Cosmological solutions to the Einstein-Vlasov system

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Kinetic Theory

The purpose of kinetic theory is to model the time evolution of a collection of particles. The particles may be entirely different objects:

- atoms and molecules in a neutral gas
- electrons and ions in a plasma
- stars in stellar dynamics
- galaxies or clusters of galaxies in cosmology

In kinetic theory the model is statistical and the particle system is described by a distribution function $f = f(t, x, p)$, which represents the density of particles with given space-time position $(t, x) \in \mathbb{R} \times \mathbb{R}^3$ and momentum $p \in \mathbb{R}^3$.

A distribution function contains a wealth of information, and macroscopical quantities are easily calculated from this function.

Example: In Minkowski space the energy density ρ is given by

$$\rho(t,x) = \int_{R^3} \sqrt{1+p^2} f(t,x,p) dp,$$

and the current $j \in R^3$ is given by

$$j(t,x) = \int_{R^3} p f(t,x,p) dp.$$

The Vlasov-Poisson system

A Newtonian gravitational system or a plasma which only interacts through its electromagnetic field is modelled in kinetic theory by the Vlasov-Poisson system

$$\partial_t f + p \cdot \nabla_x f + \beta E(t,x) \cdot \nabla_p f = 0 \tag{1}$$

$$E = \nabla \phi, \quad \Delta \phi = \rho := \int_{R^3} f(t,x,p) dp. \tag{2}$$

Here $\beta = -1$ in the gravitational case and $\beta = 1$ in the plasma case.

A fundamental result in kinetic theory is that global existence holds true for the VP system (in both the cases $\beta \pm 1$) for smooth and comapctly supported initial data:

- K. Pfaffelmoser, J. Diff. Eqns. 95, (1992).
- P.L. Lions and B. Perthame, Invent. Math., 105, (1991).

There is no analogous result for a fluid model. This global existence result in the Newtonian case is a strong motivation for coupling a kinetic matter model to the Einstein equations. The recent global existence result by S. Calogero on the Nordström-Vlasov system also strongly supports Vlasov matter as matter model in GR.

The Vlasov equation in curved spacetimes

We are considering a collisionless gas where all particles have identical rest mass $m = 1$. Since there are no collisions the particles travel along geodesics in spacetime.
The possible values of the four-momentum of the particles are thus all future-directed unit timelike vectors and they constitute a hypersurface P in the tangent bundle TM, which is called the mass shell.

$$P = \{p^a : g_{ab}p^a p^b = -m = -1\}.$$

The Vlasov equation takes the following form in local coordinates

$$\partial_t f + \frac{p^j}{p^0}\partial_{x^j} f - \frac{1}{p^0}\Gamma^j_{ab} p^a p^b \partial_{p^j} f = 0. \qquad (3)$$

Here $a, b = 0, 1, 2, 3$ and $j = 1, 2, 3$ and Γ^j_{ab} are the Christoffel symbols. It is understood that p^0 is expressed in terms of p^j and the metric g_{ab} using the relation $g_{ab}p^a p^b = -m = -1$. Hence, in Minkowski space we have

$$p^0 = \sqrt{1 + |p|^2}, \quad p \in R^3.$$

In a fixed spacetime the Vlasov equation is a linear hyperbolic equation for f and we can solve it by solving the characteristic system,

$$dX^i/ds = \frac{P^i}{P^0}, \quad (4)$$

$$dP^i/ds = -\Gamma^i_{ab}\frac{P^a P^b}{P^0}. \quad (5)$$

In terms of initial data f_0 the solution to the Vlasov equation can be written

$$f(t, x^i, p^i) = f_0(X^i(0, t, x^i, p^i), P^i(0, t, x^i, p^i)), \quad (6)$$

where $X^i(t, t, x^i, p^i) = x^i$ and $P^i(t, t, x^i, p^i) = p^i$.

The Einstein-Vlasov system

In the coordinates (x^a, p^a) on P we define the energy momentum tensor by

$$T_{ab} = -\int_{\mathbb{R}^3} f\, p_a p_b |g|^{1/2} \frac{dp^1 dp^2 dp^3}{p_0}.$$

T_{ab} **satisfies the standard energy conditions.** We can now formulate the Einstein-Vlasov system

$$G_{ab} := R_{ab} - \tfrac{1}{2} R g_{ab} = 8\pi T_{ab} + \Lambda g_{ab}, \qquad (7)$$

$$\partial_t f + \tfrac{p^j}{p^0}\partial_{x^j} f - \tfrac{1}{p^0}\Gamma^j_{ab} p^a p^b \partial_{p^j} f = 0, \qquad (8)$$

where Λ is the cosmological constant.

Cosmological spacetimes

- In cosmology the “particles” in the kinetic description are galaxies or even clusters of galaxies.
- The goal is to determine the global properties of the solutions to the Einstein-Vlasov system and a global time coordinate t must be found and the asymptotic behaviour of the solutions when t tends to its limiting values has to be analyzed. This might correspond to approaching a singularity or to a phase of unending expansion.
- The general case is at present open and all known results are for cases with symmetry.

Symmetry classes

Cosmological spacetimes are characterized by prescribing data on a *compact* spacelike hypersurface. These spacetimes admit a large number of symmetry groups in contrast to asymptotically flat spacetimes. Denote by N the dimension of the symmetry group. Studies of the Einstein-Vlasov system has been carried out when

- $N \geq 4$. These spacetimes are spatially homogenous. (Recent studies include a scalar field.)
- $N = 3$. These spacetimes are inhomogeneous but admit **no** gravitational waves.
- $N = 2$. These spacetimes are inhomogeneous and include gravitational waves.

Topology and symmetry

The topology of M is assumed to be $\mathbb{R} \times S^1 \times F$, with F a compact two-dimensional manifold (let $\tilde{F}$ denote the covering manifold of F).
In the case $N = 3$ the following situations have been investigated

- *The plane case:* $F = S^1 \times S^1 = T^2$ and $G = E_2$ acts isometrically on $\tilde{F} = R^2$.
- *The spherical case:* $F = S^2$ and $G = SO(3)$ acts isometrically and without fixed points on $S^1 \times S^2$.
- *The hyperbolic case:* $\tilde{F} = H^2/\Gamma$, with Γ a discrete group of isometries of H^2.

The case $N = 2$

In the case $N = 2$ the following situation has been investigated

- $F = S^1 \times S^1 = T^2$, and $G = U(1) \times U(1)$,

which we call the T^2-case. The well-known Gowdy spacetime is a sub-case. The T^2-symmetry case is “the most general” situation that presently has been studied for the Einstein-Vlasov system.

As will be seen below, the issue of global existence is well-understood in the T^2 case **but only when** $N \geq 3$ (partial) information on geodesic completeness and curvature blow-up has been obtained.

Time coordinates

The two principal types of time coordinate that have been studied in the inhomogeneous case are CMC- and areal time.

- A CMC time coordinate, t, is one where each hypersurface of constant time has constant mean curvature (CMC) and on each hypersurface of this kind the value of t is the mean curvature of that slice.
- In the case of areal coordinates, the time coordinate is a function of the area of the surfaces of symmetry.

Comparison of CMC- and areal time coordinates

These are both geometrically based time foliations. The advantage with a CMC approach is that the definition of a CMC hypersurface does not depend on any *symmetry assumptions* and it is possible that CMC foliations will exist for rather general spacetimes. The areal coordinate foliation, on the other hand, is adapted to the symmetry of spacetime but it has analytical advantages.

The metric in areal coordinates

In areal time coordinates the metric in the case of T^2 symmetry can be written

$$g = \mathrm{e}^{2(\eta - U)}(-\alpha dt^2 + d\theta^2) + \mathrm{e}^{-2U} t^2 [dy + H d\theta + M$$
$$+ \mathrm{e}^{2U}[dx + A dy + (G + AH) d\theta + (L + AM) dt]^2$$

Here the metric coefficients $\eta, U, \alpha, A, G, H, L$ and M depend on t and θ and $\theta, x, y \in S^1$. Note that the time coordinate is of areal type since the area element of the surfaces of symmetry is t^2 since ∂_x and ∂_y are Killing fields.

Some sub-cases

There are some possible subcases to the T^2 symmetry class.

- Gowdy symmetry which admits two Killing fields. It is obtained by letting $G = H = L = M = 0$.
- The plane case (where $N = 3$). The form of the metric in areal coordinates is obtained by letting $A = G = H = L = M = 0$ and $U = \log t/2$.

The spherical- and hyperbolic case (where $N = 3$) are however not sub cases but need to be treated separately.

Plan for the rest of the talk

I will first review the *global existence* results in the T^2 case, and then comment on the $N \geq 3$ cases where more information has been obtained about the *asymptotics*.
In the final part of my talk I will mention some results in the case of a positive cosmological constant and in the case where in addition a scalar field has been included.

The Einstein-matter constraint equations

$$
\begin{aligned}
\frac{\eta_t}{t} &= U_t^2 + \alpha U_\theta^2 + \frac{\mathrm{e}^{4U}}{4t^2}(A_t^2 + \alpha A_\theta^2) \\
&\quad + \frac{\mathrm{e}^{-2\eta}}{4}(\mathrm{e}^{4U}\Gamma^2 + t^2 H_t^2) + \mathrm{e}^{2(\eta-U)}\alpha\rho \\
\frac{\eta_\theta}{t} &= 2U_t U_\theta + \frac{\mathrm{e}^{4U}}{2t^2}A_t A_\theta - \frac{\alpha_\theta}{2t\alpha} - \mathrm{e}^{2(\eta-U)}\sqrt{\alpha}J_1 \\
\alpha_t &= 2t\alpha^2\mathrm{e}^{2(\eta-U)}(P_1 - \rho) - \alpha t\mathrm{e}^{-2\eta}(\mathrm{e}^{4U}\Gamma^2 + t^2 H_t^2)
\end{aligned}
$$

The Einstein-matter evolution equations

$$\eta_{tt} - \alpha\eta_{\theta\theta} = \frac{\eta_\theta\alpha_\theta}{2} + \frac{\eta_t\alpha_t}{2\alpha} - \frac{\alpha_\theta^2}{4\alpha} + \frac{\alpha_{\theta\theta}}{2} - U_t^2 + \alpha U_\theta^2$$

$$+\frac{\mathrm{e}^{4U}}{4t^2}(A_t^2 - \alpha A_\theta^2) - \mathrm{e}^{-2\eta}(t^2 H_t^2 - \frac{\mathrm{e}^{4U}}{2}$$

$$-\alpha \mathrm{e}^{2(\eta-U)}P_3,$$

$$U_{tt} - \alpha U_{\theta\theta} = -\frac{U_t}{t} + \frac{U_\theta\alpha_\theta}{2} + \frac{U_t\alpha_t}{2\alpha} + \frac{\mathrm{e}^{4U}}{2t^2}(A_t^2 - \alpha A_\theta^2$$

$$+\frac{\mathrm{e}^{4U-2\eta}}{2}\Gamma^2 + \frac{\mathrm{e}^{2(\eta-U)}\alpha}{2}K,$$

where $K = \rho - P_1 + P_2 - P_3$, and $\Gamma = G_t + AH_t$.

$$A_{tt} - \alpha A_{\theta\theta} = \frac{A_t}{t} + \frac{\alpha_\theta A_\theta}{2} + \frac{\alpha_t A_t}{2\alpha} + 4\alpha A_\theta U_\theta$$
$$- 4A_t U_t + t^2 \mathrm{e}^{-2\eta} \Gamma H_t + 2t\alpha \mathrm{e}^{2(\eta - 2U)} S_{23}.$$

Auxiliary equations (where $Y = A\mathrm{e}^{4U}\Gamma + t^2 H_t$)

$$\partial_\theta[\mathrm{e}^{-2\eta}\alpha^{-1/2}\mathrm{e}^{4U}\Gamma] = -2\mathrm{e}^{\eta} J_2$$
$$\partial_t[\mathrm{e}^{-2\eta}t\alpha^{-1/2}\mathrm{e}^{4U}\Gamma] = 2t\alpha^{1/2}\mathrm{e}^{\eta} S_{12}$$
$$\partial_\theta[\mathrm{e}^{-2\eta}\alpha^{-1/2}Y] = -2\mathrm{e}^{\eta} A J_2 - 2t\mathrm{e}^{\eta-2U} J_{33}$$
$$\partial_t[\mathrm{e}^{-2\eta}t\alpha^{-1/2}Y] = 2t\alpha^{1/2}\mathrm{e}^{\eta}(AS_{12} + t\mathrm{e}^{-2U} S_{13})$$

The Vlasov equation

$$
\begin{aligned}
&\frac{\partial f}{\partial t} + \frac{\sqrt{\alpha} v^1}{v^0}\frac{\partial f}{\partial \theta} - \Big[(\eta_\theta - U_\theta + \frac{\alpha_\theta}{2\alpha})\sqrt{\alpha} v^0 \\
&+ (\eta_t - U_t) v^1 + \frac{\sqrt{\alpha} U_\theta}{v^0}((v^3)^2 - (v^2)^2) \\
&- \frac{\sqrt{\alpha}\mathrm{e}^{2U} A_\theta}{t}\frac{v^2 v^3}{v^0} + \mathrm{e}^{-\eta}(\mathrm{e}^{2U}\Gamma v^2 + t H_t v^3) \Big] \frac{\partial f}{\partial v^1} \\
&- \Big[U_t v^2 + \sqrt{\alpha} U_\theta \frac{v^1 v^2}{v^0} \Big] \frac{\partial f}{\partial v^2} - \Big[(\frac{1}{t} - U_t) v^3 \\
&- \sqrt{\alpha} U_\theta \frac{v^1 v^3}{v^0} + \frac{\mathrm{e}^{2U} v^2}{t}(A_t + \sqrt{\alpha} A_\theta \frac{v^1}{v^0}) \Big] \frac{\partial f}{\partial v^3} = 0.
\end{aligned}
$$

The matter quantities

$$
\begin{aligned}
\rho(t,\theta) &= \int_{R^3} v^0 f(t,\theta,v)\, dv \\
P_k(t,\theta) &= \int_{R^3} \frac{(v^k)^2}{v^0} f(t,\theta,v)\, dv, \quad k = 1,2,3 \\
J_k(t,\theta) &= \int_{R^3} v^k f(t,\theta,v)\, dv, \quad k = 1,2,3 \\
S_{jk}(t,\theta) &= \int_{R^3} \frac{v^j v^k}{v^0} f(t,\theta,v)\, dv, \quad j,k = 1,2,3
\end{aligned}
$$

The variables v are related to p by

$$v^0 = \sqrt{\alpha}\mathrm{e}^{\eta - U} p^0,\ v^1 = \mathrm{e}^{(\eta - U)} p^1,$$
$$v^2 = \mathrm{e}^{U} p^2,\ v^3 = t\mathrm{e}^{-U} p^3,$$

where

$$p^\mu := \frac{dx^\mu}{d\tau}, \quad x^\mu = (t, \theta, x, y),$$

and τ is proper time. Since

$$g_{\mu\nu} p^\mu p^\nu = -1,$$

it follows that

$$v^0 = \sqrt{1 + (v^1)^2 + (v^2)^2 + (v^3)^2}\,.$$

Main theorems for T^2 space-times

Theorem 0.1 *Let* $(M, g_{\alpha\beta}, f)$ *be the maximal globally hyperbolic development of non-flat* C^∞ *initial data for the Einstein-Vlasov system with* T^2 *symmetry. Then* M *can be covered by compact spacelike hypersurfaces of constant area function* R *with each value in the range* $(0, \infty)$ *occurring as the value of the area function on precisely one of these hypersurfaces.*

The second theorem concerns the existence of a CMC time coordinate.

Theorem 0.2 *Let* $(M, g_{\alpha\beta}, f)$ *be the maximal globally hyperbolic development of non-flat* C^∞ *initial data for the Einstein-Vlasov system with* T^2 *symmetry. Then* M *can be covered by compact spacelike hypersurfaces of constant mean curvature with each value in the range* $(-\infty, 0)$ *occurring as the mean curvature of precisely one of these hypersurfaces.*

Some references

- H. A., A. RENDALL AND M. WEAVER, Existence of CMC and constant areal time foliations in T^2 symmetric spacetimes with Vlasov matter, *Comm. Partial Differential Equations*, 29 (2004).
- M. WEAVER., On the area of the symmetry orbits in T^2 symmetric spacetimes with Vlasov matter, *Class. Quantum Grav.*, 21 (2004).

More references

- H. A., Global foliations of matter spacetimes with Gowdy symmetry, *Comm. Math. Phys.*, 206 (1999).
- A.D. Rendall, Existence of CMC foliations in spacetimes with two-dimensional local symmetry, *Commun. Math. Phys.*, 189 (1997).
- O. HENKEL, Local prescribed mean curvature foliations in cosmological spacetimes, *Math. Proc. Cambridge Philos. Soc.*, 134 (2003).
- B. BERGER, P. CHRUSCIEL, J. ISENBERG AND V. MONCRIEF, Global foliations of vacuum spacetimes with T^2 isometry, *Ann. Phys.*, 260 (1997).

Sketch of proof in the expanding direction

“Energy” monotonicity. Let $E(t)$ be defined by

$$E(t) =$$

$$= \int_{S^1} \Big[\alpha^{-\frac{1}{2}} U_t^2 + \sqrt{\alpha} U_\theta^2 + \frac{e^{4U}}{4t^2} (\alpha^{-\frac{1}{2}} A_t^2 + \sqrt{\alpha} A_\theta^2)$$

$$+ \frac{e^{-2\eta}\alpha^{-1/2}}{4} (e^{4U}\Gamma^2 + t^2 H_t^2) + \sqrt{\alpha}\mathrm{e}^{2(\eta-U)}\rho \Big] \, d\theta.$$

Lemma 0.3 $E(t)$ *is a monotonically decreasing function in* t*, and satisfies*

$$\frac{d}{dt}E(t) = -\frac{2}{t}\int_{S^1}\Bigg[\alpha^{-1/2}U_t^2 + \frac{e^{4U}}{4t^2}\sqrt{\alpha}A_\theta^2$$
$$+\frac{e^{-2\eta}\sqrt{\alpha}}{4}(e^{4U}\Gamma^2 + 2t^2H_t^2) + \frac{\sqrt{\alpha}}{2}e^{2(\eta-U)}(\rho + P_3)\Bigg]\, d$$

Bounds on U, A, η **etc.** This lemma leads to bounds on (most of) the metric functions.

Support of the momenta: Q(t)

We define

$$Q(t) = \{|p| : \text{there exists } (t, x) \text{ such that } f(t, x, p) \neq 0$$

Thus, at time t there are no particles having momenta larger than $Q(t)$. This function plays an important role in kinetic theory in general, and to control $Q(t)$ is often the major obstacle.

Bounds on $U_t, U_\theta, \eta_t, \alpha_t$ and $Q(t)$

Below we only treat the case
$A = G = H = M = L = 0$. Let us define

$$G = \frac{1}{2}(U_t^2 + \alpha U_\theta^2),$$
$$H = \sqrt{\alpha} U_t U_\theta,$$

and

$$\chi = \frac{1}{\sqrt{2}}(\partial_t + \sqrt{\alpha}\partial_\theta)$$
$$\zeta = \frac{1}{\sqrt{2}}(\partial_t - \sqrt{\alpha}\partial_\theta)$$

By using the evolution equation for U we find

$$
\begin{aligned}
\zeta(G+H) &= L(U_t^2, U_\theta^2; \alpha_t, \alpha) + \\
&\quad + L(U_t, U_\theta)\mathrm{e}^{2(\tilde{\eta}-U)}\kappa, \\
\chi(G-H) &= L(U_t^2, U_\theta^2; \alpha_t, \alpha) + \\
&\quad + L(U_t, U_\theta)\mathrm{e}^{2(\tilde{\eta}-U)}\kappa,
\end{aligned}
$$

where $\kappa = \rho - P_1 + P_2 - P_3$.

Main inequality

Define

$$\Gamma(t) := \sup_{\theta \in S^1} G(t,\theta) + Q^2(t),$$

we will show

$$\Gamma(t) \leq C + \int_{t_0}^{t} \Gamma(s) \ln \Gamma(s) ds,$$

which then, by a Grönvall argument, leads to a bound on $\Gamma(t)$.

Symmetries$\longrightarrow$ conserved quantities

The two Killing vector fields ∂_x and ∂_y lead to conservation of

$$V^2(t)\mathrm{e}^{U(t,\Theta(t))} \text{ and } V^3(t)t\mathrm{e}^{-U(t,\Theta(t))},$$

where $V^2(t), V^3(t)$ and $\Theta(t)$ are solutions of the characteristic system associated to the Vlasov equation. Hence $|V^2(t)|$ and $|V^3(t)|$ are both uniformly bounded on $[t_0, t_1)$.

Therefore, to control the support of the momentum, $Q(t)$, it is sufficient to control

$$Q^1(t) := \sup\{|v^1| : \exists (s,\theta) \in [t_0, t] \times S^1,\ f(s,\theta,v) \neq 0\}$$

From the equations for $G + H$ and $G - H$ we can derive the inequality

$$\sup_\theta G(t) \leq C +$$

$$+ C(t) \int_{t_0}^{t} \sup_\theta G(s) + \sup_\theta \sqrt{G(s)} \ln Q^1(s) ds.$$

A sufficient estimate on Q

Lemma 0.4 *Let* $Q^1(t)$ *and* $G(t,\theta)$ *be as above. Then*

$$|Q^1(t)|^2 \leq C + D(t)\int_{t_0}^{t}[(Q^1(s))^2 + \sup_{\theta} G(s,\theta)]ds,$$

where C *is a constant and* $D(t)$ *is a uniformly bounded function on* $[t_0, t_1)$.

The proof of this lemma uses the exact structure of the Vlasov equation and a cancellation property is crucial.

The characteristic equation for V^1 reads

$$\frac{d}{ds}(V^1(s))^2 = 2(U_\theta - \eta_\theta - \frac{\alpha_\theta}{2\alpha})\sqrt{\alpha}V^1V^0$$
$$-2(\eta_t - U_t)(V^1)^2 - \frac{\sqrt{\alpha}U_\theta V^1}{V^0}((V^2)^2 - (V^3)^2).$$

A rough treatment leads to terms of the type GQ^2 which shows that a cancellation property is crucial for proving the lemma.

Bounds on ∂f, α_θ and η_θ

Recall that the solution f can be written in the form

$$f(t,\theta,v) = f_0(\Theta(0,t,\theta,v), V(0,t,\theta,v)),$$

where $\Theta(s,t,\theta,v)$, $V(s,t,\theta,v)$ are solutions of

$$\begin{aligned}
\frac{d\Theta}{ds} &= \sqrt{\alpha}\frac{V^1}{V^0}, \\
\frac{dV^1}{ds} &= -(\eta_\theta - U_\theta + \frac{\alpha_\theta}{2\alpha})\sqrt{\alpha}V^0 - (\eta_t - U_t)V^1 \\
&\quad -\sqrt{\alpha}U_\theta\frac{(V^3)^2 - (V^2)^2}{V^0}, \\
\frac{dV^2}{ds} &= \ldots \ , \quad \frac{dV^3}{ds} = \ldots
\end{aligned}$$

Hence to establish bounds on the first derivatives of f it is sufficient to bound $\partial\Theta$ and ∂V since f_0 is smooth. Here ∂ denotes the first order derivative with respect to t, θ or v.
However, second order derivatives appear, which we have no control of up to now. The following lemma solves that problem.

Lemma 0.5 *Let ∂ denote $\partial_t, \partial_\theta$ or ∂_v, and let $\Theta(s) = \Theta(s,t,\theta,v)$ and $V(s) = V(s,t,\theta,v)$. Define*

$$
\begin{aligned}
\Psi &= \alpha^{-1/2}\partial\Theta, \\
Z^1 &= \partial V^1 + \left(\frac{\eta_t V^0}{\sqrt{\alpha}} - \frac{U_t V^0}{\sqrt{\alpha}} \frac{(V^0)^2 - (V^1)^2}{(V^0)^2 - (V^1)^2} \right. \\
&\quad - \frac{U_t V^0}{\sqrt{\alpha}} \frac{((V^2)^2 - (V^3)^2)}{(V^0)^2 - (V^1)^2} \\
&\quad \left. + U_\theta \frac{V^1((V^2)^2 - (V^3)^2)}{(V^0)^2 - (V^1)^2} \right) \partial\Theta, \\
Z^2 &= \partial V^2 + V^2 U_\theta\, \partial\Theta, \\
Z^3 &= \partial V^3 - V^3 U_\theta\, \partial\Theta.
\end{aligned}
$$

Then there is a matrix $A = \{a_{lm}\}$, $l, m = 0, 1, 2, 3$, such that

$$\Omega := (\Psi, Z^1, Z^2, Z^3)^T$$

satisfies

$$\frac{d\Omega}{ds} = A\Omega,$$

and the matrix elements $a_{lm} = a_{lm}(s, \Theta(s), V^k(s))$ are all uniformly bounded on $[t_0, t_1)$.

The link between Thm 1 and 2

The mean curvature of a slice S_t in areal coordinates satisfies

$$\text{mean curvature of } S_t \leq \frac{-\sqrt{3}}{t}.$$

The past direction. Since the mean curvature of the slice is everywhere negative the result by Henkel (2003) guarantees the existence of at least one CMC hypersurface. Rendall's result (1997) then applies in the past direction.

The future direction. If p is any point of spacetime it is sufficient to show that there is a CMC hypersurface which contains p in its past. If S_1 is the Cauchy surface of constant areal time passing through p the inequality above shows that the mean curvature of this hypersurface is negative. Since S_1 is compact it achives a maximum value $H_1 < 0$. Let S_2 be the CMC surface with mean curvature $H_1/2$, then a standard argument (Marsden and Tipler 1980) shows that S_2 is strictly to the future of S_1.

Asymptotic results when $N = 2$

In the case $N = 2$ the only conclusion about the asymptotic behaviour is that the initial singularity is a crushing singularity. There are no other results about the asymptotic behaviour, i.e. also in the case of Gowdy (and even polarized Gowdy) with Vlasov matter no such results are available in contrast to the Gowdy-vacuum case where Ringström has obtained strong results.

$N = 3$: The plane, spherical and hyperbolic case

- The existence results in the T^2 case also hold in the hyperbolic case (H.A., Rein and Rendall 2003), and trivially in the plane case since it is a sub-case.
- In the spherical case the solution exists only for a finite time and there is a big crunch (Rein 1996).

Asymptotic results when $N = 3$

- The past direction: Rein (1996) has shown (in all three cases) that for sufficiently *small initial* data the initial singularity is a curvature singularity.
- The future direction: Rein (2004) has shown that under a *smallness assumption* on the initial data, future geodesic completeness holds in *the hyperbolic* case.

$N \geq 4$: The homogeneous case

There are several studies of the Einstein-Vlasov system in the homogeneous case where the main contributions are from Rendall, Tod and Uggla (1995-2000). Here there are many different cases (Bianchi I,II,... and others) and I will not give any detailed review of these. However, it should be stressed that the general conclusion from these studies is that *the choice of matter model is very important* since in all symmetry classes studied there are differencies betweeen the collisionless model and a perfect fluid, both regarding the initial singularity and the expanding phase.

The case when $\Lambda \neq 0$

Several studies including a non-zero cosmological constant have been carried out. The most interesting contribution is by Rendall and Tchapnda (2003) who studied the case with a positive cosmological constant in the plane and hyperbolic case. They show global existence *to the future* and obtain substantial information about the asymptotics:

- Future geodesic completeness is proved (recall that f.g.c. has only been proved when $\Lambda = 0$ in the hyperbolic case for restricted initial data)
- The cosmic no-hair conjectire is proved, i.e. the de Sitter solution acts as model for the dynamics since the generalized Kasner exponents tend to $1/3$ as $t \to \infty$

Other studies with $\Lambda \neq 0$

- H. LEE, Asymptotic behaviour of the Einstein-Vlasov system with a positive cosmological constant, *Math. Proc. Camb. Phil. Soc.*, 137 (2004).
- S.B. TCHAPNDA, N. NOUTCHEGUEME, The surface-symmetric Einstein-Vlasov system with cosmological constant, Preprint gr-qc/0304098.
- S.B. TCHAPNDA, Structure of solutions near the initial singularity for the surface-symmetric Einstein-Vlasov system, Preprint gr-qc/0407062.

Inclusion of a nonlinear scalar field when $N \geq 4$

Lee (2004) has investigated the *homogeneous* case including a nonlinear scalar field in addition to Vlasov matter. $T_{\alpha\beta}$ then reads

$$T_{\alpha\beta} = T_{\alpha\beta}^{\text{Vlasov}} + \nabla_\alpha\phi\nabla_\beta\phi - (\frac{1}{2}\nabla^\gamma\phi\nabla_\gamma\phi + V(\phi))g_{\alpha\beta}. \tag{9}$$

Here ϕ is the scalar field and V is a potential and the Bianchi identities lead to the following equation for the scalar field

$$\nabla^\gamma\phi\nabla_\gamma\phi = V'(\phi). \tag{10}$$

Under the assumption that V is nonnegative and C^2 global existence to the future is obtained and if the potential is restricted to the form

$$V(\phi) = V_0 e^{-c\Phi},$$

where $0 < c < 4\sqrt{\pi}$ then future geodesic completeness is proved.

Inclusion of a linear scalar field when $N = 3$

A linear scalar field + Vlasov matter has been considered in the case of plane, spherical and hyperbolic symmetry by Tegankong, Noutchegueme and Rendall (2005). They obtain *local* existence together with a continuation criterion which in addition to a bound on the support of the momenta requires bounds on the derivatives on the scalar field.