setting $Y = X - \nabla u$, we have

$$-Q + |Y|^2 \leq \operatorname{div} Y$$

For any $\psi \in C^\infty(\Sigma)$, multiply through by ψ^2,

$$\begin{aligned} -\psi^2 Q + \psi^2 |Y|^2 &\leq \psi^2 \operatorname{div} Y \\ &= \operatorname{div}(\psi^2 Y) - 2\psi\langle\nabla\psi, Y\rangle \qquad \text{(IBP)} \\ &\leq \operatorname{div}(\psi^2 Y) + 2|\psi||\nabla\psi||Y| \qquad \text{(Schwarz ineq)} \\ &\leq \operatorname{div}(\psi^2 Y) + |\nabla\psi|^2 + \psi^2|Y|^2 \qquad (2ab \leq a^2 + b^2) \end{aligned}$$

Canceling and integrating arrive at ...

$$\int_\Sigma |\nabla\psi|^2 + Q\psi^2 \geq 0 \qquad \forall\psi \in C^\infty(\Sigma)$$

where $\quad Q = \frac{1}{2}S - \mathcal{T}(U,K) - \frac{1}{2}|\chi|^2$

Consider equation

$$-\triangle\psi + Q\psi = 0$$

and corresponding eigenvalue problem,

$$-\triangle\psi + Q\psi = \lambda\psi$$

Have $\lambda_1 \geq 0$. Let f be corresponding eigenfunction; can choose $f > 0$.

Then $\tilde{g} = f^{2/(n-2)}g$ has nonnegative scalar curvature.