Dynamical systems approach to inhomogeneous cosmology

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Motivation

\# of Killing vector fields

3	G_3	Bianchi cosmologies spatially homogeneous ODEs
2	G_2	spatially inhomogeneous PDEs
1	G_1	
0	G_0	

- EFEs for Bianchi cosmologies are ODEs.
- Analyzed using dynamical systems methods.
- Try to extend the application to inhomogeneous models, in asymptotic regimes where spatial derivative terms are negligible.

  e.g. 1. Close-to-de Sitter future asymptotic regime.

  2. Close-to-flat Friedmann-Lemaître past asymptotic regime (special)

  3. Past asymp. regime in vacuum Gowdy models. (G_2 models with special group action)

- Use numerical simulation to verify assumptions and results.
- Quantitative results (growth & decay rates).

Orthonormal Frame Formalism

Orthonormal vector fields $\underset{\sim}{e}_a$, $a = 0,1,2,3.$

covectors $\underset{\sim}{\omega}^a$

Metric $ds^2 = \eta_{ab}\, \underset{\sim}{\omega}^a \underset{\sim}{\omega}^b$, $\eta_{ab} = \mathrm{diag}(-1,1,1,1)$

Commutator $[\underset{\sim}{e}_a, \underset{\sim}{e}_b] = \gamma^c{}_{ab}\, \underset{\sim}{e}_c$

commutation functions

$$\gamma^c{}_{ab} = \{ H, \sigma_{\alpha\beta}, \omega_{\alpha\beta}, \Omega^\alpha, \dot{u}_\alpha, \overbrace{n_{\alpha\beta}, a_\alpha}^{\text{spatial curvature}} \}$$

Hubble scalar | shear | vorticity | frame rotation | acceleration

Matter : $\Lambda \geq 0$ & perfect fluid with $\tilde{p} = (\gamma - 1)\tilde{\rho}$, $1 \leq \gamma \leq 2$

fluid congruence $\tilde{\underset{\sim}{u}} = \dfrac{\underset{\sim}{e}_0 + v^\alpha \underset{\sim}{e}_\alpha}{\sqrt{1-v^2}}$

Matter variables : $\{\rho, v^\alpha, \gamma, \Lambda\}$

For computation, need to use coordinates $\{t, x^i\}$

Set $\omega_{\alpha\beta} = 0$, write $\underset{\sim}{e}_0 = \dfrac{1}{N}\dfrac{\partial}{\partial t}$

$$\underset{\sim}{e}_\alpha = e_\alpha{}^i \frac{\partial}{\partial x^i}$$

$\{N, e_\alpha{}^i\}$ are frame components in $\{t, x^i\}$ coord.

Temporal gauge : Separable volume gauge : $N = \dfrac{1}{H}$

For G_2 : separable area gauge : $N = \dfrac{1}{\beta}$

$\beta = H - \frac{1}{2}\sigma_{11}$

Expansion-normalized variables

$$\{\Sigma_{\alpha\beta}, R^{\alpha}, \dot{U}_{\alpha}, N_{\alpha\beta}, A_{\alpha}\} = \{\sigma_{\alpha\beta}, \Omega^{\alpha}, \dot{u}_{\alpha}, n_{\alpha\beta}, a_{\alpha}\} / H$$

$$\{\Omega, \Omega_{\Lambda}\} = \{\rho, \Lambda\} / 3H^2$$

$$E_{\alpha}{}^{i} = e_{\alpha}{}^{i} / H$$

$$\partial_a = \frac{1}{H} \underset{\sim}{e}_a \Rightarrow \partial_0 = \partial_t \quad , \quad \partial_{\alpha} = E_{\alpha}{}^{i} \partial_{x_i}$$

For G_2: normalize with β.

Define $r_{\alpha} = - \dfrac{\partial_{\alpha} H}{H}$ and either evolve it ($\partial_0 r_{\alpha}$) or determine it from constraint equations

Define $q + 1 = - \dfrac{\partial_0 H}{H}$ for convenience.

For G_2: define r & q for β.

We obtain a system of PDEs. The evolution eqs are first order in ∂_t & second order in ∂_{α}.

The constraint eqs are first order in ∂_{α}.

Deriving decay rates

Toy model
$$\begin{cases} E'(t) = -(1-F^2)E \\ F'(t) = -(3-F^2)F - E^2 \end{cases}$$

Assume E & $F \to 0$ as $t \to \infty$, and $E > 0$.

Find the decay rates. Guess: $E \sim e^{-t}$
$F \sim e^{-2t}$

Derivation

First Round

Since $F \to 0$, we have $F^2 \leq \varepsilon$, $\forall \varepsilon > 0$, for $t \geq t_0$.

$$\Rightarrow \quad E' \leq (-1+\varepsilon)E \qquad \forall\ t \geq t_0.$$
$$\Rightarrow \quad E' + (1-\varepsilon)E \leq 0$$
$$\Rightarrow \quad [E e^{(1-\varepsilon)(t-t_0)}]' \leq 0$$

Integrate from t_0 to t,

$$\Rightarrow \quad E e^{(1-\varepsilon)(t-t_0)} - E_0 \leq 0$$
$$\Rightarrow \quad E \leq E_0 e^{-(1-\varepsilon)(t-t_0)}$$

i.e. E decays faster than $e^{-(1-\varepsilon)(t-t_0)}$.

For F, $F=0$ is not an invariant set, so consider $|F|$.

$$|F|' = \begin{cases} F' & \text{for } F>0 \quad \Rightarrow |F|' = (-3+F^2)|F| - E^2 \\ (-F)' & \text{for } F<0 \quad \Rightarrow |F|' = (-3+F^2)|F| + E^2. \\ \text{undefined} & \text{for } F=0 \end{cases}$$

So $|F|' \leq (-3+F^2)|F| + E^2$.

$$\Rightarrow \quad |F|' \leq (-3+\varepsilon)|F| + E_0^2 e^{-2(1-\varepsilon)(t-t_0)} \qquad \forall\ t \geq t_0.$$

$$\Rightarrow \quad \left[|F| e^{(3-\varepsilon)(t-t_0)}\right]' \leq E_0^2 e^{(1+\varepsilon)(t-t_0)}$$

Integrate from t_0 to t and rearrange.

$$\Rightarrow \quad |F| \leq |F|_0 e^{(-3+\varepsilon)(t-t_0)} + \frac{E_0^2}{1+\varepsilon}\left[e^{(-2+2\varepsilon)(t-t_0)} - e^{(-3+\varepsilon)(t-t_0)}\right.$$

dominant term

i.e. $|F|$ decays faster than $e^{(-2+2\varepsilon)(t-t_0)}$.

Second Round

$$E' = -E + F^2 E$$

$$\Rightarrow -Ce^{-3(1-\varepsilon)(t-t_0)} \leq E' + E \leq Ce^{-3(1-\varepsilon)(t-t_0)}$$

for some $C>$
$\forall\, t \geq t_0$

$$\Rightarrow -Ce^{(-2+3\varepsilon)(t-t_0)} \leq \left[Ee^{t-t_0}\right]' \leq Ce^{(-2+3\varepsilon)(t-t_0)}$$

Integrate from t to ∞. Note $Ee^{t-t_0} \to \hat{E}$ as $t \to \infty$.

$$\Rightarrow \quad \left|E - \hat{E}e^{-(t-t_0)}\right| \leq \frac{C}{2-3\varepsilon} e^{-3(1-\varepsilon)(t-t_0)}$$

the leading term $\qquad\qquad$ higher order term

Next,

$$F' = (-3+F^2)F - E^2$$

$$\Rightarrow \quad F' + 3F + \hat{E}^2 e^{-2(t-t_0)} = F^3 - E^2 + \hat{E}^2 e^{-2(t-t_0)}$$

and

$$|\text{RHS}| \leq C_1 e^{-6(1-\varepsilon)(t-t_0)} + C_2 e^{(-4+3\varepsilon)(t-t_0)}$$
$$\leq C_3 e^{(-4+3\varepsilon)(t-t_0)}$$

So

$$-C_3 e^{(-1+3\varepsilon)(t-t_0)} \leq \left[Fe^{3(t-t_0)}\right]' + \hat{E}e^{t-t_0} \leq C_3 e^{(-1+3\varepsilon)(t-t_0)}$$

Integrate from t to ∞

$$\Rightarrow \quad \left|F - \hat{F}e^{-3(t-t_0)} + \hat{E}^2 e^{-2(t-t_0)}\right| \leq \frac{C_3}{1-3\varepsilon} e^{(-4+3\varepsilon)(t-t_0)}$$

linear term $\qquad$ dominant term (non-linear) $\qquad$ higher order term

Repeat Round 2 to refine the higher order term.

Final result:

$$E = \hat{E}e^{-t} + O(e^{-3t})$$
$$F = -\hat{E}^2 e^{-2t} + \hat{F}e^{-3t} + O(e^{-4t})$$

Recap for integration method.

Round 1 yields a big-O description

Round 2 yields the leading order terms and the integration constants, plus a smaller big-O term.

Repeat Round 2 to refine the big-O term.

Advantage: rigorous.

Limitation: So far only works for decays with a fixed rate.

Extension to systems of PDEs

Need to assume

1. $\frac{\partial}{\partial x^i}(\ )$ are bounded as $t \to \infty$
2. $\frac{\partial}{\partial x^i} O(f(t)) = O(f(t))$ e.g. $e^{-t}\sin x$ ✓
   $e^{-t}\sin(xe^{t})$ ✗

Application in cosmology

1. Close-to-de Sitter future asymp. regime.
2. Close-to-flat Friedmann-Lemaître past asymp. regime

Close-to-de Sitter future asymp. regime

Results

$$(E_\alpha{}^i, A^\alpha, N^{\alpha\beta}) = (\hat{E}_\alpha{}^i, \hat{A}^\alpha, \hat{N}^{\alpha\beta})e^{-t} + O(e^{-3t})$$

as $t \to \infty$

$$r_\alpha = -\tfrac{1}{2}\hat{E}_\alpha{}^i \partial_i \hat{\Omega}_k\, e^{-3t} + O(e^{-(1+3\gamma)t} + e^{-5t})$$

$$\Sigma_{\alpha\beta} = -3\,\underset{\substack{\uparrow\\ \text{dominant nonlinear term}}}{\hat{S}_{\alpha\beta}\, e^{-2t}} + \hat{\Sigma}_{\alpha\beta}\, e^{-3t} + O(e^{-4t})$$

$$\Omega_\Lambda = 1 - \hat{\Omega}_k e^{-2t} + O(e^{-3\gamma t} + e^{-4t})$$

$$\Omega = \begin{cases} \hat{\Omega} e^{-3\gamma t} + O(e^{(3\gamma-8)t}) & \text{for } 1 \le \gamma < \frac{4}{3} \\ \hat{\Omega} e^{-4t} + O(e^{-5t}) & \text{for } \gamma = \frac{4}{3} \\ \hat{\Omega} e^{-4t} + O(e^{-5t} + e^{-(4+r)t}) & \text{for } \frac{4}{3} < \gamma < 2 \end{cases}$$

$$r = \frac{2(3\gamma-4)}{2-\gamma}$$

$$v^\alpha = \begin{cases} \hat{v}^\alpha e^{-t} + O(e^{-3t}) & \text{for } \gamma = 1 \\ \hat{v}^\alpha e^{(3\gamma-4)t} + O(e^{-t} + e^{3(3\gamma-4)t}) & \text{for } 1 < \gamma < \frac{4}{3} \\ \hat{v}^\alpha + O(e^{-t}) & \text{for } \gamma = \frac{4}{3} \\ \hat{v}^\alpha + O(e^{-t} + e^{-rt}), \quad \hat{v}^\alpha \hat{v}_\alpha = 1 & \text{for } \frac{4}{3} < \gamma < 2 \end{cases}$$

(extreme tilt)

$$1 - v^2 = e^{-rt}\left[\widehat{1-v^2} + O(e^{-t} + e^{-rt})\right] \quad \text{for } \tfrac{4}{3} < \gamma < 2$$

$$H = \sqrt{\tfrac{\Lambda}{3}}\left[1 + \tfrac{1}{2}\hat{\Omega}_k e^{-2t} + \tfrac{1}{2}\hat{\Omega} e^{-3\gamma t} + O(e^{-4t})\right]$$

$\hat{\Omega}_k$ & $\hat{S}_{\alpha\beta}$ depend on $\hat{N}^{\alpha\beta}$ and $\hat{A}^\alpha$.

Isotropic singularity
(Close-to-flat Friedmann-Lemaître)

Results

$$(E_\alpha{}^i, A^\alpha, N^{\alpha\beta}) = (\hat{E}_\alpha{}^i, \hat{A}^\alpha, \hat{N}^{\alpha\beta})\,\eta + O(\eta^3)$$

$$\eta = e^{\frac{1}{2}(3\gamma-2)t} \to 0 \quad \text{as} \quad t \to -\infty$$

$$r_\alpha = -\tfrac{1}{2}\hat{E}_\alpha{}^i \partial_i \hat{\Omega}_k \cdot \eta^3 + O(\eta^5 + \eta^{3+s})$$

$$s = \frac{4}{3\gamma-2}$$

$$\Sigma_{\alpha\beta} = -\frac{6}{3\gamma+2}\hat{S}_{\alpha\beta}\,\eta^2 + O(\eta^4)$$

Unstable $\hat{\Sigma}_{\alpha\beta}$ term has been killed.

$$\Omega_\Lambda = \frac{\Lambda}{3\hat{H}^2}\,\eta^{\frac{6\gamma}{3\gamma-2}}\left[1 - \hat{\Omega}_k\,\eta^2 + O(\eta^4 + \eta^{2+s})\right]$$

$$\Omega = 1 - \hat{\Omega}_k\,\eta^2 + O(\eta^4 + \eta^{2+s})$$

$$v_\alpha = \frac{1}{3\gamma+2}\hat{E}_\alpha{}^i \partial_i \hat{\Omega}_k \cdot \eta^3 + O(\eta^5 + \eta^{3+s})$$

$$H = \hat{H}\,\eta^{\frac{-3\gamma}{3\gamma-2}}\left[1 + \tfrac{1}{2}\hat{\Omega}_k\,\eta^2 + O(\eta^4 + \eta^{2+s})\right]$$

$\hat{H}$ const.

$\hat{\Omega}_k$ & $\hat{S}_{\alpha\beta}$ depend on $\hat{N}^{\alpha\beta}$ and $\hat{A}^\alpha$.

Asymptotic silence

- $E_\alpha{}^i \to 0$ exponentially in the above examples.
- Combined with bounded $\partial_{x^i}(\ )$ terms, this makes spatial derivative terms negligible.
- Event horizon forms in the close-to-de Sitter regime.
- Particle horizon forms in the close-to-flat FL regime.
- Formation of horizon prevents communication in the asymptotic regime, hence the name "asymptotic silence".
- We tentatively define asymptotic silence as $E_\alpha{}^i \to 0$
- Loose end: $E_\alpha{}^i$ has to vanish <u>fast enough</u> in order to form horizon.
- Even when horizon forms, $\partial_{x^i}(\ )$ may be large, or even unbounded. (**spiky structures**)

Spikes in Gowdy models

(vacuum G_2 models with special group action)

Variables: $\{E_1{}^1, \Sigma_+, \Sigma_-, \Sigma_\times, N_-, N_\times\}$. ($\beta$-normalized)

$$E_1{}^1 = e^{2t} \to 0 \text{ as } t \to -\infty.$$

Relation to metric:

$$ds^2 = -e^{2f}dt^2 + e^{2g}dx^2 + e^{2t}\left[e^P(dy + Q\,dz)^2 + e^{-P}dz^2\right]$$

$\beta = e^{-f}$

$E_1{}^1 = e^{f-g}$

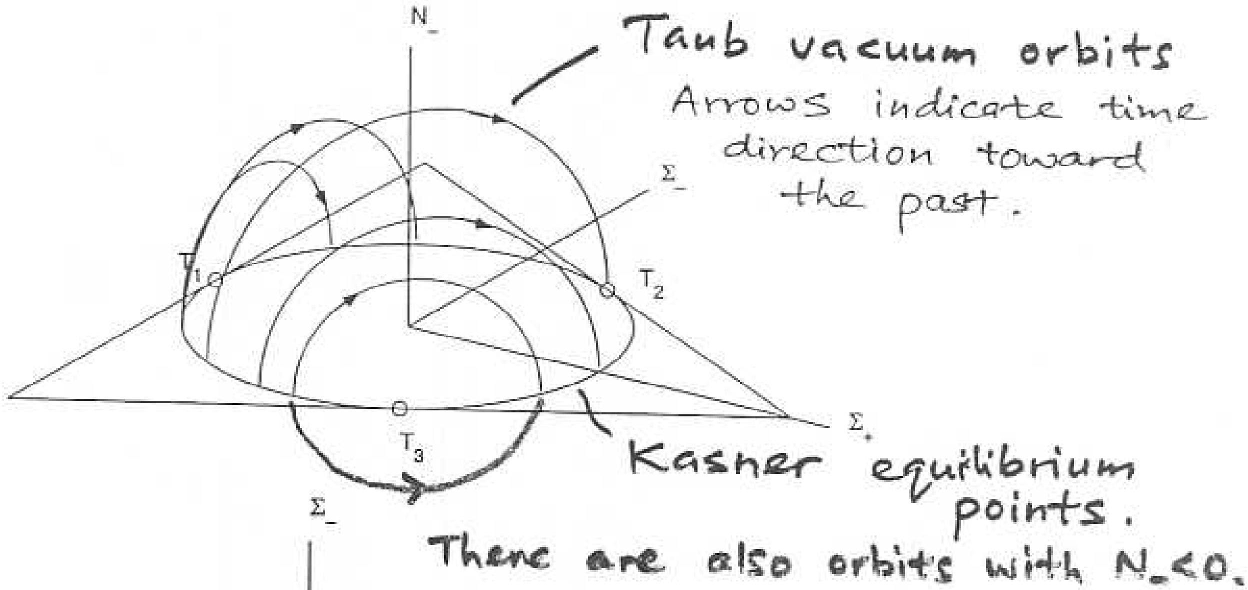
$1 - 3\Sigma_+ = \partial_t g$

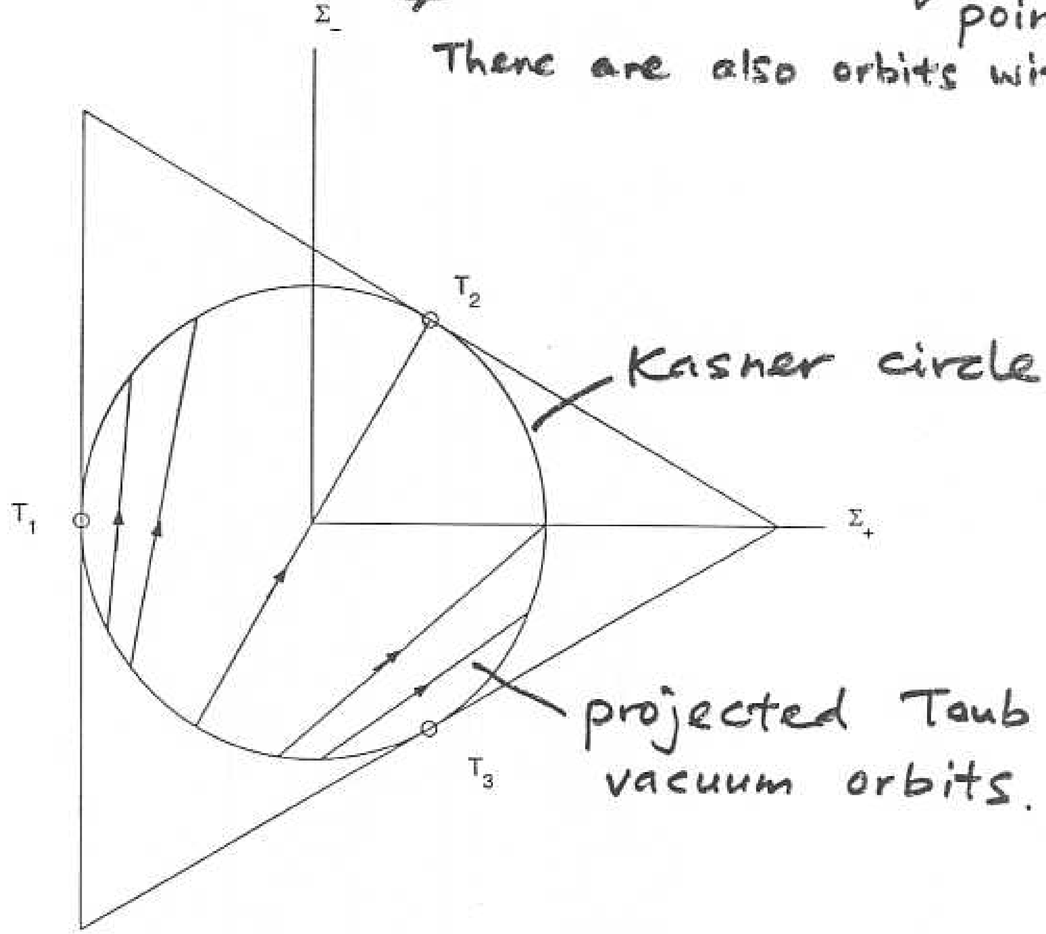
$$(\Sigma_-, N_\times) = \tfrac{1}{2\sqrt{3}}\left(\partial_t P, -E_1{}^1\partial_x P\right)$$

$$(\Sigma_\times, N_-) = \tfrac{1}{2\sqrt{3}}e^P\left(\partial_t Q, E_1{}^1\partial_x Q\right).$$

Background dynamics at early times ($t \to -\infty$) is that of Bianchi II cosmologies.

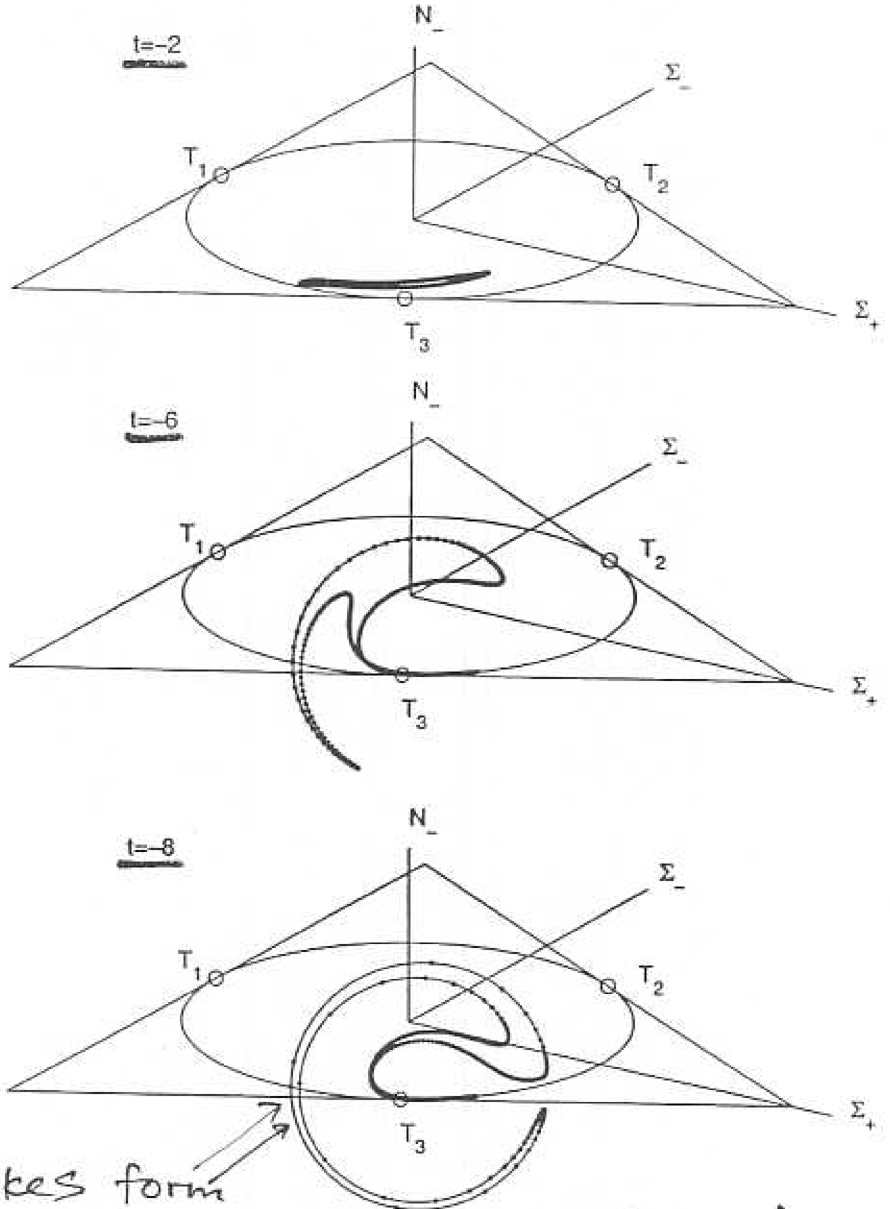
Taub vacuum orbits in Hubble-normalized Bianchi II state space





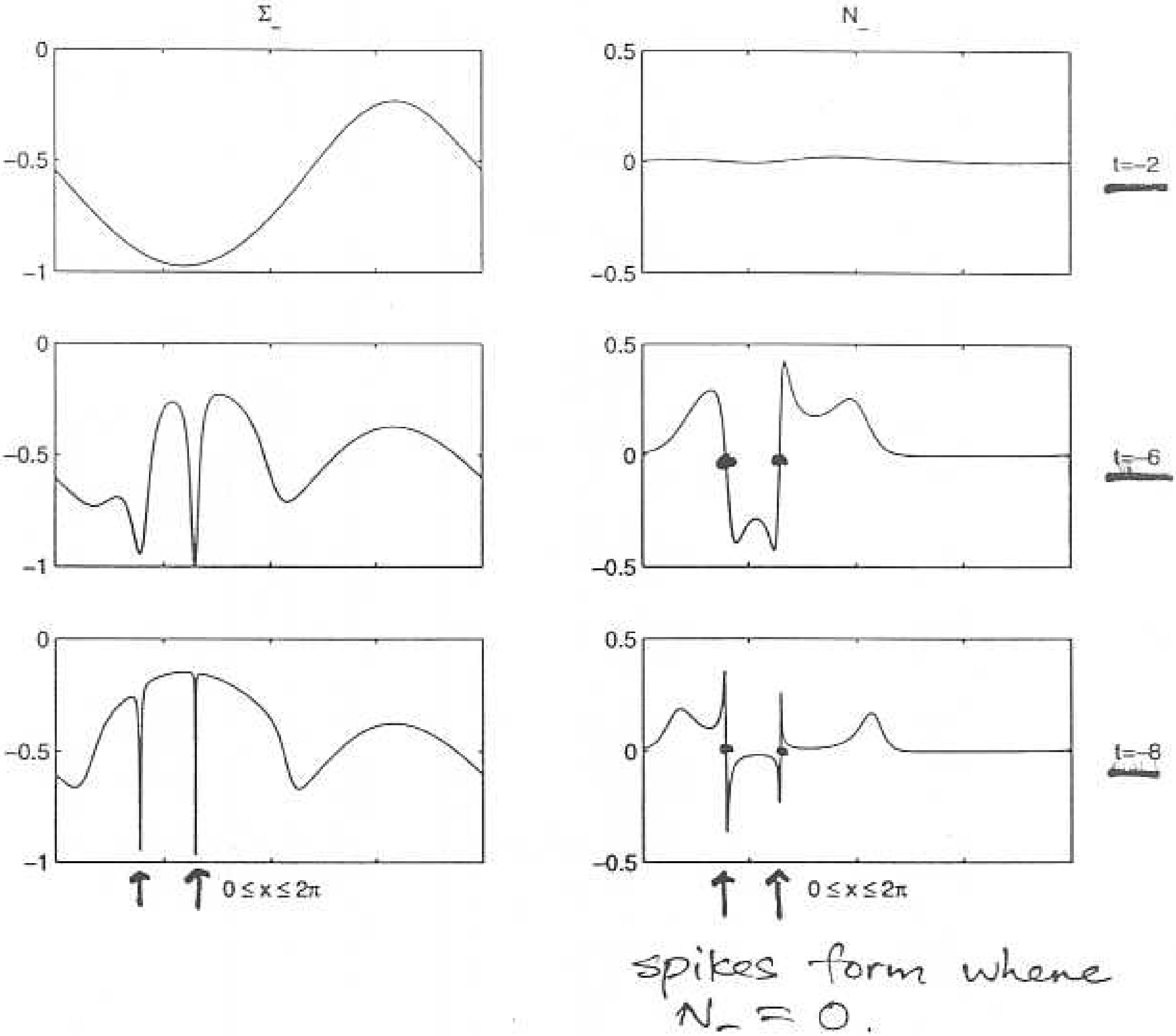
Spikes in Gowdy models.

(visualized in Hubble-normalized Bianchi II state space. Each dot represent an observer with fixed x coordinate)



Spikes form at two locations (where $N_- = 0$) where the observers are unable to follow the Taub vacuum orbits.

Corresponding snapshots for Σ_- & N_-.



Spikes in generic G_2 models are conjectured to be recurring, due to Mixmaster dynamics. Some numerical evidence is available.

Numerical study of inhomogeneous effects

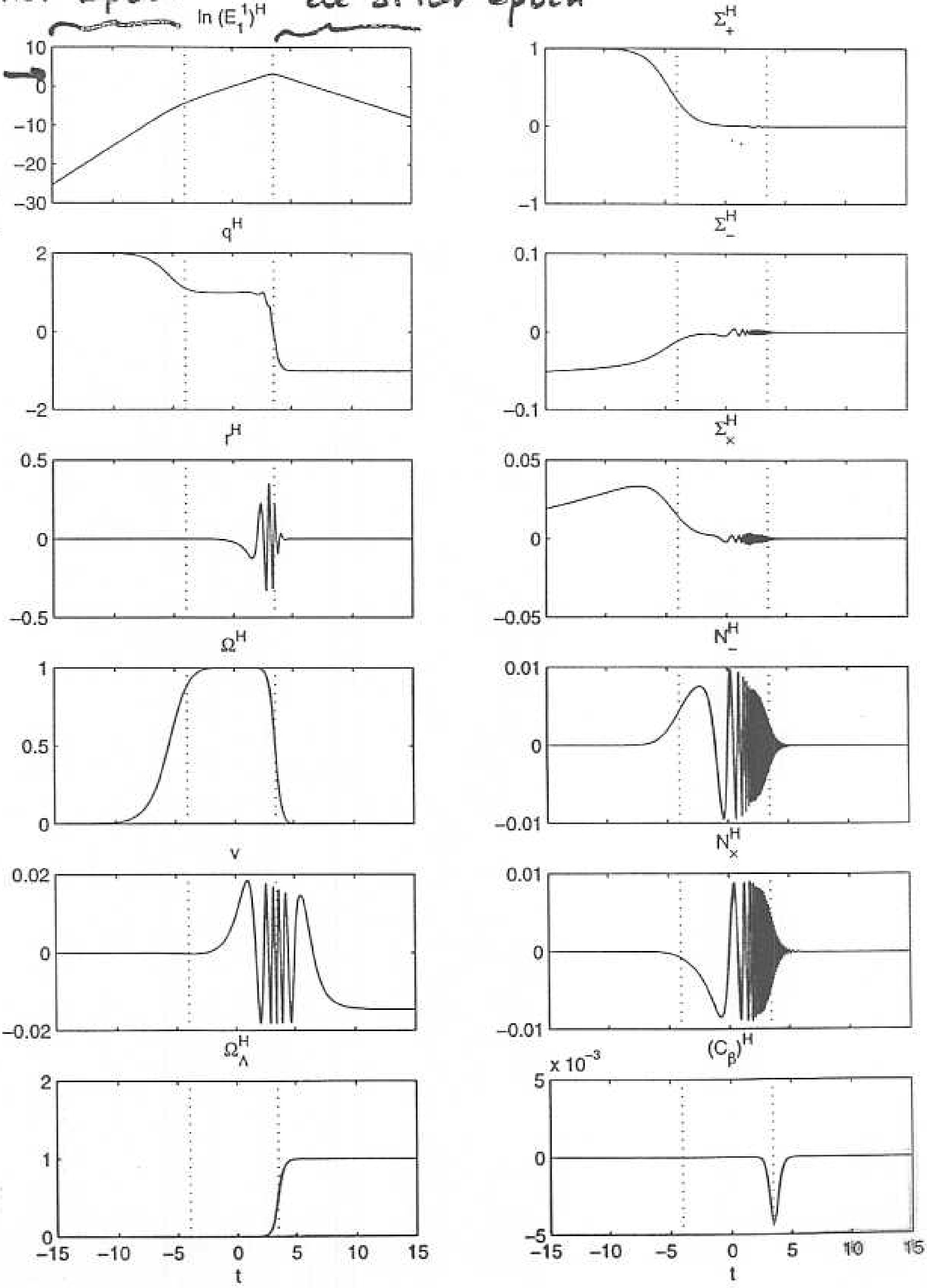
e.g. close-to-flat Friedmann-Lemaître intermediate epoch

flat FL epoch

Kasner epoch

de Sitter epoch

max inhom. effect at late FL.



Fuchsian methods

- Rigorous method used to construct the form of typical solutions in an asymptotic regime.
- Developed by Alan Rendall & S. Kichenassamy.
- Have been applied to Gowdy models, and G_0 models with $\Lambda > 0$.

Concluding remarks

- New heuristic, rigorous and numerical tools for analyzing inhomogeneous cosmological models extend our understanding gained from more traditional dynamical systems analysis of homogeneous models.
- Need to make the new tools more robust for wider cosmological and astrophysical applications.

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