Vacuum Gravitational Collapse in $4+1$ Dimensions

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This talk is about recent joint work with Tadek Chmaj and Bernd Schmidt
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Outline

- Introduction and setup
- What we have done
- What we are doing
- Directions for future

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Introduction

- Motivation: weak cosmic censorship conjecture - a physically realistic generic gravitational collapse cannot result in a naked singularity
- Limitation: with current techniques the problem is tractable only in $1+1$ dimensions (spherical symmetry)
- There is no vacuum collapse in spherical symmetry (Birkhoff's theorem) $\Rightarrow$ one has to couple matter fields to generate dynamics
- Simple choice of matter: massless scalar field
  - Christodoulou: proof of weak cosmic censorship
  - Choptuik: discovery of critical phenomena
- Key idea: in higher dimensions one can evade Birkhoff's theorem by allowing deformations of spatial spheres orthogonal to the (t, r) surface.

  Such squashed spheres are well known – for example, on S^3 we can have a homogeneous metric with symmetry (isotropy) broken from $SO(4)$ to $SU(2)$ (Bianchi IX).

Ansatz

$$ds^2 = -U(t,r)dt^2 + V(t,r)dr^2 + \sum_{k=1}^{3} L_k^2(t,r)\sigma_k^2,$$

where

$$\sigma_1 + i\,\sigma_2 = e^{i\psi}(\cos\theta\; d\phi + i\; d\theta), \quad \sigma_3 = d\psi - \sin\theta\; d\phi.$$

- If all three L_k are equal we have the standard spherically symmetric ansatz ($SO(4)$ symmetry) - Birkhoff's theorem applies $\Rightarrow$ the only solutions are Minkowski and Schwarzschild
- If L_k are different, Birkhoff's theorem is not valid $\Rightarrow$ there exist nontrivial vacuum solutions with gravitational radiation
- Special case (biaxial): $L_1 = L_2 \neq L_3$ ($SU(2) \times U(1)$ symmetry)

Using the volume radial coordinate $r = (vol(S^3)/2\pi^2)^{1/3}$, we write the metric as

$$ds^2 = -Ae^{-2\delta}dt^2 + A^{-1}dr^2 + \frac{1}{4}r^2\left[e^{2B}(\sigma_1^2 + \sigma_2^2) + e^{-4B}\sigma_3^2\right],$$

where A, δ, and B are functions of t and r.

Field equations

We define the mass function $m(t,r)$ by $A = 1 - m(t,r)/r^2$.
The hamiltonian constraint $(G_{00} = 0)$

$$m' = 2r^3\left(e^{2\delta}A^{-1}\dot{B}^2 + AB'^2\right) + \frac{2}{3}r\left(3 + e^{-8B} - 4e^{-2B}\right)$$

The momentum constraint $(G_{01} = 0)$

$$\dot{m} = 4r^3 A\dot{B}B'$$

The slicing condition $(G_{00} + G_{11} = 0)$

$$\delta' = -2r\left(e^{2\delta}A^{-2}\dot{B}^2 + B'^2\right)$$

The evolution equation for B $(G_{55} - G_{44} = 0)$

$$\left(e^{\delta}A^{-1}r^3\dot{B}\right)^{\cdot} - \left(e^{-\delta}Ar^3B'\right)' + \frac{4}{3}e^{-\delta}r\left(e^{-2B} - e^{-8B}\right) = 0$$

Birkhoff's theorem:

$$B = 0 \Rightarrow \begin{cases} \delta = 0, m = 0 & \text{Minkowski} \\ \delta = 0, m = const > 0 & \text{Schwarzschild-Tangherlini} \end{cases}$$

Boundary conditions

- Regularity at the center

$$B(t,r) \sim b(t)r^2, \quad m(t,r) \sim a(t)r^6, \quad \delta(t,0)=0 \ \ \text{(time normalization)}$$

- From the hamiltonian constraint

$$m' = 2r^3\left(e^{2\delta}A^{-1}\dot{B}^2 + AB'^2\right) + \frac{2}{3}r\left(3 + e^{-8B} - 4e^{-2B}\right),$$

  it follows that $m(t,r)$ is monotone increasing with r. For asymptotically flat spacetimes $m_\infty = \lim_{r\to\infty} m(t,r)$ exists and is constant in time.

- Scaling invariance implies there are no regular asymptotically flat static solutions besides Minkowski (Lichnerowicz theorem)

- There do exist regular static solutions which are not asymptotically flat (Taub-NUT)

- Numerics indicates that there no regular self-similar solutions (S. Szybka)

Numerical results

- Method: free evolution scheme
- Typical initial data:

$$B(0,r) = p\left(\frac{r}{r_0}\right)^4 e^{-(r-r_0)^4/s^4}, \quad P(0,r) = e^{\delta}\frac{\dot{B}}{A}(0,r) = rB'(0,r)/r_0$$

- Same picture of all families of initial data:
  - small data ($p < p^*$) - dispersion
  - large data ($p > p^*$) - collapse to Schwarzschild
  - critical data ($p = p^*$) - discretely self-similar codimension-one attractor

Animations: dispersion - $P(t,r)$, collapse - $B(t,r)$, collapse - $m(t,r)$

Convergence to Schwarzschild

Linearization around Schwarzschild gives

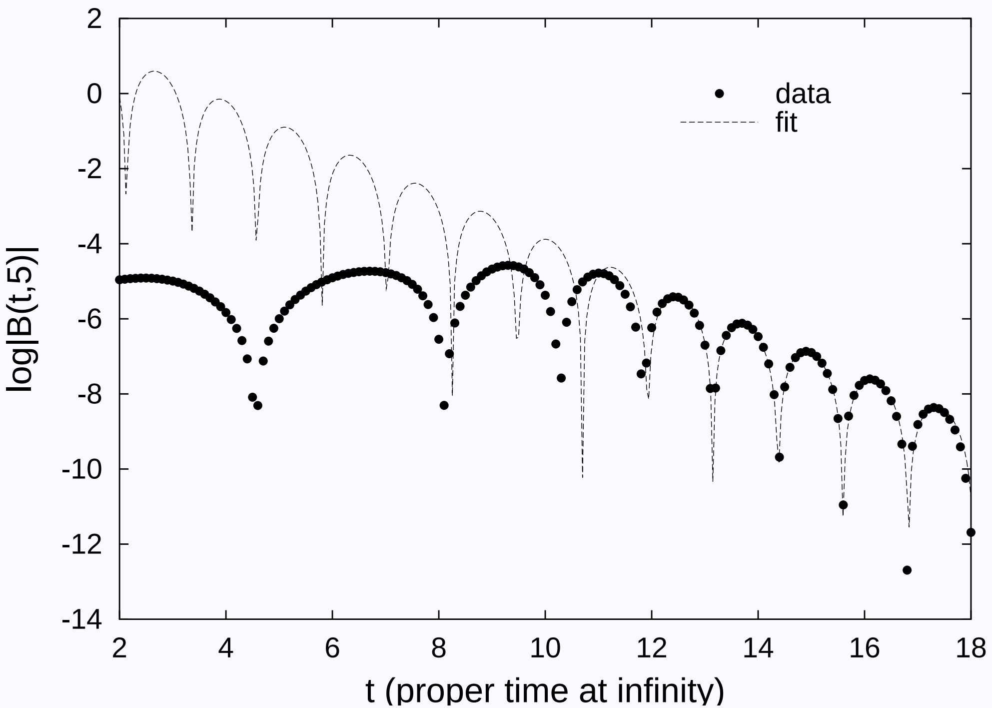
$$\delta\ddot{B} - \frac{1}{r^3}A_0(r^3 A_0 \delta B')' + \frac{8A_0}{r^2}\delta B = 0, \quad A_0 = 1 - \frac{1}{r^2}$$

Using the tortoise coordinate $x = r + \frac{1}{2}\ln\frac{r-1}{r+1}$ and $\delta B(t,r) = e^{-ikt}u(x)$ we get the Schrödinger equation on the real line $-\infty < x < \infty$

$$-\frac{d^2u}{dx^2} + V(r(x))u = k^2 u, \quad V(r) = \frac{1}{4}\left(1 - \frac{1}{r^2}\right)\left(\frac{35}{r^2} + \frac{9}{r^4}\right)$$

- There are no unstable modes (Gibbons&Hartnoll, Ishibashi&Kodama)
- Asymptotic convergence to Schwarzschild is expected to proceed via ringdown (intermediate times) and tails (late times)
- Quasinormal modes (i.e. solutions satisfying the outgoing wave conditions $u \sim e^{\pm ikx}$ for $x \to \pm\infty$) have been computed by Cardoso et al. The least damped mode (for $l = 2$) has the eigenvalue $k = 1.51 - 0.36\,i$ (in units r_h^{-1})
- Tails behave as t^{-7} (for $l = 2$) - *irrespectively* of the presence of a black hole (the same fall-off to Minkowski)

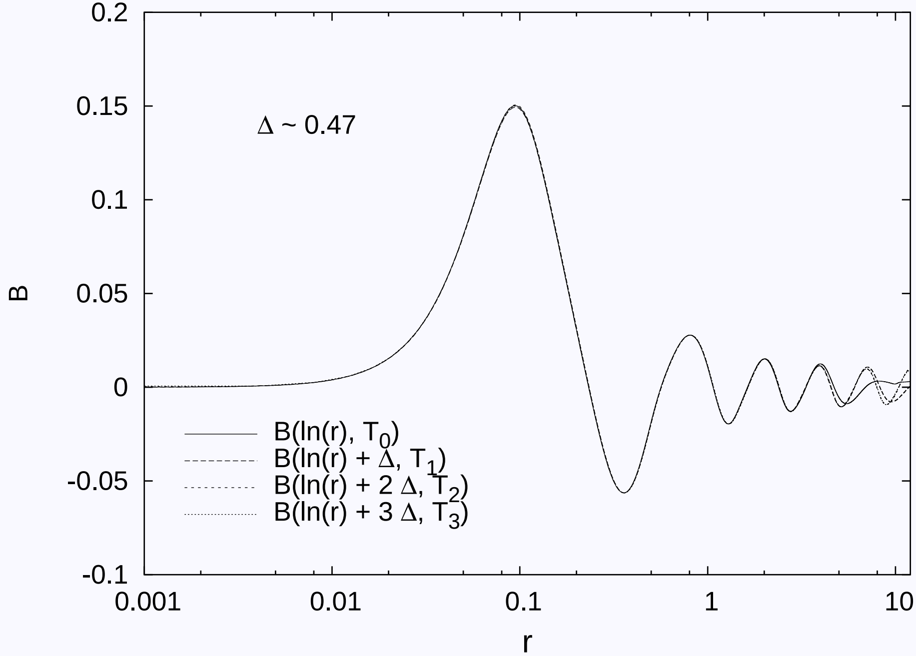
Quasinormal ringing of Schwarzschild



Weak damping of the fundamental QNM + fast fall-off of the tail
= ideal theoretical laboratory for studying the ringdown

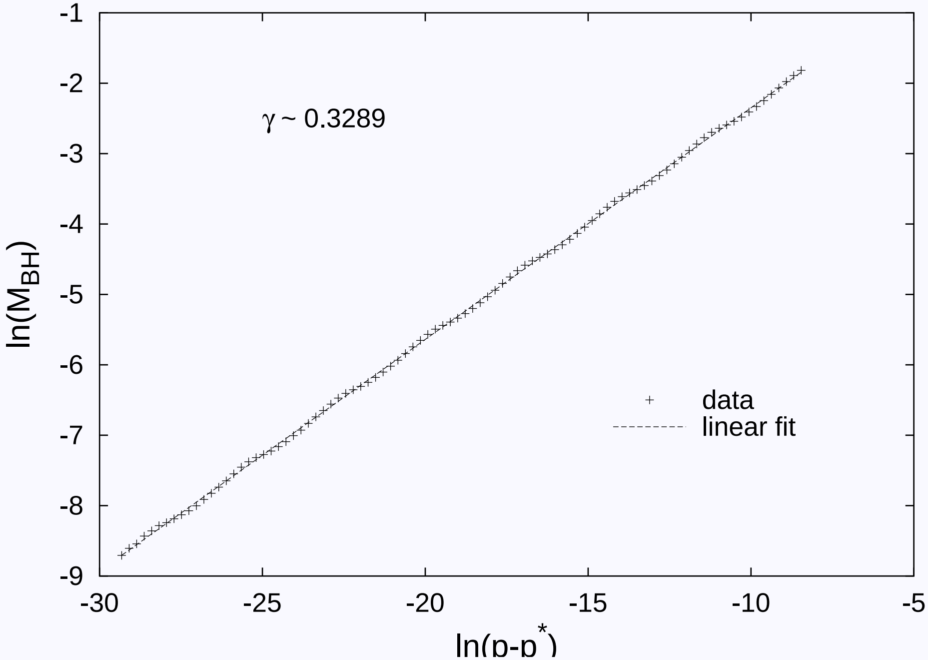
Critical behavior

Animation: $B(t, r)$



Discrete self-similarity: $B(\ln(r) + n\Delta, T_n) = B(\ln(r), T_0)$

Black-hole mass scaling



$$M_{BH} \sim (p^* - p)^\gamma, \qquad \gamma = 2/\lambda$$

$\lambda \approx 6.08$ is the eigenvalue of the growing mode (Note: $\lambda\Delta \approx 2.86$ is small)

Nonlinear stability of Taub-NUT

As pointed out by Gary Gibbons, some gravitational instantons (4-dimensional Riemannian Ricci-flat metrics) can be interpreted as static solutions of our equations. One of them is the Taub-NUT metric

$$ds^2 = -dt^2 + \left(1+\frac{m}{\rho}\right)d\rho^2 + \left(1+\frac{m}{\rho}\right)\rho^2(\sigma_1^2+\sigma_2^2) + \left(1+\frac{m}{\rho}\right)^{-1}\sigma_3^2$$

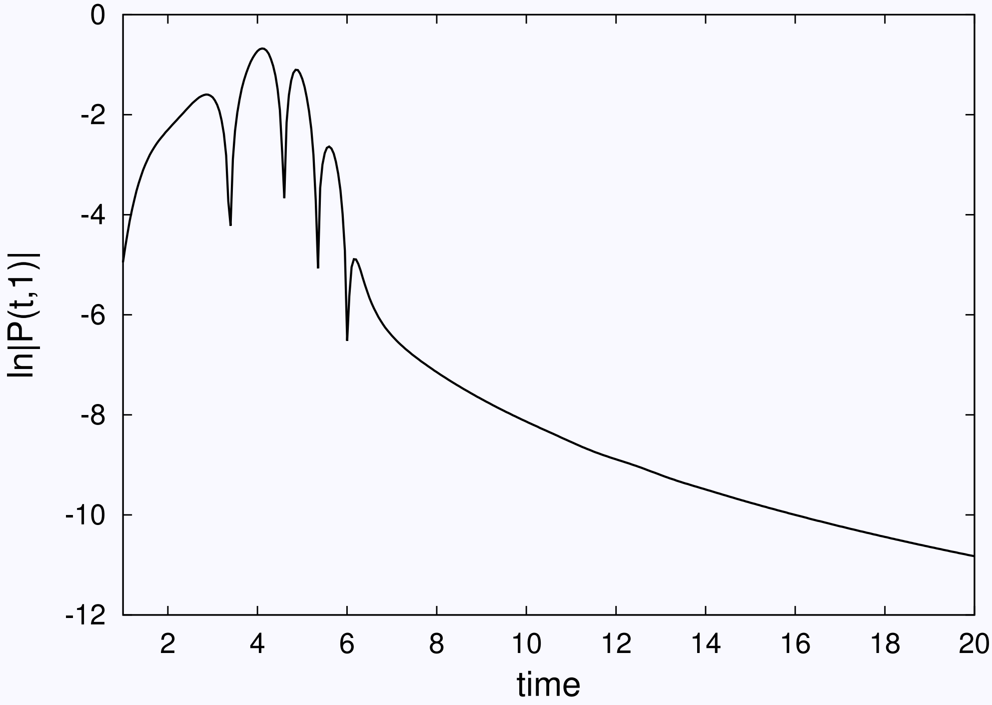
Transforming this into our variables we obtain (for $m=1$)

$$A = \frac{(1+\frac{4}{3}\rho)^2}{(1+\rho)^{8/3}}, \quad B = \frac{1}{3}\ln(1+\rho), \quad e^{\delta} = A^{1/2},$$

where ρ is related to our coordinate r by the formula $r = 2\rho^{1/2}(1+\rho)^{1/6}$.

- TN solution is non-singular but it is *not* asymptotically flat
- TN solution is linearly stable (Hawking&Pope, zero mode has no nodes)
- How about nonlinear stability? animation: perturbation of TN

Return to equilibrium for Taub-NUT



Full (triaxial) ansatz ($SU(2)$ symmetry)

$$ds^2 = -Ae^{-2\delta}dt^2 + A^{-1}dr^2 + \frac{1}{4}r^2\left[e^{2B}\sigma_1^2 + e^{2C}\sigma_2^2 + e^{-2(B+C)}\sigma_3^2\right]$$

The vacuum Einstein equations:

$$\begin{aligned}
rA' &= -2A + \frac{2}{3}\left(2e^{2(B+C)} - e^{-4(B+C)} + 2e^{-2B} + 2e^{-2C} - e^{4B} - e^{4C}\right) + \\
&\quad -\frac{2}{3}r^2\left[e^{2\delta}A^{-1}(\dot{B}^2 + \dot{C}^2 + \dot{B}\dot{C} + A(B'^2 + C'^2 + B'C')\right], \\
\dot{A} &= -\frac{2}{3}rA(2\dot{B}B' + 2\dot{C}C' + \dot{B}C' + \dot{C}B'), \\
\delta' &= -\frac{2}{3}r\left[e^{2\delta}A^{-2}(\dot{B}^2 + \dot{C}^2 + \dot{B}\dot{C}) + B'^2 + C'^2 + B'C'\right],
\end{aligned}$$

$$\left(e^{\delta}A^{-1}r^3\dot{B}\right)^{\cdot} = (e^{-\delta}Ar^3B')' + \frac{4}{3}e^{-\delta}r\left(2e^{4B} + 2e^{-2B} - e^{2(B+C)} - e^{-4(B+C)} - e^{4C} - e^{-2C}\right),$$

$$\left(e^{\delta}A^{-1}r^3\dot{C}\right)^{\cdot} = (e^{-\delta}Ar^3C')' + \frac{4}{3}e^{-\delta}r\left(2e^{4C} + 2e^{-2C} - e^{2(B+C)} - e^{-4(B+C)} - e^{4B} - e^{-2B}\right).$$

Numerical results for the triaxial ansatz (preliminary)

- Biaxial behavior is stable
- Qualitatively the same picture as in the biaxial case:
  - Convergence to Minkowski (for small data) and to Schwarzschild (for large data) has the same form as in the biaxial case
    - for small B and C the two degrees of freedom decouple
- Critical behavior: Probably the same critical solution but larger parameter space needs to be explored to verify this (the biaxial configuration acts as an attractor) Animation: BC critical evolution

Final remarks

- The model provides a simple setting for studying the weak cosmic censorship in vacuum - one could try to repeat Christodoulou's analysis with the field B playing the role of the massless scalar field
- The model incorporates many known explicit solutions (cf. Gary Gibbons' talk) and allows to study dynamical connections between them
- There are natural generalizations to higher dimensions - similar models exist on any odd-dimensional sphere (chapter 7 in Besse - classification of transitive actions of compact Lie groups on spheres). For instance

$$S^{2n+1} = SU(n+1)/SU(n) \quad \text{or} \quad S^{4n+3} = Sp(n+1)/Sp(n)$$

Example: On $S^7 = Sp(2)/Sp(1) = SO(5)/SO(3)$ we have the following pattern of symmetry breaking

$$SO(8) \to SO(6) \to SO(5) \times SU(2) \to SO(5) \times U(1) \to SO(5)$$