Black Hole Stability

(w. S. Hartnoll, hep-th/0206202)

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 g_{mn}(y) dy^m dy^n$$

$$R_{mn} = k(n-3) g_{mn}; \qquad f = k - \frac{2c}{r^{n-3}}$$

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}; \quad h^\mu{}_\mu = 0; \quad h^{\mu\nu}{}_{;\nu} = 0$$

$$\delta R_{\mu\nu} = -\frac{1}{2} \Delta_L h_{\mu\nu}$$

$$\Delta_L h^{\mu\nu} = -\nabla^2 h_{\mu\nu} + 2 R_{\mu\alpha\nu\beta} h^{\alpha\beta} + R_{\mu\alpha} h^\alpha_\nu + R_{\nu\alpha} h^\alpha_\mu$$

set $h_{\mu\nu} = 0$ unless $\mu, \nu = m, n$

$$\Delta_L h_{mn} = \frac{1}{f} \ddot{h}_{mn} - f h''_{mn} + \frac{6-n}{r} h'_{mn} - \frac{4f}{r^2} h_{mn} + \frac{1}{r^2} \tilde{\Delta}_L h_{mn}$$

$\tilde{\Delta}_L$: Lichnerowicz operator wrt g_{mn}

$$\tilde{\Delta}_L h_{mn} = \lambda h_{mn} \qquad \text{tensor harmonics}$$

$$h_{\mu\nu} = e^{-i\omega t} \Phi(r)\, h_{mn}(y)$$

$$\left[-\frac{d^2}{dr^{*2}} + V(r)\right]\Phi = \omega^2 \Phi \quad ; \quad dr^* = \frac{dr}{f}$$

$$V(r) = \frac{\lambda f}{r^2} + \frac{n-6}{2}\frac{f'f}{r} + \frac{(n-2)^2 + 10(n-2) + 8}{4r^2} f^2$$

$$\sim \frac{\nu^2 - 1/4}{r^2} \qquad \text{as } r \to \infty$$

$$\nu = \frac{1}{2}\sqrt{(7-n)^2 - 4(4-\lambda)}$$

- asymptotic soln $\quad r^{1/2} K_\nu(\omega r) \quad$ Bessel
- nr. horizon $\quad e^{\pm i\omega r_*} \quad$ oscillates

could match soln. decaying at ∞ with ~~oscillating~~ decaying soln at ∞ if ν imaginary in which case $K_\nu(\omega r)$ oscillates nr. zero.

$$\Rightarrow \text{stability requires } \lambda_{min} > 4 - \frac{(7-n)^2}{4}$$

Lichnerowicz on S^{n-2}

$$\lambda = (k+2)(k+n-1)-2 \quad , \quad k=0,1,2,\ldots$$

$$\Rightarrow \lambda \geq 2(n-2) \Rightarrow \text{stability}$$

Tensor Harmonics on S^{n-2}

embed isometrically $S^{n-2} \to \mathbb{E}^{n-1}$, x^{α}, $\alpha = 1 \ldots n-1$

$$h_{\alpha\beta} = T_{\alpha\mu,\beta\nu,\tau_1\ldots\tau_k} \, x^{\mu} x^{\nu} x^{\tau_1} \ldots x^{\tau_k}$$

s.t.
1) skew on $\alpha\mu$, $\beta\nu$
2) sym wrt $\alpha\mu \leftrightarrow \beta\nu$
3) symm on $\tau_1 \tau_2 \ldots \tau_k$
4) trace free on any pair

$$\Rightarrow h_{\alpha\beta} x^{\beta} = 0 \Rightarrow h_{\alpha\beta} \text{ lives on } S^{n-2}$$

irrep of $SO(n-1)$

$$\begin{array}{|c|c|c|c|c|c|}
\hline
\alpha & \beta & \tau_1 & \tau_2 & \ldots & \tau_k \\
\hline
\mu & \nu & \multicolumn{4}{c}{} \\
\cline{1-2}
\end{array}$$

Special Case : S^3

$$SO(4) = SU(2)_L \times SU(2)_R / \mathbb{Z}_2$$

$$so(4) = \underset{\text{self-dual}}{\Lambda^2_+(\mathbb{E}^4)} \oplus \underset{\text{anti-self-dual.}}{\Lambda^2_-(\mathbb{E}^4)}$$

Tensor harmonics carry $(p+5, p)$ or $(p, p+5)$ rep. $((n_L, n_R)$ has dim $n_L \times n_R$, $n_L, n_R \in \mathbb{N})$

Example $p = 1$ $(5,1)$ or $(1,5)$

invariant under $SU(2)_L$ or $SU(2)_R$

consider $L_{\alpha\beta} = -L_{\beta\alpha} \in \Lambda^2_+(\mathbb{E}^4) = su(2)_R$

i.e. $L_{\alpha\beta} \in (1,3)$ & is left invariant.

$L_{\alpha\mu} X^\mu$ is left-invariant 1-form on S^3

$$h_{\alpha\beta} = L_{\alpha\mu} L_{\beta\nu} X^\mu X^\nu \in (1,1) \oplus (1,5)$$

i.e. $\sigma_1 \otimes \sigma_2$, $\sigma_2 \otimes \sigma_2$ & $\sigma_3 \times \sigma_3$

represent "longest possible gravitational waves that will fit in a closed S^3 universe"

$d\sigma_1 = \sigma_2 \wedge \sigma_3$, left-invariant one-forms on $SU(2)$

Bianchi $\underline{IX}$ metrics are non-linear extension of these tensor modes

<u>BCS</u> <u>ansatz</u> ("Triaxial Bianchi $\underline{IX}$")

$$ds_5^2 = -A e^{-2\delta} dt^2 + \frac{dr^2}{A}$$
$$+ \frac{1}{4} r^2 \left(e^{2B} \sigma_1^2 + e^{2C} \sigma_2^2 + e^{-2(B+C)} \sigma_3^2 \right)$$

A, δ, B, C functions of r & t

$\sigma_1, \sigma_2, \sigma_3$ left-invariant one-forms on $SU(2)$

<u>Special</u> <u>case</u> $B = C$ ("Biaxial Bianchi $\underline{IX}$" or "Generalised Taub-NUT")

$$\boxed{\begin{aligned} ds_5^2 &= -A e^{-2\delta} dt^2 + \frac{dr^2}{A} \\ &+ \frac{1}{4} r^2 \left(e^{2B} (d\theta^2 + \sin^2\theta\, d\phi^2) + e^{-2B} (d\psi + \cos\theta\, d\phi)^2 \right) \end{aligned}}$$

ψ, θ, ϕ Euler angles : $\psi \in (0, 4\pi]$
$\phi \in (0, 2\pi]$
$\theta \in [0, \pi]$

e.o.m.

$$A' = -\frac{2A}{r} + \frac{1}{3r}\left(8e^{-2B} - 2e^{-8B}\right) - 2r\left(e^{2\delta}\frac{\dot{B}^2}{A} + AB'^2\right)$$

$$\dot{A} = -4r\, A\dot{B}B'$$

$$\left(e^{2\delta}A^{-1}r\dot{B}\right)^{\cdot} - \left(e^{-\delta}Ar\, B'\right)' + \frac{4}{3}e^{-\delta}r\left(e^{-2B} - e^{-8B}\right) = 0$$

$$\delta' = -2r\left(e^{2\delta}A^{-2}\dot{B}^2 + B'^2\right)$$

They find critical behavior at the threshold of gravitational collapse to the Tangherlini-metric

$$\boxed{ds_5^2 = -\left(1 - \frac{2c}{r^2}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2c}{r^2}\right)} + \frac{r^2}{4}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right)}$$

Claim: i) The system above has many exact & non-trivial solutions, including time-dependent solutions

ii) dimensional reduction on $\frac{\partial}{\partial\psi}$ $\rightarrow$ time-dependent Kaluza-Klein monopoles & black holes.

Ultra static solutions

$$ds^2 = -dt^2 + \frac{dr^2}{A} + \frac{r^2}{4}\left(e^{2B}(\sigma_1^2+\sigma_2^2) + e^{-4B}\sigma_3^2\right)$$

$A = A(r)$, $B = B(r)$: spatial metric is Ricci flat

1) General solution is known ("Taub-NUT")

$\exists$ 4 & only 4 complete non-singular solns.

1) $\mathbb{E}^4$ on $\mathbb{R}^4$ AE

2) Eguchi-Hanson on $T^*(S^2)$ ALE

3) Self-dual Taub-NUT on $\mathbb{E}^4$ ALF

4) Taub-Bolt on $\mathbb{CP}^2 \setminus \{pt\}$ ALF

1), 2) & 3) are stable against linearised perturbations

$$\Delta_L h > 0$$

3) is unstable, $\exists$ a negative mode

N.B 5) $S^1 \times \mathbb{E}^3$ is a degenerate case which is also classically stable

$$ds^2 = -dt^2 + \frac{r^2-n^2}{r^2-2mr-n^2}\,dr^2 + (r^2-n^2)(\sigma_1^2+\sigma_2^2) + 4n^2\left(\frac{r^2-2mr-n^2}{r^2-n^2}\right)\sigma_3^2$$

- $m = n \Rightarrow$ self-dual Taub-NUT
- $m = \frac{5n}{4} \Rightarrow$ Taub-Bolt
- Eguchi-Hanson ; funny limit

$$\boxed{ds^2 = -dt^2 + \frac{d\rho^2}{1-\frac{a^4}{\rho^4}} + \frac{1}{4}\rho^2(\sigma_1^2+\sigma_2^2) + \frac{1}{4}\rho^2\left(1-\frac{a^4}{\rho^4}\right)\sigma_3^2}$$

To remove coord sing. at $\rho = a$ choose

$\psi \in (0, 2\pi]$

$\Rightarrow$ metric tends to $\mathbb{E}^4/\pm 1$ ALE

$g_{mn} \Leftrightarrow S^3/\pm 1 \equiv \mathbb{RP}^3 = SO(3)$

Exact time-dependent solns.

(gws + pope + lü : hep-th/0501117)

$$\boxed{ds^2 = -dt^2 + \left(t + \frac{1}{2r}\right)^{-1} \frac{1}{4}(d\psi + \cos\theta\, d\phi)^2 + \left(t + \frac{1}{2r}\right)\left(dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right)}$$

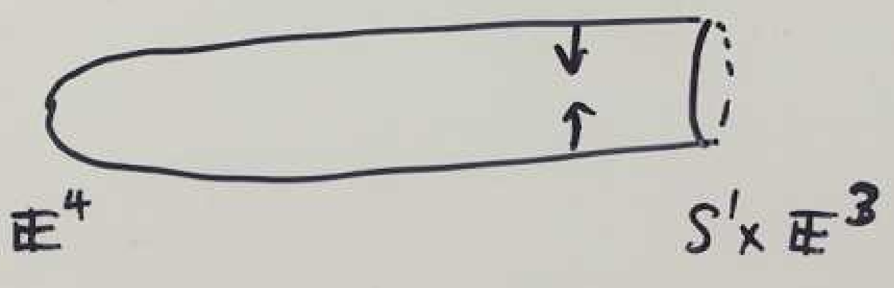
A (linear!) combination of self-dual

Taub-NUT

$$ds^2 = -dt^2 + \left(\frac{1}{2R} + \frac{1}{2r}\right)^{-1} \frac{1}{4}(d\psi + \cos\theta\, d\phi)^2 + \left(\frac{1}{2R} + \frac{1}{2r}\right)\left(dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right)$$

& Kasner on $\mathbb{R}^3 \times S^1$

$$ds^2 = -dt^2 + \frac{1}{4t} d\psi^2 + t\left(dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right)$$



Kaluza-Klein circle collapses as $t \to \infty$

reversing time, this is a decompactifying solution

Kaluza-Klein Interpretation

$$\boxed{\begin{aligned} ds_5^2 &= e^{\frac{4\sigma\kappa}{\sqrt{3}}}\left(dx^5 + 2\kappa A_\mu dx^\mu\right)^2 + e^{-\frac{2\sigma\kappa}{\sqrt{3}}} g_{\mu\nu}\, dx^\mu dx^\nu \\ &\frac{R}{4\kappa^2} - \frac{1}{4} e^{2\kappa\sigma\sqrt{3}} F_{\mu\nu}^2 - \frac{1}{2} g^{\mu\nu} \partial_\mu\sigma\, \partial_\nu\sigma \end{aligned}}$$

σ is "dilaton" : $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

set $\kappa = 1$, $x^5 = \psi$: $A_\mu dx^\mu = \frac{1}{2}\cos\theta\, d\phi$

assume spherical symmetry in 4-dimensions &
we recover the BCS ansatz with

$$\sigma = \sqrt{3}\left(-B + \tfrac{1}{2}\ln(r/2)\right)$$

$\Rightarrow$ time-dependent magnetically charged black holes
coupled to a massless scalar field.

Time-independent solns (Gibbons & Wiltshire)
with regular horizons depend on 2 parameters
(P, M) $\quad 0 \le |P| \le 2M$.

$P = 0$ Schwarzschild$_4$
$P = 2M$ Taub-NUT

regular horizon, scalar hair is "secondary"

Electric-Magnetic Duality

$$\boxed{\begin{aligned} g_{\mu\nu} &\to g_{\mu\nu} \\ \sigma &\to -\sigma \\ F_{\mu\nu} &\to e^{2\pi\sigma\sqrt{3}} * F_{\mu\nu} \end{aligned} \qquad 4\text{-dims.}}$$

- now lift to 5-dims. One gets new 5-diml. solns, but outside the BCS ansatz
- The time dependent Taub-NUT soln. was obtained this way by dualizing a boost-invariant pp-wave in 5-dimensions.
- This is analogous to the exact Husain-Martinez-Nunez (gr-qc/9402021) solution of gravity + scalar which may be obtained in a similar way.