STABILITY OF MOTS + LOCAL EXISTENCE OF HORIZONS (MOTT)

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based on joint work with
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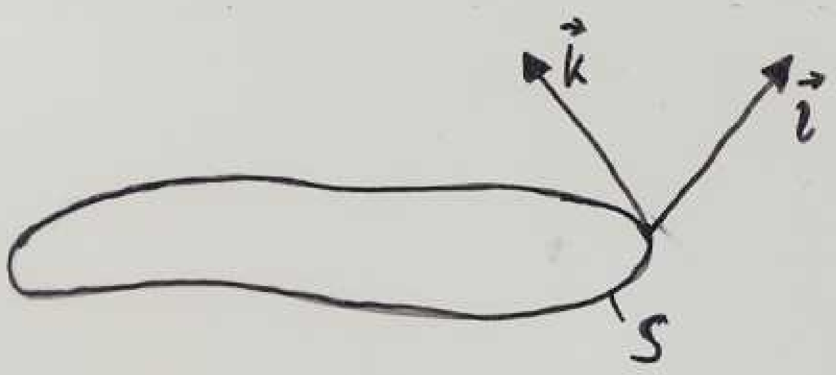
I Heuristics
II Stability
III Local Existence
IV Related matters

Heuristics

4-dim Manifold $(\mathcal{M}, g_{\mu\nu})$ $(-,+,+,+)$

Everything smooth (unless stated otherwise)



$\vec{k}, \vec{l}$ null, ⊥ to S, future directed.

call $\vec{l}$ outgoing (our choice!)

$\Theta := \Theta_{\vec{l}} = \mathrm{div}_S \vec{l}$

outer trapped: $\Theta < 0$ on S

weakly outer trapped: $\Theta \leq 0$ on S

marginally outer trapped: (MOTS) $\Theta \equiv 0$ on S

marginally outer trapped tube (MOTT):

a 3 surface foliated by MOTSs.

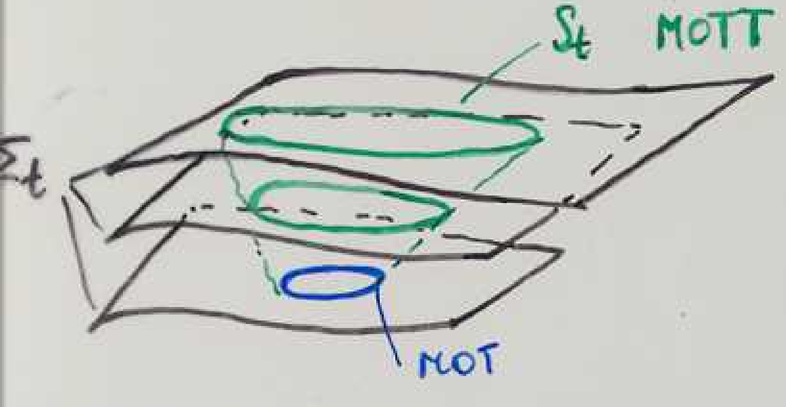
Σ_t a foliation of spacetime by spacelike or null surf.

A MOTT respects the foliation:

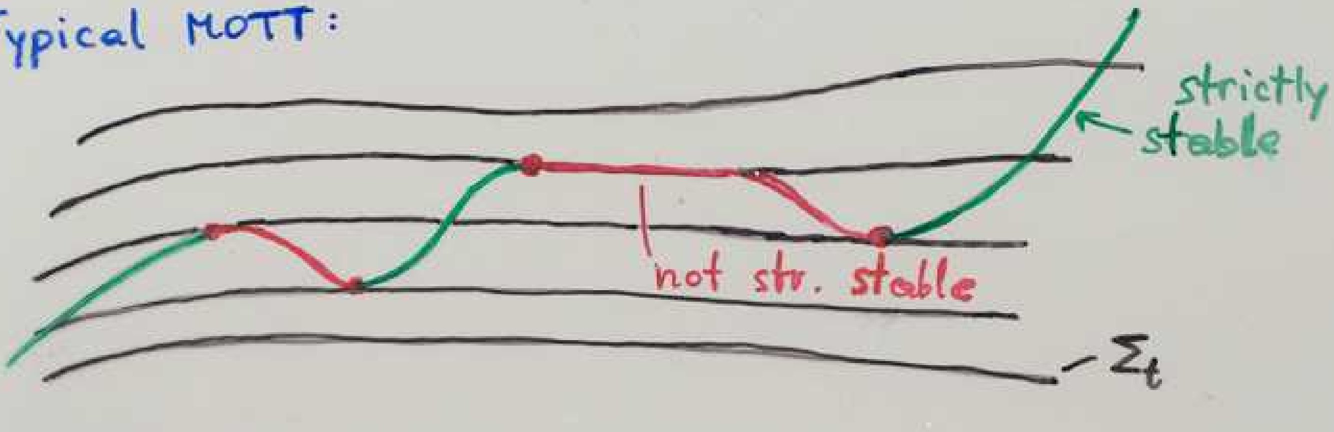
Its MOT leaves lie in the Σ_t.

"Propagation" of a MOTS in spacetime $S \to \{S_t\}$

Question:	Answer:
∃ always a MOTT through a given MOTS ?	?
Are there "normally" many MOTT through a given MOTS ?	Yes (Vaidya); restrictions → Ashtekar + Galloway
Is there always a MOTT which respects a given foliation ?	No (in general) Yes, if "strictly stable"
Causal properties of MOTT ?	spacelike or null (stability, genericity)
Topology of MOTS	S^2 or torus (stability, genericity)

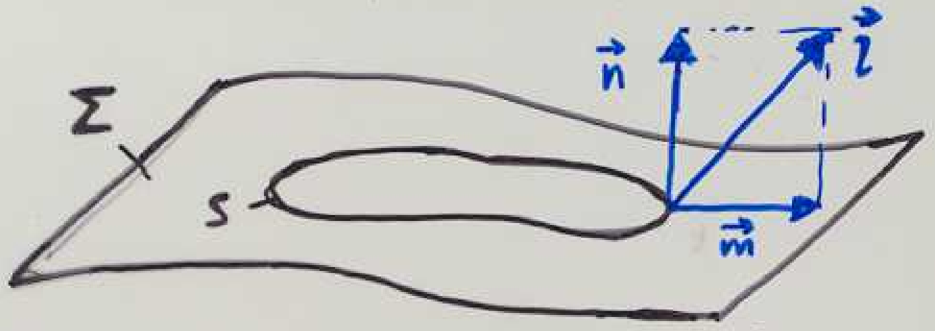


"Typical MOTT:



Stability

Stability of MOTS w.r. to a spacelike 3-surface → $\exists$ of MOTT
→ topology
→ boundary of region with TS



$\vec{l} = \vec{m} + \vec{n}$ $\quad |\vec{m}| = 1$ $\quad |\vec{n}| = 1$ $\quad \vec{l}$ outward, $\vec{n}$ future → $\vec{m}$ outward

Variation of geom. object on a 2-surface along a normal vector $\vec{p} = \Psi \vec{m}$ $\quad \Psi$ ... function

EX: Θ; mean curv., area

Let $p^i \frac{\partial}{\partial x^i} = \frac{\partial}{\partial \tau}$ $\qquad \delta_{\vec{p}} = \frac{\partial}{\partial \tau}$ $\qquad$ or: $\delta_{\vec{p}} = \underset{\text{Lie deriv.}}{\mathcal{L}_{\vec{p}}}$

Remark: $\delta_{\Psi \vec{m}} \neq \Psi \delta_{\vec{m}}$ unless acting on scalars on Σ, or unless $\Psi = \text{const.}$ on S

"Stability operator" for $\Theta[S]$, at $\Theta = 0$, w.r. to Σ.

$$\delta_{\Psi \vec{m}} \Theta = L_{\Sigma} \Psi = -\Delta_S \Psi + 2 s^A D_A \Psi +$$
$$+ \left(\frac{1}{2} R_S - s_A s^A + D_A s^A - \frac{1}{2} K^{\mu}_{AB} K^{\nu AB} l_\mu l_\nu - G_{\alpha\beta} l^\alpha n^\beta \right) \Psi$$

where: $G_{\alpha\beta}$ ... Einstein; D_A, R_S, Δ_S ... on S, w.r. to S

$K^{\mu}{}_{AB} l_\mu = \nabla_A l_B$ ("shear" of $\vec{l}$) $\qquad s_A = -\frac{1}{2} k_\alpha \nabla_A l^\alpha$ ("torsion")

Stability

If torsion, no good self-adjoint extension

Nevertheless:

Thm (Krein-Rutman) (c.f. Smoller "shock waves..." 11.C.)

Evans, "PDE"

$$M = -\Delta_S + t^A D_A + t$$

$$M\phi = \lambda\phi$$

The eigenvalue λ with lowest real part is real, and the eigenfunction $\phi > 0$.

Lemma $\lambda = \sup_{\psi > 0} \inf_{x \in S} \frac{M\psi}{\psi}$ but: $\lambda \neq \inf_{\psi} \frac{(\psi, M\psi)}{(\psi,\psi)}$

Variation of area: $\vec{p} = \psi \vec{m}$

$$\delta^2_{\vec{p}} A = \delta^2_{\vec{p}} \int_S \text{Vol.} = \int \psi\, \delta_{\vec{p}} \underset{\substack{\uparrow \\ \text{mean curv.}}}{K} \overset{\substack{S \text{ minimal} \\ \downarrow}}{=} \int \psi \underset{\substack{\uparrow \\ \text{self adj.}}}{\overset{\circ}{L}_\Sigma} \psi \overset{!}{\geq} \overset{\circ}{\lambda} \cdot |\psi|^2$$

stability of min. surf.: $\delta^2_{\vec{p}} A \geq 0 \ \forall \vec{p} \Leftrightarrow \overset{\circ}{\lambda} \geq 0$

general: sign of $\delta_{\vec{p}} \Theta$, sign of λ

$\uparrow$

"outermost" MOT

$$\left.\begin{array}{l}\text{Hawking} \\ \text{R. Newman (CQG. 87)}\end{array}\right\} \rightarrow \text{Topology.}$$

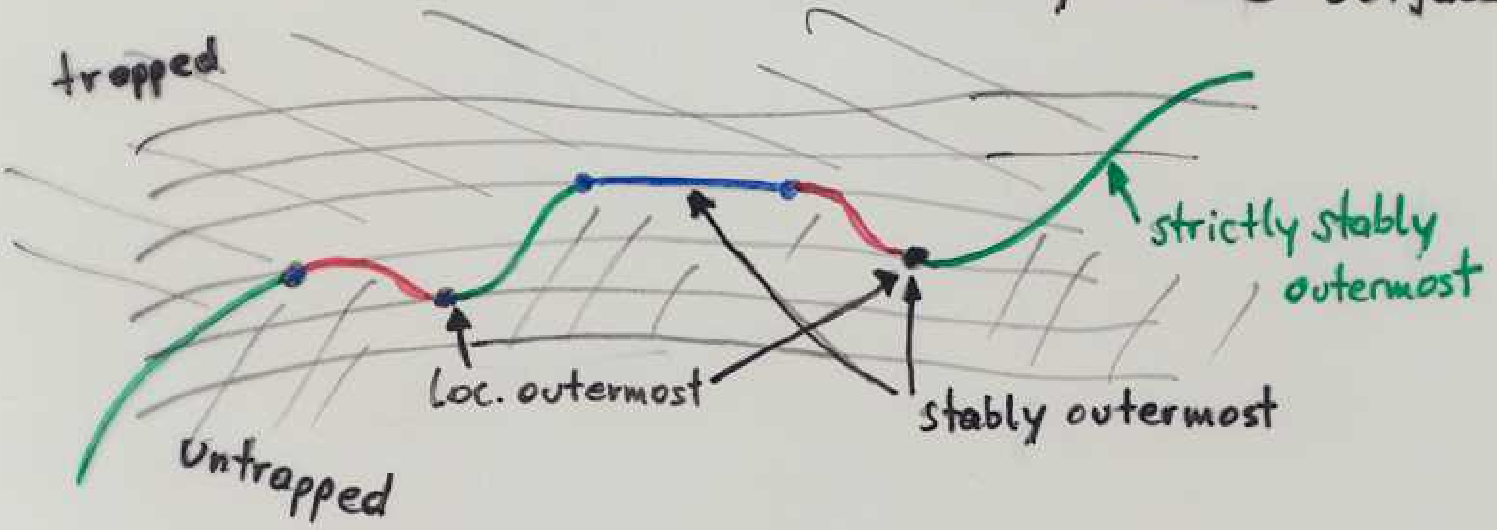
Stability

A MOTS is

stably outermost: $\exists\ \psi \geq 0\quad \psi \not\equiv 0$ s.t. $\delta_{\psi\vec{m}}\Theta \geq 0$

strictly stably outermost: stably + $\delta_{\psi\vec{m}}\Theta \not\equiv 0$

locally outermost: $\exists$ neighbourhood of S whose exterior does not contain any $\Theta \leq 0$ surface.



Thm: , **Thm**:

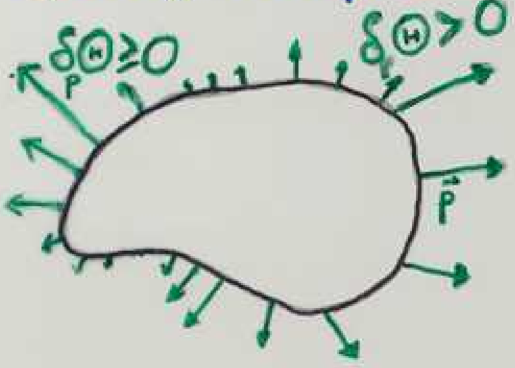
$$\begin{tikzcd}
\text{strictly stably o.} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & \text{locally o.} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{stably o.} \arrow[d, Leftrightarrow] \\
\lambda > 0 & \lambda \geq 0 & \lambda \geq 0
\end{tikzcd}$$

Thm: strictly stably outermost $\Rightarrow$ in a neighbourhood of S, surfaces with $\Theta \leq 0$ or $\Theta \geq 0$ do not cross S.

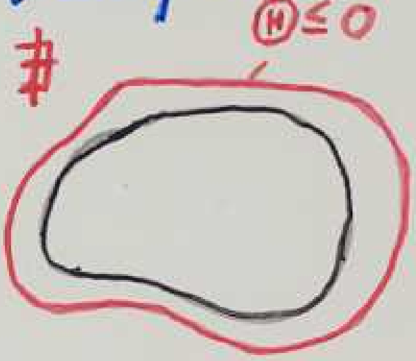
Thm: ($\approx$ Kriele + Hayward, JMP 97)
If the set in Σ containing $\Theta \leq 0$ surfaces has a smooth boundary, it is strictly stably outermost MOTS.

stability

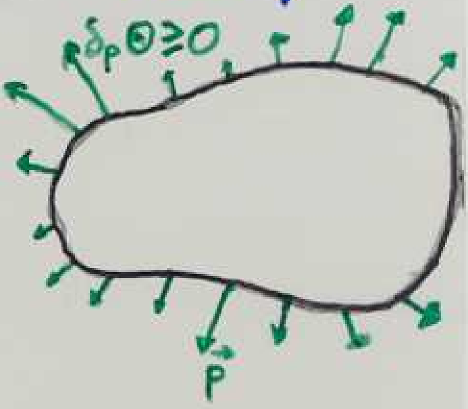
strictly stably out.

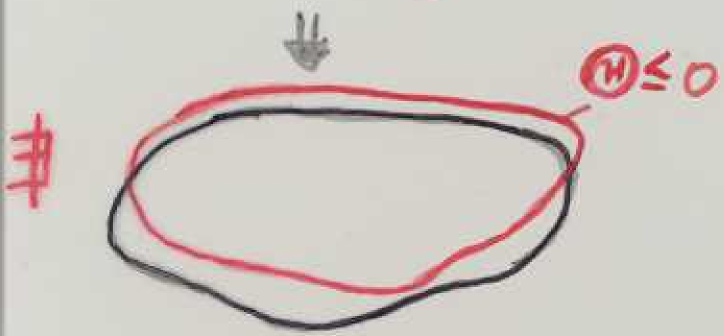


locally outermost



stably out.

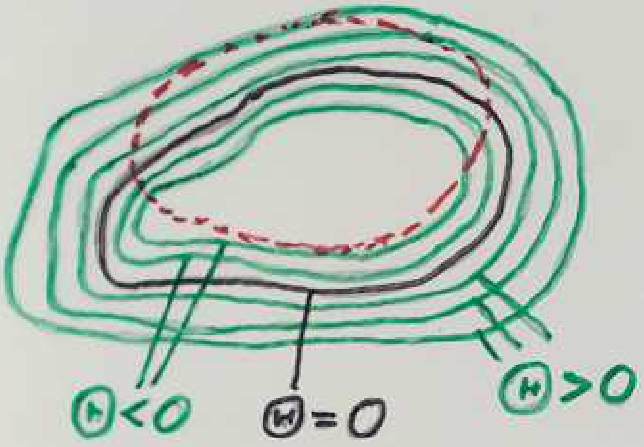




$\lambda > 0$

$\lambda \geq 0$

$\lambda \geq 0$



$$\Theta[f] = a^{AB}(x^A, f, \partial f)\,\partial_A \partial_B f + b(x^A, f, \partial f)$$

Local existence

Thm.

Ass.: $(\mathcal{M}, g_{\mu\nu})$ foliated by smooth, spacelike or null Σ_t.

Σ_0 contains a smooth, strictly stably outermost MOTS S.

Concl.: S is part of a smooth MOTT respecting Σ_t.

This MOTT exists at least as long as the MOTS are strictly stably outermost in Σ_t.

Ass.: Null energy condition $G_{\alpha\beta} l^\alpha l^\beta \geq 0$ near S.

Concl.: The MOTT is spacelike or null near S.

Ass. Genericity condition: $W = G_{\mu\nu} l^\mu l^\nu + K^{\mu}_{AB} K^{\nu AB} l_\mu l_\nu \geq 0$ and $W > 0$ somewhere on S

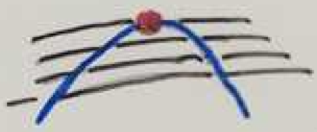
Concl.: The MOTT is spacelike everywhere near S.

If S_t is not strictly stably o. for some t (but smooth)

$\leftrightarrow$ The MOTT is tangent to Σ_t (if smooth)

Remarks:

- Smoothness at turning point?



- existence at (beyond) turning point?

→ Higher variations

- Change of foliation → stability changes
- MOTT can be timelike in "unstable" region!

Proof of Thm.

$\nexists$ a (nice, hyperbolic) evolution equation for MOTS!

Implicit function Thm.: $B_1, B_2, B_3 \ldots$ Banach

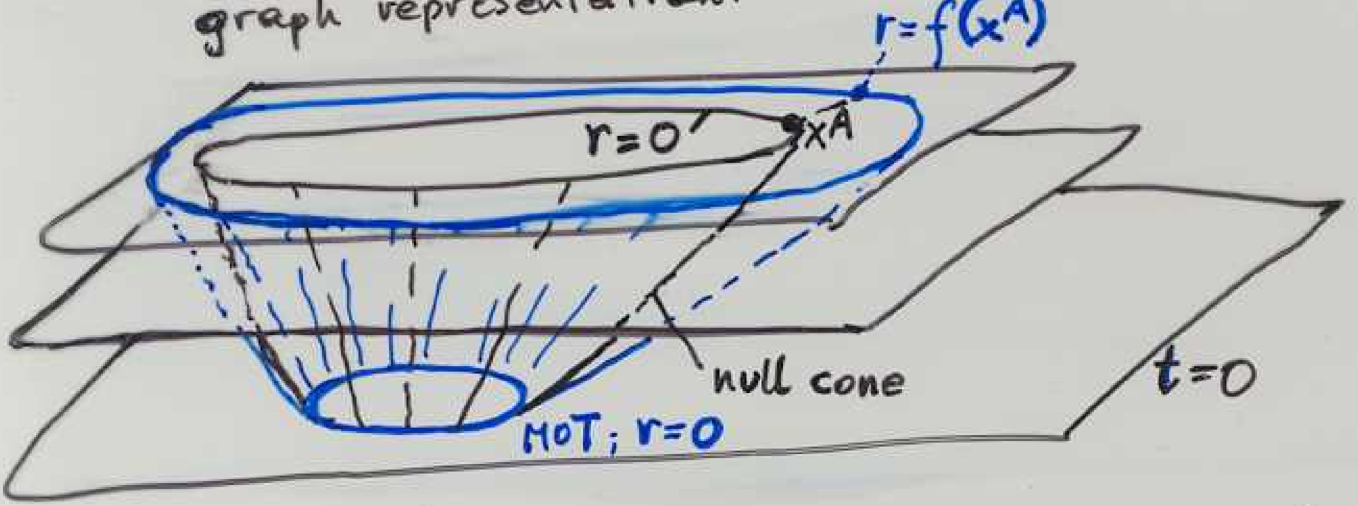
$F: U_1 \times U_2 \to U_3 \ldots C^\infty$; $F(x_0, y_0) = 0$ $D_y F|_{x_0, y_0} \ldots$ invertible

Then: $\exists\, C^\infty$ $J: \tilde{U}_1 \to \tilde{U}_2$ $\tilde{U}_i \subset U_i$ with

$$J(x_0) = y_0 \quad F(x, J(x)) = 0.$$

Here: $B_1 \ldots \mathbb{R}$ (time) $B_2, B_3 \ldots C^{k,\alpha}$ functions on S

graph representation:



$$F(t,y) = \Theta[t,f] = a^{AB}(t, x^A, f, \partial f)\partial_A \partial_B f + b(t, x^A, f, \partial f)$$

For small t, f: $\Theta[t,f]$ satisfies requ.

$$D_y \Theta[0,0] . \beta = \underset{\substack{\uparrow \\ \text{stability op.}}}{L_\Sigma} \beta \qquad \lambda > 0 \Rightarrow D_y \Theta \text{ invertible.}$$

Causal character: maximum principle.

related stuff

"Schoen's thm." (a la L. Andersson, Encycl. of Math.Phys.)

Σ... initial data, DEC.

N_+ with $\Theta_+ > 0$ and N_- with $\Theta_- < 0$.

Then $\exists$ (finitely many) MOTS $\{S_a\}$ in between.

Proof: Jang's equ.

Could be used for $\exists$ + causal behaviour.

Topology: (Hawking, R. Newman)

S strictly stable, DEC → sphere

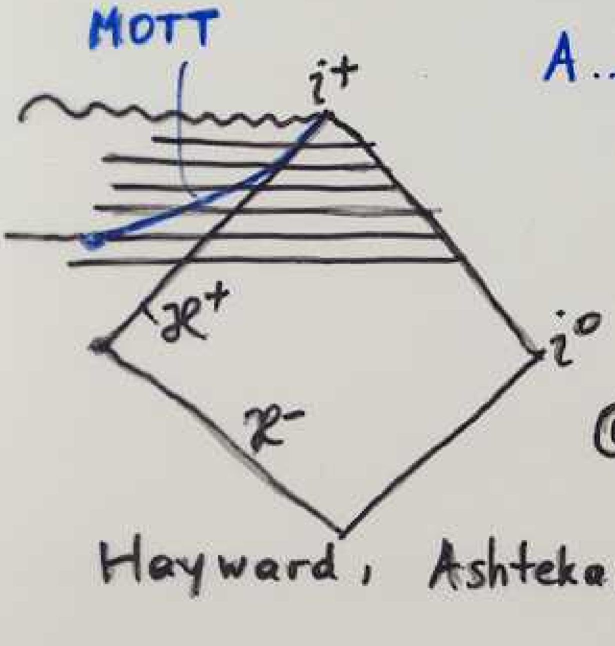
S stable, DEC → torus, ("degenerate"), sphere

$$0 \leq \lambda = \int_S \phi^{-1} L_2 \phi \leq \int R_S = 4\pi\chi$$

Penrose inequ.

$M_{ADM}, M_{Bondi}: \geq \sqrt{\frac{A}{4\pi}}$

A... area of min. surface or MOTS



$$M_H = \sqrt{\frac{A}{16\pi}} \left[1 - \frac{1}{16\pi} \int_S \Theta_l \Theta_k \right]$$

$$\Theta_L = 0 \Rightarrow M_H(t) = \sqrt{\frac{A(t)}{16\pi}}$$

Hayward, Ashtekar + Krishnan: $\frac{dA(t)}{dt} \geq 0$

$\overset{?}{\Rightarrow} M_{Bondi}: \geq \sqrt{\frac{A}{4\pi}}$