Decay of radiation on black hole spacetimes

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Outline of the talk

1. Waves on Minkowski space
2. Wave-tails on the exterior of a Schwarzschild black hole
3. The nonlinear problem: Wave-tails for a collapsing self-gravitating scalar field
4. Fast decay for initial-boundary value problems with black holes
5. Proving stability via decay: a semilinear problem

4-dimensional Minkowski space: $\mathbb{R}^{3+1}$

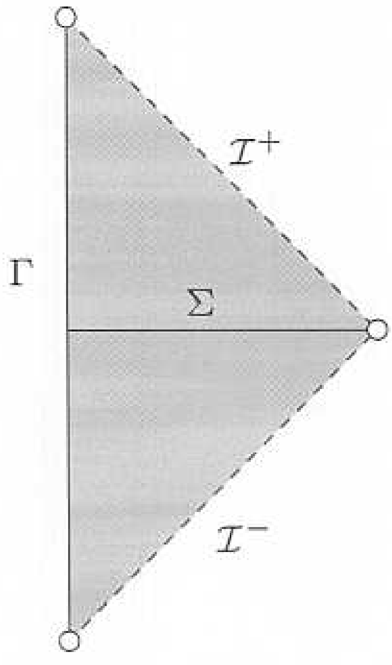
Metric: $-dt^2 + dx_1^2 + dx_2^2 + dx_3^2$.

The space is *spherically symmetric*, i.e. $SO(3)$ acts by isometry.

Define $\mathcal{Q} = \mathbb{R}^{3+1}/SO(3)$.

Can globally conformally map into a bounded domain in 2-dimensional Minkowski space $\mathbb{R}^{1+1}$.

The image of such a map is depicted below:



Glossary

r: *area-radius* function $r : \mathcal{Q} \to \mathbb{R}$ defined by $r(p) = \sqrt{Area(p)/4\pi}$.

$\mathcal{I}^+$: *future null infinity*, future endpoints of null rays in $\mathcal{Q}$ such that $r \to \infty$ to the future.

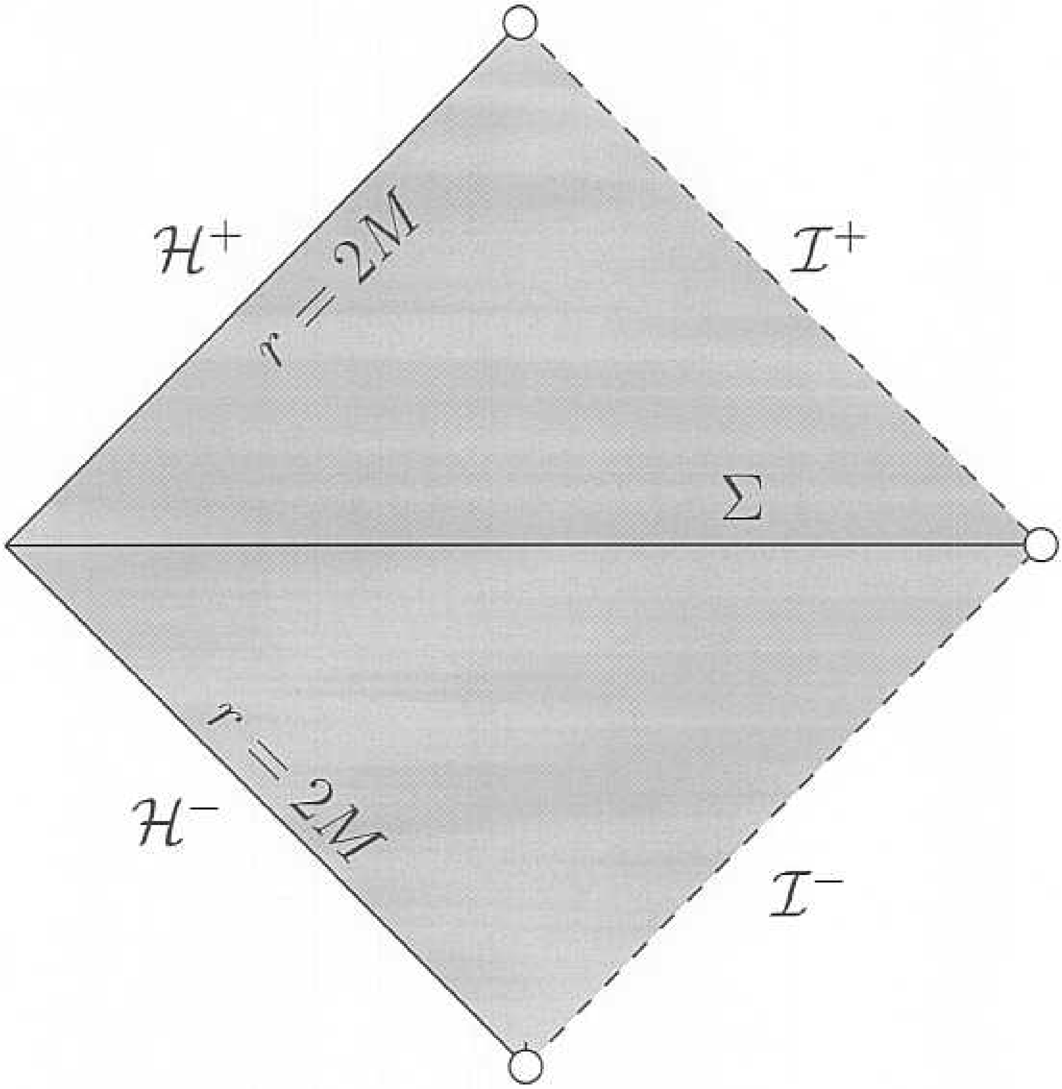
Γ: the *centre* of the group action.

Strong Huygens' Principle

Theorem. *Let ϕ be a (spherically symmetric) solution of the wave equation $\Box\phi = 0$ such that ϕ, $\nabla\phi$ are of compact support on a Cauchy surface Σ. Then $r\phi$ is compactly supported along $\mathcal{I}^+$.*

Proof. Write the wave equation $\partial_u\partial_v(r\phi) = 0$.
□

Schwarzschild exterior $M > 0$



Problem: understand the asymptotic behaviour in u and v of solutions to the wave equation $\Box_g \phi = 0$, in particular, along $\mathcal{I}^+$ and $\mathcal{H}^+$.

Theorem 1 (M. D.–I. Rodnianski). *Let ϕ, $\nabla\phi$ be compactly supported on Σ. Define u so that $\partial_u r = -1$ along $\mathcal{I}^+$, and $\partial_v r \sim 1$ along an (arbitrarily chosen) outgoing null ray $u = 1$. Then for any $\epsilon > 0$, the following decay rates hold:*

$$|r\phi|(u,\infty) \leq Cu^{-2}, |r\partial_u\phi|(u,\infty) \leq C_\epsilon u^{-3+\epsilon} \tag{1}$$

along $\mathcal{I}^+$, and

$$|\phi|(\infty,v) \leq C_\epsilon v^{-3+\epsilon}, |\partial_v\phi|(\infty,v) \leq C_\epsilon v^{-3+\epsilon} \tag{2}$$

along $\mathcal{H}^+$.

The decay rates above are known as *Price's law.* On a heuristic level, they were discovered by Price in 1972, and historically have played an important role in convincing us that black holes are "stable".

There is now a long tradition in the study of these decay rates in the physics literature, e.g. Ching *et al.* (1995), and also from the numerical point of view, e.g. Marsa and Choptuik (1996).

Previous rigorous results: Kay and Wald (1987), Twainy (1989), Machedon-Stalker (2002)

In this problem, the "usual" techniques to understand decay rates have been Fourier analytic and/or spectral theoretic.

If one is primarily interested in the self-gravitating problem, these techniques are not sufficiently robust...

Proof of Theorem 1

1. **Energy conservation**: Schwarzschild possesses a timelike Killing field T in its exterior $J^-(\mathcal{I}^+) \cap J^+(\mathcal{I}^-)$. Contract with the energy momentum tensor of ϕ and apply Noether's theorem.

2. **Red-shift effect**: Define $\alpha = r(\partial_u r)^{-1}\partial_u \phi$, $\theta = r\partial_v \phi$, and write $\Box_g \phi = 0$ as the first order system

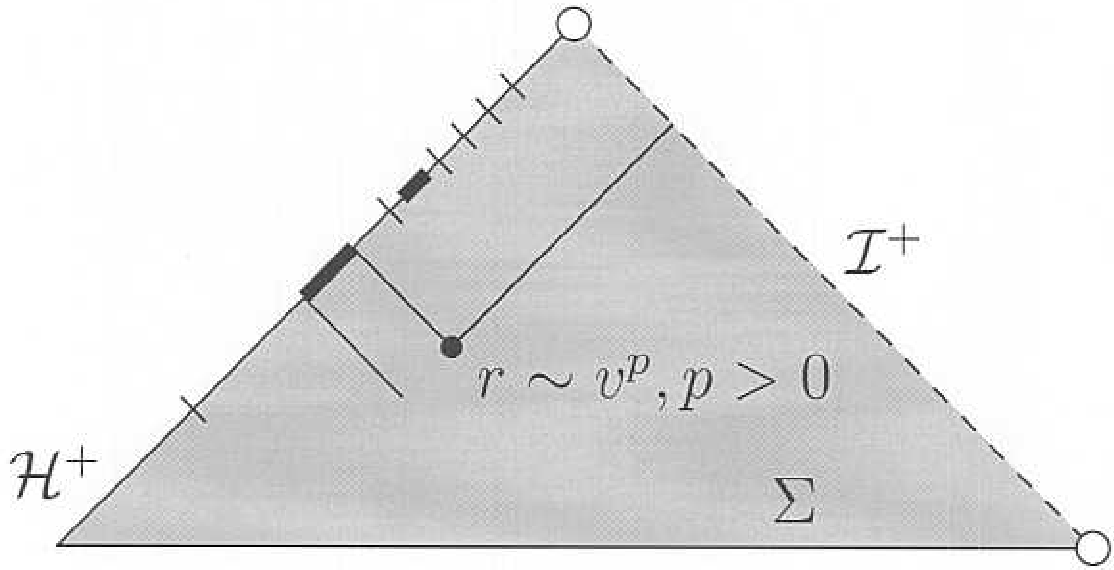
$$\partial_v \alpha = -r^{-1}\theta - \alpha \frac{M}{r^2}$$

$$\partial_u \theta = -r^{-1}\partial_u r(1 - 2Mr^{-1})\alpha$$

3. **An almost Riemann invariant**

$$\partial_u \partial_v (r\phi) = -2M(1 - 2Mr^{-1})r^{-2}\phi$$

4. **Global causal geometry**



$\mathcal{I}^+$
$r \sim v^p, p > 0$
$\mathcal{H}^+$
Σ

Non-linear problem (Gravitational Collapse)

We consider the initial value problem for

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 2T_{\mu\nu}$$

$$\Box_g \phi = 0$$

$$\nabla^{\alpha} F_{\alpha\beta} = 0, \nabla_{[\alpha} F_{\beta\gamma]} = 0$$

$$T_{\mu\nu} = \phi_{;\mu}\phi_{;\nu} - \frac{1}{2} g_{\mu\nu} \phi^{;\alpha}\phi_{;\alpha} + F_{\mu\alpha}F^{\alpha}_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta}F^{\alpha\beta}$$

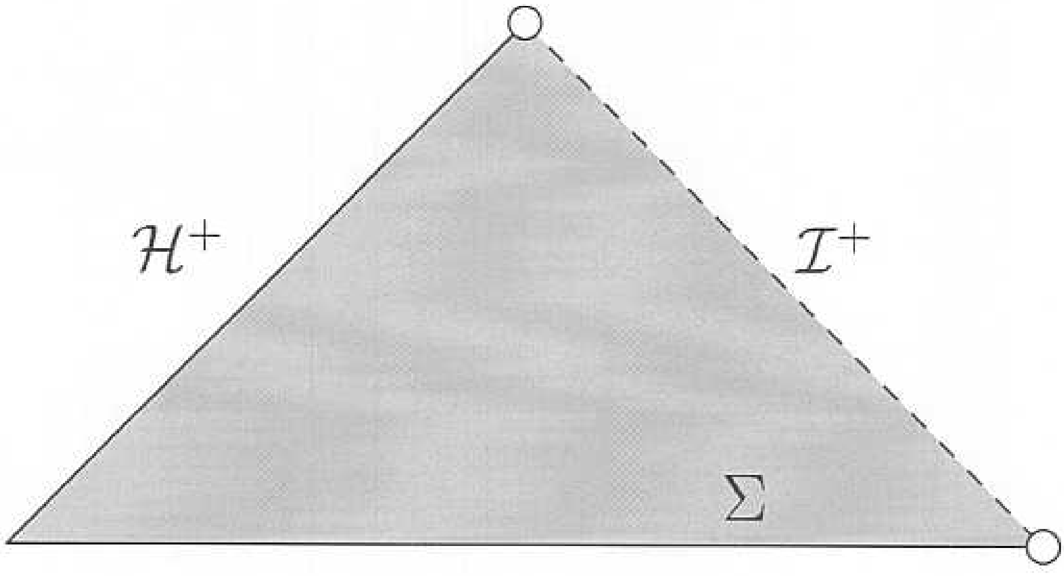
with spherically symmetric asymptotically flat initial data.

Motivation for this system:

1. Coupling with ϕ allows the system to radiate to infinity despite spherical symmetry (cf. Birkhoff's Theorem)
2. $F_{\mu\nu}$–"Poor man's" angular momentum. Decouples, effect on the rest of the system encorporated by a constant e, called *charge*.

Let $\mathcal{Q}$ denote the 2-dimensional quotient of the maximal Cauchy development. We can depict $\mathcal{Q}$ as a Penrose diagram, and since the data is asymptotically flat, we can always define a little piece of $\mathcal{I}^+$ ("endpoints" of null rays in the quotient such that $r \to \infty$).

Theorem 2 (M. D.–I. Rodnianski). *Suppose $\mathcal{Q} \setminus J^-(\mathcal{I}^+) \neq \emptyset$. Then null infinity is complete, and $J^-(\mathcal{I}^+)$ has "Penrose diagram":*



If the black hole is "non-extremal in the limit" then defining coordinates u and v as in Theorem 1, the decay rates (1) *and* (2) *hold.*

Remarks

1. Christodoulou (1987): $\phi(\infty, v) \to 0$ along $\mathcal{H}^+$, $r\phi(u, \infty) \to 0$ along $\mathcal{I}^+$, no rate.
2. The existence of a single trapped surface (a point $p \in \mathcal{Q}$ such that $\partial_u r < 0$, $\partial_v r < 0$) is a sufficient condition for $\mathcal{Q} \setminus J^-(\mathcal{I}^+) \neq \emptyset$.
3. In the case $e = 0$, Christodoulou (1999) proved that trapped surfaces form generically, and used this to prove *weak cosmic censorship*.
4. In the case $e \neq 0$, the condition $\mathcal{Q} \setminus J^-(\mathcal{I}^+) \neq \emptyset$ *always* holds for complete asymptotically flat initial data.

Proof

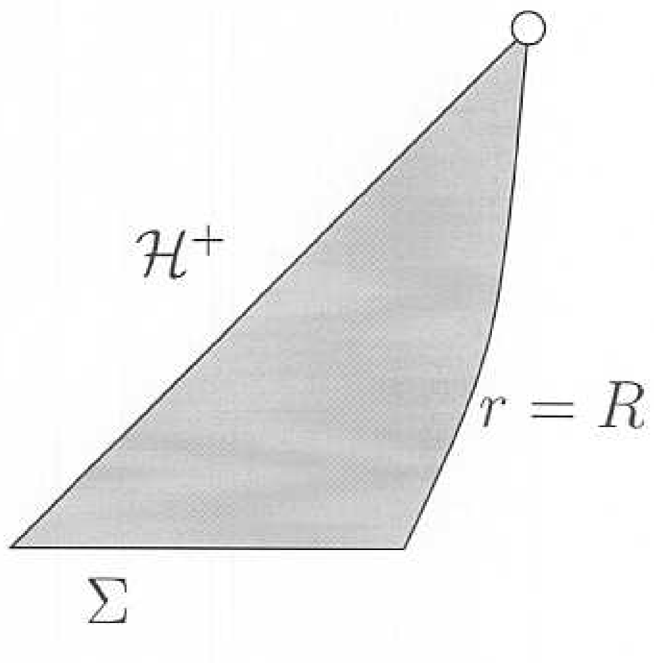
It turns out that despite the fact that we no longer have a timelike Killing vector field, the coupling with gravity provides precisely the same energy estimate as in the linear case, where the energy flux through a null segment in the quotient is realized as the difference between the value of the renormalized Hawking mass $m + \frac{e^2}{r}$, where $m = \frac{r}{2}(1 - \partial^a r \partial_a r)$.

First of all, this allows us easily to obtain that Property 1 of my previous talk here holds, and from this, one obtains the first part of the theorem, as well as a *Penrose inequality* for the area radius function along the event horizon.

Now, all the elements needed for the method of the proof of Theorem 1 are in place. So we continue as before...

Initial-boundary value formulation

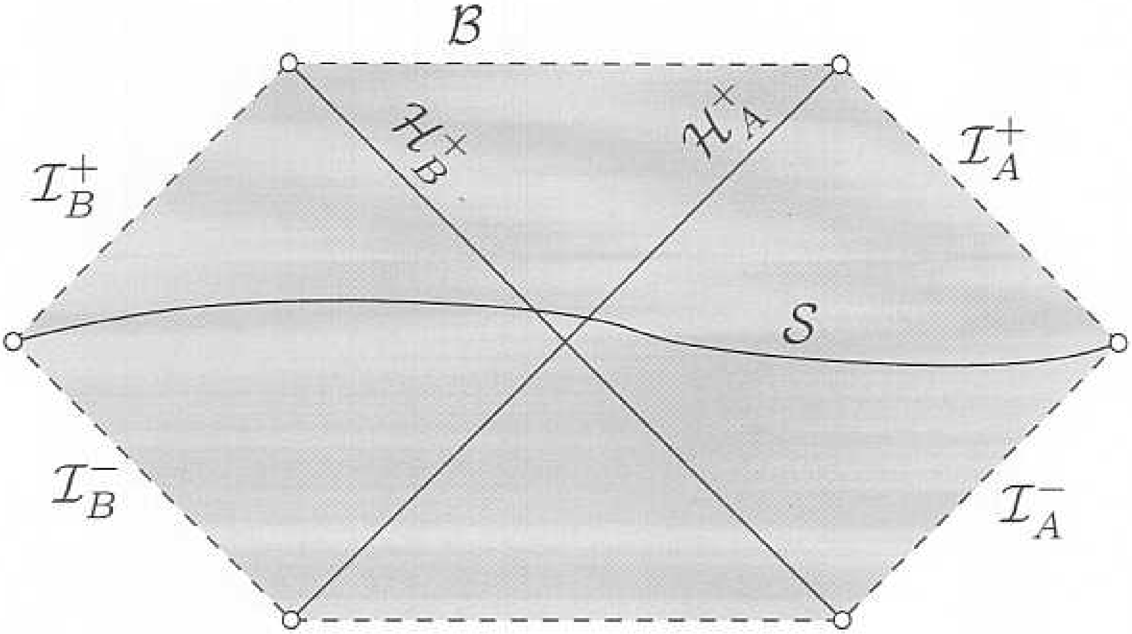
Theorem 3 (M. D.–I. Rodnianski). *If the Cauchy problem is replaced by an appropriate initial-boundary value problem,*



for **any** *boundary condition ensuring that the Hawking mass is non-increasing along $r = R$, then $\partial_v \phi$ decays in v along $\mathcal{H}^+$ faster than any polynomial rate.*

A semilinear problem

Consider the equation $\Box_g \phi = \phi^p$ for $p > 4$ on a fixed Schwarzschild background:



Let ϕ_0 and ϕ_1 be continuous, compactly supported, spherically symmetric and sufficiently small functions on $\mathcal{S}$, and let $(\phi, \mathcal{D})$ denote the solution of the Cauchy problem, where $\mathcal{D}$ is the maximal globally hyperbolic domain in which ϕ is defined.

Theorem 4 (M. D.–I. Rodnianski). $\mathcal{D} \subset J^-(\mathcal{I}^+) \cap J^+(\mathcal{I}^-)$, *i.e.,* ϕ *does not blow up in the domain of outer communications. Moreover, with respect to* v *defined before, we have decay* $|\phi| \leq C(\max\{v, 1\})^{-1}$.

Remarks. **No energy estimate**, so need decay of the linear problem to yield regularity of the non-linear problem, via Duhammel's principle. Because one does not have uniform decay in t, this is tricky...