Global Classical Solutions to the 3D Nordström-Vlasov System (math-ph/0507030)

Simone Calogero

Departamento de Matemática para a Ciência e Tecnologia
Campus de Azurém da Universidade do Minho
4800-058 Guimarães, Portugal

E-mail: calogero@mct.uminho.pt

Mean Field Approximation Kinetic Models

- Large collection of particles (with unit mass and charge) in which collisions are sufficiently rare to be neglected

- Interaction is only through a mean field generated by the particles altogether

- $f(t, x, p) \equiv$ probability density to find a particle at time t in the position x with momentum p

- Applications: Plasma Physics (Electromagnetic Field), Galactic Dynamics (Gravitational Field)

Non-Relativistic:

Vlasov-Poisson system

$$\partial_t f + p \cdot \nabla_x f - \nabla_x U \cdot \nabla_p f = 0$$
$$\Delta_x U = 4\pi\gamma \int f \, dp$$

$\gamma = -1$: Electrostatic field (Plasma physics)
$\gamma = 1$: Newtonian gravity (Stellar dynamics)

Relativistic:

Vlasov-Maxwell system $\gamma = -1$

Einstein-Vlasov system $\gamma = 1$

Non-Relativistic:

Vlasov-Poisson system

$$\partial_t f + p \cdot \nabla_x f - \nabla_x U \cdot \nabla_p f = 0$$
$$\Delta_x U = 4\pi\gamma \int f \, dp$$

$\gamma = -1$: Electrostatic field (Plasma physics)

$\gamma = 1$: Newtonian gravity (Stellar dynamics)

Relativistic:

Vlasov-Maxwell system $\gamma = -1$

Einstein-Vlasov system $\gamma = 1$

Nordström-Vlasov system $\gamma = 1$

Main assumptions for the Nordström-Vlasov system

- The gravitational forces are mediated by a scalar field ϕ and the effect of such forces is to conformally rescale the metric of the (four dimensional) spacetime according to the relation
$$g = e^{2\phi}\eta, \qquad (1)$$
where η is the Minkowski metric.

- The particle density density is constant along the geodesic flow of the metric (1).

The Nordström-Vlasov system

Upon a conformal tranformation, the Nordström-Vlasov system can be put in the form

$$\partial_t^2\phi - \Delta_x\phi = -\mu, \quad \mu(t,x) = \int f(t,x,p)\,\frac{dp}{\sqrt{1+|p|^2}},$$
$$Sf - \Big[(S\phi)\,p + (1+|p|^2)^{-1/2}\nabla_x\phi\Big]\cdot\nabla_p f = 4f\,S\phi\,,$$

where

$$\widehat{p} = \frac{p}{\sqrt{1+|p|^2}}, \quad S = \partial_t + \widehat{p}\cdot\nabla_x.$$

Interpretation of the solution: *Given a solution* (f,ϕ)*, the space-time metric is* $g = e^{2\phi}\mathrm{diag}(-1,1,1,1)$ *and the physical particle density (i.e., the one which is defined on the mass shell of* g *and it is constant on the geodesic flow of* g*) is* $f_{\mathrm{ph}}(t,x,p) = f(t,x,e^{\phi}p)e^{-4\phi}$.

Vlasov-Maxwell system

$$\partial_t f + \widehat{p} \cdot \nabla_x f + (E + \widehat{p} \times B) \cdot \nabla_p f = 0,$$

$$\partial_t E - \nabla \times B = -j, \quad \partial_t B + \nabla \times E = 0,$$

$$\nabla \cdot E = \rho, \quad \nabla \cdot B = 0,$$

$$\rho(t,x) = \int f(t,x,p)\, dp, \quad j(t,x) = \int \widehat{p}\, f(t,x,p)\, dp.$$

Remark: Global existence of classical solutions to the Cauchy problem in 3 dimensions for the Vlasov-Maxwell system is an open problem.

Theorem 1 *Given* $f_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ *and* $\phi_0, \phi_1 : \mathbb{R}^3 \to \mathbb{R}$ *satisfying*

$$f_0 \in C_c^1,\ \ \phi_0 \in C_b^3 \cap H^1,\ \ \phi_1 \in C_b^2 \cap L^2,$$

there exists a unique global solution (f, ϕ) *of the Nordström-Vlasov system in the class*

$$(f, \phi) \in C^1([0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3) \times C^2([0, \infty) \times \mathbb{R}^3),$$

such that $(f, \phi)_{|t=0} = (f_0, \phi_0)$ *and* $(\partial_t \phi)_{|t=0} = \phi_1$.

Local existence and continuation criterium

Theorem 2 *(i) Given* $f_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to [0,\infty)$ *and* $\phi_0, \phi_1 : \mathbb{R}^3 \to \mathbb{R}$ *satisfying*

$$f_0 \in C^1_c,\ \phi_0 \in C^3_b,\ \phi_1 \in C^2_b,$$

there exists $T > 0$ *and a unique solution* (f,ϕ) *of the Nordström-Vlasov system in the class*

$$(f,\phi) \in C^1([0,T) \times \mathbb{R}^3 \times \mathbb{R}^3) \times C^2([0,T) \times \mathbb{R}^3),$$

such that $(f,\phi)_{|t=0} = (f_0,\phi_0)$ *and* $(\partial_t\phi)_{|t=0} = \phi_1$.

(ii) Let $T_{\max}$ *be the maximal time of existence and denote*

$$\mathcal{P}(t) = \sup_{0\leqslant s<t} \{e^{\phi}\sqrt{1+|p|^2} : f(s,x,p) \neq 0,\ \text{for some } x \in \mathbb{R}^3\}.$$

Then $\mathcal{P}(T_{\max}) < \infty \Rightarrow T_{\max} = \infty$, *i.e., the solution is global.*

Strategy of the proof

Consider the characteristics of the Vlasov equation,

$$\dot{x} = \widehat{p}, \quad \dot{p} = -(S\phi)\, p - (1 + |p|^2)^{-1/2} \nabla_x \phi.$$

We denote by $(X, P)(s)$ the (backward) characteristic curve satisfying $(X, P)(t) = (x, p)$. Then we use that, along characteristics,

$$\frac{d}{ds} e^{2\phi}(1 + |p|^2) = 2e^{2\phi} \partial_s \phi. \qquad (2)$$

The aim is to transform (2) in a Grönwall's type inequality by estimating $e^{2\phi}\partial_s\phi$ in terms of $\mathcal{P}(t)$. An estimate like

$$|e^{2\phi}\partial_t\phi| \leqslant C(t)\mathcal{P}(t)^2 \log \mathcal{P}(t)$$

would be enough, since it implies

$$\mathcal{P}(t)^2 \leqslant C + \int_0^t \mathcal{P}(s)^2 \log \mathcal{P}(s)\, ds$$

and so $\mathcal{P}(t) \leqslant C(t)$. However we are not able to obtain such a pointwise estimate for $\partial_t \phi$. Rather we have to use the integral version of (2), namely

$$e^{2\phi}(1 + |p|^2) = e^{2\phi_0(X(0))}(1 + |P(0)|^2) + 2\int_0^t e^{2\phi}\partial_s\phi(s, X(s))\, ds. \tag{3}$$

Eventually the quantity we shall estimate is the time integral in the right hand side of (3).

Conservation of energy

Local energy identity

$$\partial_t e + \nabla_x \cdot \mathfrak{p} = 0,$$

where

$$e(t,x) = \int \sqrt{1+|p|^2}\, f\, dp + \frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\nabla_x \phi)^2,$$

$$\mathfrak{p}(t,x) = \int p\, f\, dp - \partial_t \phi \nabla_x \phi.$$

Non-local energy identities

$$\int\int \sqrt{1+|p|^2}\, f\, dp\, dx + \int \left[\frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\nabla_x \phi)^2\right] dx = \text{const.},$$

$$\int_{|x-y|\leqslant t} (e+\mathfrak{p}\cdot\omega)(t-|x-y|, y)\, dy = \int_{|x-y|\leqslant t} e(0,y)\, dy, \qquad \omega = \frac{(y-x)}{|x-y|}.$$

Note also that

$$e+\mathfrak{p}\cdot\omega = \int(\sqrt{1+|p|^2}+\omega\cdot p)\, f\, dp + \frac{1}{2}(\omega\wedge\nabla_x\phi)^2 + \frac{1}{2}(\partial_t\phi - \omega\cdot\nabla_x\phi)^2.$$

Hence

$$\|\partial_t\phi(t)\|_{L^2(\mathbb{R}^3)} + \|\nabla_x\phi(t)\|_{L^2(\mathbb{R}^3)} \leqslant \text{const.},$$

$$\|(\omega\wedge\nabla_x\phi)(t,x)\|_{L^2(\Lambda_{t,x})} + \|(\partial_t\phi - \omega\cdot\nabla_x\phi)(t,x)\|_{L^2(\Lambda_{t,x})} \leqslant \text{const.},$$

where $\mathbb{R}^4 \supset \Lambda_{t,x} = \{(t-|x-y|, y) : |x-y| \leqslant t\}_{y\in\mathbb{R}^3}$ is the past light cone with vertex at (t,x) and base on $t=0$.

L^∞ estimates

Lemma 1

$$(1) \quad e^{\phi}(t) \leq C(t),$$

$$(2) \quad e^{\phi}|\phi| \leq C(t),$$

$$(3) \quad e^{-4\phi} f(t) \leq C.$$

Proof: Let $\phi = \phi_{\text{hom}} + \psi$, where ψ is the solution of $\Box\phi = -\mu$ with zero data and ϕ_{hom} solves $\Box\phi = 0$ with data (ϕ_0, ϕ_1). Since $\mu \geqslant 0$, then $\psi \leqslant 0$ and this implies (1) and (2). For (3) we observe that the Vlasov equation can be rewritten

$$\frac{d}{ds}\left[e^{-4\phi(s,X(s))} f(s, X(s), P(s))\right] = 0.$$

By Kirchhoff's formula,

$$\begin{aligned}\partial_t\phi &= \partial_t\phi_{\text{hom}} - t^{-1}\int_{|x-y|=t}\int f^{\text{in}}(y,p)\frac{dp}{\sqrt{1+|p|^2}}dS_y \\ &\quad - \int_{|x-y|\le t}\int \partial_t f(t-|x-y|,y,p)\frac{dp}{\sqrt{1+|p|^2}}\frac{dy}{|x-y|}.\end{aligned}$$

In the last integral we split $\partial_t f$ in a component along the free transport operator and a component along the light cone

$$\begin{aligned}(\partial_t f)(t-|x-y|,y,p) = (1+\omega\cdot\widehat{p})^{-1}\Big[\; &(Sf)(t-|x-y|,y,p) \\ &-\widehat{p}\cdot\nabla_y[f(t-|x-y|,y,p)]\Big].\end{aligned}$$

By Vlasov, $Sf = \Big[(S\phi)\,p + (1+|p|^2)^{-1/2}\nabla_x\phi\Big]\cdot\nabla_p f - 4f\,S\phi$. Substituting, integrating by parts in y and p and rearranging the terms proves the following

The representation formula for $\partial_t\phi$

Proposition 1 *The function $\partial_t\phi$ satisfies the identity*

$$\partial_t\phi(t,x) = (\partial_t\phi)_D + \sum_{i=0}^{5} \mathcal{Z}_i,$$

where

$$(\partial_t\phi)_D = \partial_t\phi_{\mathrm{hom}} - \frac{1}{t}\int_{|x-y|=t}\int \frac{f_0(y,p)}{(1+\omega\cdot\widehat{p})}\frac{dp}{\sqrt{1+|p|^2}}\,dS_y,$$

$$\mathcal{Z}_0 = -2\int_{|x-y|\leqslant t}(\partial_t\phi)\mu(t-|x-y|,y)\,\frac{dy}{|x-y|},$$

$$\mathcal{Z}_1 = \int_{|x-y|\leqslant t}\int \frac{f(t-|x-y|,y,p)}{(\sqrt{1+|p|^2}+\omega\cdot p)}\,dp\,\frac{dy}{|x-y|^2},$$

$$\mathcal{Z}_2 = -\int_{|x-y|\leqslant t}\int \frac{f(t-|x-y|,y,p)}{\sqrt{1+|p|^2}(\sqrt{1+|p|^2}+\omega\cdot p)^2}\,dp\,\frac{dy}{|x-y|^2},$$

$$\mathcal{Z}_3 = 2\int_{|x-y|\leqslant t}(\partial_t\phi-\omega\cdot\nabla_x\phi)\int \frac{(\omega\cdot\widehat{p})f(t-|x-y|,y,p)}{(\sqrt{1+|p|^2}+\omega\cdot p)}\,dp\,\frac{dy}{|x-y|},$$

$$\mathcal{Z}_4 = \int_{|x-y|\leqslant t}(\partial_t\phi-\omega\cdot\nabla_x\phi)\int \frac{f(t-|x-y|,y,p)}{\sqrt{1+|p|^2}(\sqrt{1+|p|^2}+\omega\cdot p)^2}\,dp\,\frac{dy}{|x-y|},$$

$$\mathcal{Z}_5 = -2\int_{|x-y|\leqslant t}(\omega\wedge\nabla_x\phi)\cdot\int \frac{(\omega\wedge\widehat{p})f(t-|x-y|,y,p)}{(\sqrt{1+|p|^2}+\omega\cdot p)}\,dp\,\frac{dy}{|x-y|}.$$

Lemma 2 *For* $R > 1$, $a, b \geqslant 0$, *denote*

$$\mathcal{B}_{a,b}(R) = \int_{|p| \leqslant R} (\sqrt{1 + |p|^2} + \omega \cdot p)^{-a} (1 + |p|^2)^{-b} dp.$$

Then

$$\mathcal{B}_{a,b}(R) \leqslant CR^{(2-2b)} \log R, \quad \text{if } a = 1,\ b < 1;$$
$$\mathcal{B}_{a,b}(R) \leqslant CR^{(3-2b-a)}, \quad \text{if } a < 1,\ b < \frac{3-a}{2};$$
$$\mathcal{B}_{a,b}(R) \leqslant CR^{(1+a-2b)}, \quad \text{if } a > 1,\ b < \frac{1+a}{2}.$$

Proof: The proof is by evaluating the integral in polar coordinates.

Let's go back to our main goal, which is to transform the identity

$$e^{2\phi}(1+|p|^2) = e^{2\phi_0(X(0))}(1+|P(0)|^2) + 2\int_0^t e^{2\phi}\partial_s\phi(s,X(s))\,ds.$$

in a Grönwall inequality. For this purpose we first estimate $\partial_t\phi$.

Proposition 2

$$|\partial_t\phi(t,x)| \leqslant 2\int_{|x-y|\leqslant t} |\partial_t\phi|\mu(t-|x-y|,y)\,\frac{dy}{|x-y|} + C(t)\mathcal{P}(t)^2\log\mathcal{P}(t).$$

Proof: From the integral representation formula we have

$$|\partial_t\phi(t,x)| \leqslant C(t) + 2\int_{|x-y|\leqslant t} |\partial_t\phi|\mu(t-|x-y|,y)\frac{dy}{|x-y|} + \sum_{i=1}^{5}|\mathcal{Z}_i|.$$

Let us show the estimate for $\mathcal{Z}_1$ and $\mathcal{Z}_3$:

$$\mathcal{Z}_1 = \int_{|x-y|\leqslant t} \int \frac{f(t-|x-y|, y, p)}{(\sqrt{1+|p|^2} + \omega \cdot p)} \, dp \, \frac{dy}{|x-y|^2},$$

$$\mathcal{Z}_3 = 2 \int_{|x-y|\leqslant t} (\partial_t \phi - \omega \cdot \nabla_x \phi) \int \frac{(\omega \cdot \widehat{p}) f(t-|x-y|, y, p)}{(\sqrt{1+|p|^2} + \omega \cdot p)} \, dp \, \frac{dy}{|x-y|},$$

The domain of integration in the variable p can be chosen as $\{|p| \leqslant e^{-\phi} \mathcal{P}(t)\}$ by the definition of $\mathcal{P}(t)$.

Estimate for $\mathcal{Z}_1$: By the L^∞ estimates and the calculus Lemma,

$$
\begin{aligned}
|\mathcal{Z}_1| &= \int_{|x-y|\leqslant t}\int_{|p|\leqslant e^{-\phi}\mathcal{P}(t)} \frac{f(t-|x-y|,y,p)}{(\sqrt{1+|p|^2}+\omega\cdot p)}\,dp\,\frac{dy}{|x-y|^2} \\
&= \int_{|x-y|\leqslant t}\int_{|p|\leqslant e^{-\phi}\mathcal{P}(t)} \frac{e^{4\phi}e^{-4\phi}f(t-|x-y|,y,p)}{(\sqrt{1+|p|^2}+\omega\cdot p)}\,dp\,\frac{dy}{|x-y|^2} \\
&\leqslant C\int_{|x-y|\leqslant t} e^{4\phi}\mathcal{B}_{1,0}\left(e^{-\phi}\mathcal{P}(t)\right)(t-|x-y|,y)\frac{dy}{|x-y|^2} \\
&\leqslant C(t)\mathcal{P}(t)^2\int_{|x-y|\leqslant t} e^{2\phi}\log\left(e^{-\phi}\mathcal{P}(t)\right)(t-|x-y|,y)\frac{dy}{|x-y|^2} \\
&\leqslant C(t)\mathcal{P}(t)^2\int_{|x-y|\leqslant t}\left(e^{2\phi}|\phi|(t-|x-y|,y)+e^{2\phi}\log\mathcal{P}(t)\right)\frac{dy}{|x-y|^2} \\
&\leqslant C(t)\mathcal{P}(t)^2\log\mathcal{P}(t).
\end{aligned}
$$

Estimate for $\mathcal{Z}_3$: By the Cauchy-Schwarz inequality,

$$\begin{aligned}
|\mathcal{Z}_3| &= \left| 2\int_{|x-y|\leqslant t} (\partial_t\phi - \omega\cdot\nabla_x\phi) \int \frac{(\omega\cdot\widehat{p}) f(t-|x-y|,y,p)}{(\sqrt{1+|p|^2}+\omega\cdot p)}\,dp\,\frac{dy}{|x-y|} \right| \\
&\leqslant 2\int_{|x-y|\leqslant t} |\partial_t\phi - \omega\cdot\nabla_x\phi| \int \frac{f(t-|x-y|,y,p)}{(\sqrt{1+|p|^2}+\omega\cdot p)}\,dp\,\frac{dy}{|x-y|} \\
&\leqslant C\int_{|x-y|\leqslant t} |\partial_t\phi - \omega\cdot\nabla_x\phi| e^{4\phi} \mathcal{B}_{1,0}\left(e^{-\phi}\mathcal{P}(t)\right)(t-|x-y|,y)\frac{dy}{|x-y|} \\
&\leqslant C\left(\int_{|x-y|\leqslant t} |\partial_t\phi - \omega\cdot\nabla_x\phi|^2 (t-|x-y|,y)\right)^{1/2} \\
&\quad \times \left(\int_{|x-y|\leqslant t} e^{8\phi}\left[\mathcal{B}_{1,0}\left(e^{-\phi}\mathcal{P}(t)\right)\right]^2 (t-|x-y|,y)\frac{dy}{|x-y|^2}\right)^{1/2} \\
&\leqslant C(t)\mathcal{P}(t)^2 \log\mathcal{P}(t).
\end{aligned}$$

Recall the integral equation

$$e^{2\phi}(1+|p|^2) = e^{2\phi_0(X(0))}(1+|P(0)|^2) + 2\int_0^t e^{2\phi}\partial_s\phi(s,X(s))\,ds.$$

Using the estimate just proved, i.e.,

$$|\partial_t\phi(t,x)| \leqslant 2\int_{|x-y|\leqslant t} |\partial_t\phi|\mu(t-|x-y|,y)\,\frac{dy}{|x-y|} + C(t)\mathcal{P}(t)^2\log\mathcal{P}(t).$$

we obtain

$$\begin{aligned} e^{2\phi}(1+|p|^2) &\leqslant C + 2\int_0^t e^{2\phi}|\partial_s\phi(s,X(s))|\,ds \\ &\leqslant C + C(t)\int_0^t \mathcal{P}(s)^2\log\mathcal{P}(s)\,ds + 4I_0(|\partial_t\phi|\mu,t), \end{aligned}$$

where

$$I_0(\partial_t\phi\mu,t) = \int_0^t e^{2\phi(s,X(s))}\int_{|X(s)-y|\leqslant s}\partial_t\phi\mu(s-|X(s)-y|,y)\,\frac{dy}{|X(s)-y|}\,ds.$$

We rewrite $I_0(g,t)$ as

$$I_0(g,t) = \int_0^t \mathcal{I}_0(g,\tau,t)\,d\tau,$$

where

$$\mathcal{I}_0(g,\tau,t) = \int_\tau^t e^{2\phi(s,X(s))} \int_{|y|=s-\tau} \frac{g(\tau, X(s)-y)}{(s-\tau)}\,dS_y\,ds.$$

We estimate $\mathcal{I}_0$ using a method due to Pallard*

Lemma 3 *For all* $0 \leqslant \tau \leqslant t$*,*

$$\mathcal{I}_0(g,\tau,t) \leqslant C(t)\frac{\|g(\tau)\|_{L^2}}{\sqrt{t-\tau}} \int_\tau^t \log \mathcal{P}(s)\,ds.$$

*C. Pallard: *On the boundness of the momentum support of solutions to the relativistic Vlasov-Maxwell system*. Indiana Univ. Math. J. (to appear)

Proof: We first rewrite $\mathcal{I}_0$ in spherical coordinates:

$$\mathcal{I}_0(g,\tau,t) = \int_\tau^t e^{2\phi(s,X(s))} \int_0^\pi \int_0^{2\pi} g(\tau, X(s) - (s-\tau)\omega)(s-\tau) \sin\theta \, d\varphi \, d\theta \, ds.$$

Now, Pallard showed that the transformation of variables $(s,\theta,\varphi) \to X(s) - (s-\tau)\omega$ is a C^1 diffeomorphism with Jacobian

$$J = (\dot{X}(s)\cdot\omega - 1)(s-\tau)^2 \sin\theta = (\widehat{P}(s)\cdot\omega - 1)(s-\tau)^2 \sin\theta.$$

Hence, applying Cauchy-Schwarz's inequality,

$$\begin{aligned} \mathcal{I}_0(g,\tau,t) \ &\leqslant \ \left(\int_\tau^t \int_0^\pi \int_0^{2\pi} g^2(\tau, X(s) - (s-\tau)\omega) |J| \, d\varphi \, d\theta \, ds \right)^{1/2} \\ &\quad \times \left(\int_\tau^t e^{4\phi(s,X(s))} \int_0^\pi \int_0^{2\pi} \frac{\sin\theta}{(1 - \widehat{P}(s)\cdot\omega)} \, d\varphi \, d\theta \, ds \right)^{1/2} \\ &\leqslant \ \|g(\tau)\|_{L^2} \left(\int_\tau^t e^{4\phi(s,X(s))} \int_0^\pi \int_0^{2\pi} \frac{\sin\theta}{(1 - \widehat{P}(s)\cdot\omega)} \, d\varphi \, d\theta \, ds \right)^{1/2}. \end{aligned}$$

We estimate the angular integral as

$$\begin{aligned}\int_0^{\pi}\int_0^{2\pi}\frac{\sin\theta}{(1-\widehat{P}(s)\cdot\omega)}\,d\varphi\,d\theta\,ds &= 2\pi\int_{-1}^{1}\frac{du}{(1-|\widehat{P}(s)|u)}\\ &\leqslant C\left(1-\log(1-|\widehat{P}(s)|)\right)\\ &\leqslant C\left(|\phi|+\log\mathcal{P}(s)\right).\end{aligned}$$

We finally obtain

$$\begin{aligned}\mathcal{I}_0(g,\tau,t) &\leqslant C(t)\|g(\tau)\|_{L^2}\left(\int_\tau^t\left(e^{\phi}|\phi|+\log\mathcal{P}(s)\right)ds\right)^{1/2}\\ &\leqslant C(t)\frac{\|g(\tau)\|_{L^2}}{\sqrt{t-\tau}}\int_\tau^t\log\mathcal{P}(s)\,ds,\end{aligned}$$

which concludes the proof of the lemma. □

The proof of Theorem 1 is now almost complete. Recall that $g = \partial_t \phi \mu$ and so

$$\|\partial_t \phi \mu(\tau)\|_{L^2} \leqslant \|\mu(\tau)\|_\infty \|\partial_t \phi(\tau)\|_{L^2} \leqslant C(t)\mathcal{P}(\tau)^2, \qquad \tau \leqslant t.$$

Thus

$$\mathcal{I}_0(|\partial_t \phi|\mu, \tau, t) \leqslant C(t)\frac{\mathcal{P}(\tau)^2}{\sqrt{t-\tau}} \int_\tau^t \log \mathcal{P}(s)\, ds.$$

Recall that

$$e^{2\phi}(1+|p|^2) \leqslant C + C(t)\int_0^t \mathcal{P}(s)^2 \log \mathcal{P}(s)\, ds + 4 I_0(|\partial_t \phi|\mu, t),$$

where

$$I_0(\partial_t \phi \mu, t) = \int_0^t \mathcal{I}_0(\partial_t \phi \mu, \tau, t)\, d\tau.$$

Now I_0 is bounded by

$$\begin{aligned} I_0(|\partial_t\phi|\mu, t) &\leqslant C(t)\int_0^t\int_\tau^t \frac{\mathcal{P}(\tau)^2}{\sqrt{t-\tau}}\log\mathcal{P}(s)\,ds\,d\tau \\ &= C(t)\int_0^t\int_0^s \frac{\mathcal{P}(\tau)^2}{\sqrt{t-\tau}}\log\mathcal{P}(s)\,d\tau\,ds \\ &\leqslant C(t)\int_0^t \mathcal{P}(s)^2\log\mathcal{P}(s)\,ds. \end{aligned}$$

Finally, going back to

$$e^{2\phi}(1+|p|^2) \leqslant C + C(t)\int_0^t \mathcal{P}(s)^2\log\mathcal{P}(s)\,ds + 4I_0(|\partial_t\phi|\mu, t)$$

we obtain the Grönwall inequality

$$\mathcal{P}(t)^2 \leqslant C(t)\left(1+\int_0^t \mathcal{P}(s)^2\log\mathcal{P}(s)\,dt\right),$$

whence $\mathcal{P}(t) \leqslant C(t)$. By the continuation criterium this concludes the proof of the main theorem.