Accelerated expanding models with Vlasov matter

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Outline

1. Vlasov matter
2. accelerated expanding cosmology
   - a positive cosmological constant
   - a nonlinear scalar field : various potentials
   - exponential expansion, power-law expansion, intermediate inflation
3. spatially homogeneous (anisotropic) spacetimes
4. inhomogeneous spacetimes

Vlasov matter

- Particle systems are modeled by a distribution function $f(t, x, p)$
- f represents the density of particles with given spacetime position (t, x) and momentum p

What kind of particle systems ?

- Collisionless particles : collisions between particles are rare to be neglected
- No direct interactions between particles not like other equations of kinetic theory (cf. Boltzmann equation)
- This particle system is described by the Vlasov equation
- Collisionless matter, Vlasov matter

Vlasov matter + Fields

- The time evolutions of particle systems are determined by the interactions between particles
- Each particle is acted by self-induced fields which are generated by all particles together
- These fields depend on the physical situation being modeled

❑ Plasma physics : eletromagnetic fields

➥ *Vlasov-Poisson, relativistic Vlasov-Poisson, Vlasov-Maxwell*

❑ Stellar dynamics : gravitational fields

➥ *Vlasov-Poisson, Nordström-Vlasov, Einstein-Vlasov*

Vlasov equation

✓ Vlasov equation : $$\partial_t f + \frac{p^a}{p^0}\partial_{x^a} f - \frac{p^\mu p^\nu}{p^0}\Gamma^a_{\mu\nu}\partial_{p^a} f = 0$$

❑ Mass shell condition : $g_{\mu\nu}p^\mu p^\nu = -m^2$
→ $p^0 = (m^2 + g_{ij}p^i p^j)^{1/2}$

❑ Initial data $f_0 := f(t_0, x, p)$ is a nonnegative function with compact support in (x, p)

✓ the energy-momentum tensor : $$T_{\mu\nu} = \int \sqrt{-g}\frac{p_\mu p_\nu}{p^0} f\, dp$$

Accelerated **expanding cosmology**

- The early universe closed to big bang *(inflation)*

- The present era *(quintessence)* : the observations of supernovae of type Ia

- A positive cosmological constant – *exponential expansion*

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

- A nonlinear scalar field : choice of the potential $V(\phi)$

$$T_{\alpha\beta} = \nabla_\alpha \phi \nabla_\beta \phi - \left[\frac{1}{2}(\nabla_\gamma \phi \nabla^\gamma \phi) + V(\phi)\right] g_{\alpha\beta}$$

Spatially homogeneous spacetimes

- Spacetime : a manifold $\mathbb{R} \times G$

  G : simply connected three-dimensional Lie group

- Expanding cosmology : Bianchi models (I-VIII)

  cf. Bianchi type IX, Kantowski-Sachs models

- *Accelerated* expanding cosmology

  ① a positive cosmological constant – *exponential expansion* (Wald, Lee)

  ② a nonlinear scalar field

  - exponential potential – *power-law expansion* (Kitada/Maeda, Lee)
  - potential with a positive lower bound – *exponential expansion* (Rendall)
  - *intermediate inflation* – (Rendall)

homogeneous spacetimes with Λ

[Wald (1983)]

- At late times Bianchi models except IX with Λ expand exponentially : inflation
- the spacetime becomes isotropic at late time :
  de Sitter solution (the cosmic no hair conjecture)

(a) the dominant energy condition

$T_{\alpha\beta}v^\alpha w^\beta \geq 0$,

v^α and w^β : future pointing timelike vectors

(b) the strong energy condition

$R_{\alpha\beta}v^\alpha v^\beta \geq 0$ for any timelike vector v^α

Einstein-Vlasov system with Λ

- spacetime : a manifold $G \times I$,

  G : simply connected three dimensional Lie group

  I : an open interval

  Initial data is given on the hypersurface $G \times \{t_0\}$.

- metric :

$$ds^2 = -dt^2 + g_{ij}(t)e^i \otimes e^j$$

  $\{e_i\}$: a left invariant frame, $\{e^i\}$: the dual coframe

- The evolution equations :

$$\partial_t g_{ij} = -2k_{ij}$$

$$\partial_t k_{ij} = R_{ij} + (\text{tr } k)k_{ij} - 2k_{il}k^l_j - 8\pi T_{ij} - 4\pi T_{00}g_{ij} + 4\pi(\text{tr } k)g_{ij} - \underline{\Lambda g_{ij}}$$

- The constraints

Einstein-Vlasov system with Λ

- the Vlasov equation :

$$\partial_t f + \left\{2k^i_j p^j - (1 + g_{rs}p^r p^s)^{-1/2}\,\gamma^i_{mn}p^m p^n\right\}\partial_{p^i} f = 0$$

  - f depends only on t and p^i
  - the Ricci rotation coefficients

$$\gamma^i_{mn} := 2^{-1}g^{ik}(-C^l_{nk}g_{ml} + C^l_{km}g_{nl} + C^l_{mn}g_{kl})$$

  - C^i_{jk} : the structure constants of the Lie algebra of G

Einstein-Vlasov system with Λ

- the components of the energy-momentum tensor:

$$T_{00}(t) = \int f(t,p)(1 + g_{rs}p^r p^s)^{1/2} (\det g)^{1/2}\, dp$$

$$T_{0i}(t) = \int f(t,p) p_i (\det g)^{1/2}\, dp$$

$$T_{ij}(t) = \int f(t,p) p_i p_j (1 + g_{rs}p^r p^s)^{-1/2} (\det g)^{1/2}\, dp$$

Here $p := (p^1, p^2, p^3)$ and $dp := dp^1\, dp^2\, dp^3$.

Mass shell condition : $g_{\mu\nu}p^\mu p^\nu = -1 \;\rightarrow\; p^0 = (1 + g_{rs}p^r p^s)^{-1/2}$

Global existence of solution of the EV with Λ

- *Initial data* g_{ij}, k_{ij} *and* f *at time* t_0
- *The Vlasov equation has Bianchi symmetry*
- $f(t_0, p)$ *: a nonnegative* C^1 *function with compact support*

Then there exists a unique C^1 *solution* (g_{ij}, k_{ij}, f) *of the Einstein-Vlasov system, for all time* t*.*

Asymptotic behaviours of solutions

- Metric :

$$g_{ij}(t) = e^{2\gamma t}(\mathcal{G}_{ij} + \mathcal{O}(e^{-\gamma t}))$$

where $\gamma^2 = \Lambda/3$. $\mathcal{G}_{ij}$ is independent of t.

- mean curvature : tr $k = -(3\Lambda)^{1/2} + \mathcal{O}(e^{-2\gamma t})$

- The generalized Kasner exponents

Let λ_i be the eigenvalues of k_{ij} with respect to g_{ij}.

Define *the generalized Kasner exponents* by

$$p_i := \frac{\lambda_i}{\sum_l \lambda_l} = \frac{\lambda_i}{\text{tr } k}$$

$$\boxed{p_i(t) = 1/3 + \mathcal{O}(e^{-\gamma t})}$$

$\Rightarrow$The spacetime isotropizes at late times.

Asymptotic behaviours of solutions

- The deceleration parameter q

The deceleration parameter is related to the mean curvature.

$$\partial_t(\text{ tr } k) = -(1+q)(\text{ tr } k)^2$$

In the inflationary models, the deceleration parameter is negative. Especially exponential expansions lead to the value -1 for this parameter.

$$\boxed{q = -1 + \mathcal{O}(e^{-2\gamma t})}$$

The energy-momentum tensor

In an orthonormal frame $\{\hat{e}_i\}$

$$\widehat{T}_{00} \equiv \rho(t) = \int f(t,\hat{p})(1+|\hat{p}|^2)^{1/2}\, d\hat{p} \qquad \text{(the energy density)}$$

$$\widehat{T}_{0i} \equiv J_i(t) = \int f(t,\hat{p})\hat{p}_i\, d\hat{p} \qquad \text{(the current density)}$$

$$\widehat{T}_{ij} \equiv S_{ij}(t) = \int f(t,\hat{p})\hat{p}_i\hat{p}_j(1+|\hat{p}|^2)^{-1/2}\, d\hat{p}$$

$$\rho(t) = \mathcal{O}(e^{-3\gamma t}), \quad J_i(t) = \mathcal{O}(e^{-4\gamma t}), \quad S_{ij}(t) = \mathcal{O}(e^{-5\gamma t})$$

$\Rightarrow$ The solutions of EV with Λ are approximated by vacuum Einstein solutions

$$\frac{J_i(t)}{\rho(t)} = \mathcal{O}(e^{-\gamma t}), \quad \frac{S_{ij}(t)}{\rho(t)} = \mathcal{O}(e^{-2\gamma t})$$

$\Rightarrow$ The solutions of EV with Λ resemble the non-tilted dust-like solutions

($J_i(t) = S_{ij}(t) = 0$)

The geodesic completeness

- *Initial data* g_{ij}, k_{ij} *and* f *at time* t_0
- *The Vlasov equation has Bianchi symmetry*
- $f(t_0, p)$ *: a nonnegative* C^1 *function with compact support*

Then the spacetime is future complete.

$$\frac{d\tau}{dt} = (p^0)^{-1} = (m^2 + g_{ij}p^i p^j)^{-1/2}$$

where τ is an affine parameter

Spatially homogeneous spacetimes

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- Expanding cosmology : Bianchi models (I-VIII)

  cf. Bianchi type IX, Kantowski-Sachs models

- *Accelerated* expanding cosmology

  ① a positive cosmological constant – *exponential expansion* (Wald, Lee)

  ② a nonlinear scalar field

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  – potential with a positive lower bound – *exponential expansion* (Rendall)

  – *intermediate inflation* – (Rendall)

the energy-momentum tensor

$$\mathcal{T}_{\alpha\beta} = T_{\alpha\beta} + \nabla_\alpha\phi\nabla_\beta\phi - \left[\frac{1}{2}\nabla^\gamma\phi\nabla_\gamma\phi + V(\phi)\right]g_{\alpha\beta}$$

- ϕ : a scalar field (dark energy)
- $V(\phi)$: a nonnegative potential
- $T_{\alpha\beta}$: the energy-momentum tensor of the Vlasov matter

✔ ϕ satisfies $\nabla_\alpha\nabla^\alpha\phi = V'(\phi)$

✔ ϕ satisfies the dominant energy condition

✔ $T_{\alpha\beta}$ satisfies the dominant and strong energy conditions

Einstein-Vlasov system with a scalar field

- The spacetime metric (Gaussian time coordinate) :
  $ds^2 = -dt^2 + g_{ij}(t)e^i \otimes e^j$
  $\{e_i\}$: a left invariant frame, $\{e^i\}$: the dual coframe

- The Vlasov equation :
  $\partial_t f + \{2k^i_j p^j - (1 + g_{rs}p^r p^s)^{-1/2}\gamma^i_{mn}p^m p^n\}\partial_{p^i} f = 0$
  - p^i : spatial components of the momentum in the frame e_i
  - the Ricci rotation coefficients

$$\gamma^i_{mn} := 2^{-1}g^{ik}(-C^l_{nk}g_{ml} + C^l_{km}g_{nl} + C^l_{mn}g_{kl})$$

  - C^i_{jk} : the structure constants of the Lie algebra of G

Einstein-Vlasov system with a scalar field

- The energy momentum tensor
- Evolution equations :

$$\partial_t g_{ij} = -2k_{ij}$$
$$\partial_t k_{ij} = R_{ij} + (k_{lm}g^{lm})k_{ij} - 2k_{il}k^l_j - 8\pi T_{ij} - 4\pi T_{00}g_{ij} + 4\pi(T_{lm}g^{lm})g_{ij} - 8\pi V(\phi)g_{ij}$$
$$\partial_t \phi = \psi$$
$$\partial_t \psi = (k_{ij}g^{ij})\psi - V'(\phi)$$

  - ϕ depends on only t

- The constraints

Global existence of solutions

Theorem 1

- *Initial data* g_{ij}, k_{ij}, ϕ, ψ *and* f *at time* t_0
- *The Vlasov equation has Bianchi symmetry*
- $f(t_0, p)$ *: a nonnegative* C^1 *function with compact support*
- $V(\phi)$ *: a nonnegative* C^2 *function*

Then there exists a unique C^1 *solution of the Einstein-Vlasov system for all time.*

Exponential potential

- The potential of the scalar field $V(\phi)$ is of form

$$V(\phi) = V_0 e^{-\lambda\kappa\phi}$$

where V_0 is a positive constant, $0 < \lambda < \sqrt{2}$ and $\kappa^2 = 8\pi$.

➠ upper bound : power-law expansion (Halliwell)

➠ $\lambda = 0$: the model with a positive cosmological constant (Lee)

Geodesic completeness

Theorem 2

- *Initial data* g_{ij}, k_{ij}, ϕ, ψ *and* f *at time* t_0
- *The Vlasov equation has Bianchi symmetry*
- $f(t_0, p)$ *: a nonnegative* C^1 *function with compact support*
- $V(\phi) = V_0 e^{-\lambda\kappa\phi}$ *where* $0 < \lambda < \sqrt{2}$

  where V_0 *is a positive constant and* $\kappa^2 = 8\pi$*.*

Then the spacetime is future complete.

Asymptotics

- $k_{ij}g^{ij} = -\frac{6}{\lambda^2}t^{-1} + \text{ error}$

  cf) $k_{ij}g^{ij} = -(3\Lambda)^{1/2} + \mathcal{O}(e^{-2\gamma t})$

- $\phi = \frac{2}{\lambda\kappa}\ln t + \text{ error}$

- $g_{ij}(t) = t^{4/\lambda^2}(\mathcal{G}_{ij} + \text{ error })$

  cf) $g_{ij}(t) = e^{2\gamma t}[\mathcal{G}_{ij} + \mathcal{O}(e^{-\gamma t})]$

- $q = -1 - \frac{\lambda^2}{6} + \text{ error}$

  cf) $q = -1 + \mathcal{O}(e^{-\gamma t})$

- $p_i(t) = \frac{1}{3} + \mathcal{O}(t^{-\xi/2})$

  cf) $p_i(t) = \frac{1}{3} + \mathcal{O}(e^{-\gamma t})$

- the energy-momentum tensor

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  – *intermediate inflation* – (Rendall)

Potential with a positive lower bound [Rendall (2004)][a]

- Consider a solution of the Einstein equations of Bianchi type I-VIII
  + a nonlinear scalar field with potential V of class C^2
  + other matter satisfying the dominant and strong energy conditions
- the solution is initially expanding ($H > 0$) and exists globally to the future
- (1) $V(\phi) \geq V_0$ for a constant $V_0 > 0$
  (2) V' is bounded on any interval on which V is bounded
  (3) V' tends to a limit, finite or infinite as ϕ tends to ∞ or $-\infty$

➡ $\sigma_{ab}\sigma^{ab}$, R, ρ^M, $H^2 - (8\pi/3)[\dot{\phi}^2/2 + V(\phi)]$ decay exponentially

➡ $V(\phi)$ converges to some constant V_1

➡ $V'(\phi) \to 0$

➡ $H \to (8\pi V_1/3)^{1/2}$

[a] "Asymptotics of solutions of the Einstein equations with positive cosmological constant"

Potential with a positive lower bound [Rendall (2004)][a]

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  (3) V' tends to a limit, finite or infinite as ϕ tends to ∞ or $-\infty$

- $\phi \to \phi_1$ as $t \to \infty$ where $V''(\phi_1) > 0$

➡ $\dot{\phi}$, $V(\phi) - V(\phi_1)$, $H - H_1$ decay exponentially,
where $H_1 = (8\pi V(\phi_1)/3)^{1/2}$

➡ $g_{ab} \sim \exp[2H_1 t]$

[a] "Asymptotics of solutions of the Einstein equations with positive cosmological constant"

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  ✔ *intermediate inflation* – (Rendall)

Intermediate inflation [Rendall (2005)][a]

- Consider a solution of the Einstein equations of Bianchi type I-VIII

  + a nonlinear scalar field with potential V of class C^2

  + other matter satisfying the dominant and strong energy conditions
- the solution is initially expanding ($H > 0$) and exists globally to the future
- $\dot{\phi} > 0$ at some time
- (1) $V(\phi)$ is positive with $V(\phi) \to 0$ as $\phi \to \infty$

  (2) $V'(\phi) < 0$

  (3) $V'(\phi)/V(\phi) \to 0$ as $\phi \to \infty$

➡ $3H^2/8\pi V(\phi) \to 1$ as $t \to \infty$

➡ $\sigma_{ab}\sigma^{ab}/H^2$, R/H^2, ρ^M/H^2 tend to zero, as $t \to \infty$

➡ $\dot{H} + H^2 \geq 0$ for large t (equiv. to $\ddot{a} > 0$)

[a] "Intermediate inflation and the slow-roll approximation"

Intermediate inflation [Rendall (2005)][a]

- Consider a solution of the Einstein equations of Bianchi type I-VIII
  + a nonlinear scalar field with potential V of class C^2
  + other matter satisfying the dominant and strong energy conditions
- the solution is initially expanding ($H > 0$) and exists globally to the future
- $\dot{\phi} > 0$ at some time
- (1) $V(\phi)$ is positive with $V(\phi) \to 0$ as $\phi \to \infty$
  (2) $V'(\phi) < 0$
  (3) $V'(\phi)/V(\phi) \to 0$ as $\phi \to \infty$
  (4) V''/V' is bounded as $\phi \to \infty$
  ➡ $3H\dot{\phi}/V' \to -1$ as $t \to \infty$
  ➡ $\ddot{\phi}/3H\dot{\phi} \to 0$ as $t \to \infty$

[a] "Intermediate inflation and the slow-roll approximation"

slow-roll approximation

Under the assumption on the potential that V''/V' is bound, the term $\ddot{\phi}$ in the equation of motion of the scalar field

$$\ddot{\phi} = -3H\dot{\phi} - V'(\phi)$$

becomes negligible at late times.

➡ $3H\dot{\phi}/V' \to -1$ as $t \to \infty$

➡ $\ddot{\phi}/3H\dot{\phi} \to 0$ as $t \to \infty$

Surface symmetry

❑ Spacetime $\mathbb{R} \times S^1 \times F$ where F is a two dimensional compact manifold

❑ Areal coordinate :

$$ds^2 = -e^{-2\mu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + t^2(d\theta^2 + \sin^2_\kappa \theta d\varphi^2)$$

where $r \in [0,1]$, and the functions μ and λ are periodic in r with period 1

- spherical symmetry ($\kappa = 1$) : two dimensional sphere
  $\sin_\kappa \theta = \sin \theta$, $(\theta, \varphi) \in [0, \pi] \times [0, 2\pi]$
- hyperbolic symmetry ($\kappa = -1$) : hyperbolic space
  $\sin_\kappa \theta = \sinh \theta$, $(\theta, \varphi) \in [0, \infty) \times [0, 2\pi]$
- plane symmetry ($\kappa = 0$) : two dimensional torus (flat torus)
  $\sin_\kappa \theta = 1$, $(\theta, \varphi) \in [0, 2\pi] \times [0, 2\pi]$

Surface symmetry

❏ Areal coordinate :

$$ds^2 = -e^{-2\mu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + t^2(d\theta^2 + \sin^2_\kappa \theta d\varphi^2)$$

where $r \in [0, 1]$, and the functions μ and λ are periodic in r with period 1

❏ Λ : plane and hyperbolic symmetries [Tchapnda & Rendall (2003)][a]

- global existence of solutions / the area radius goes to infinity
- the spacetimes are future geodesically complete
- the expansion becomes isotropic and exponential at late time

[a] "Global existence and asymptotic behaviour in the future for the Einstein-Vlasov system with positive cosmological constant"

Surface symmetry

❑ Areal coordinate :

$$ds^2 = -e^{-2\mu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + t^2(d\theta^2 + \sin^2_\kappa \theta d\varphi^2)$$

where $r \in [0,1]$, and the functions μ and λ are periodic in r with period 1

❑ a linear scalar field

- [Tegankong, Noutchegueme, Rendall (2004)] :
  local existence of solutions, continuation criteria
- [Tegankong (2005)] :
  global existence,
  geodesic completeness (plane symmetry, w/o Vlasov)
- [Tegankong, Rendall (preprint)] :
  initial singularities