Linear equivalence and ODE-equivalence for coupled cell networks

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Theory and Applications of Coupled Cell Networks

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Coupled cell networks

Stewart, Golubitsky, Pivato, and Török (2003, 2005) formalize the concept of a coupled cell network as a schematic representation of a coupled cell system:

- The *architecture* of a coupled cell network is defined in terms of a *graph* that indicates which cells have the same phase space, which cells are coupled to which, and which couplings are the same.

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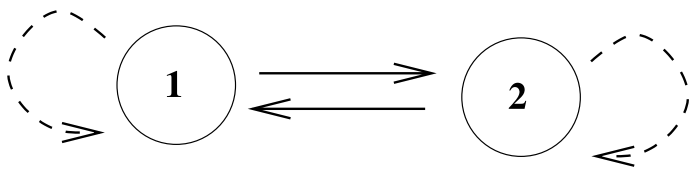
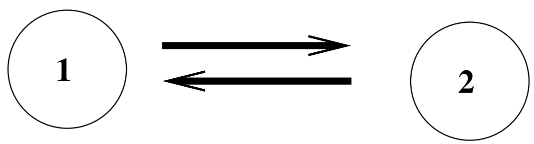
Examples





Coupled cell networks

- Associated with each network is a class of differential equations on the total phase space P of the coupled cell system, which correspond to *admissible vector fields* on P: the ODEs that are compatible with the network architecture and the choice of cell phase spaces.

Coupled cell networks

The architecture of the network imposes constraints on the dynamics – many new phenomena become ‘generic’ for a given architecture (Golubitsky, Nicol and Stewart, 2004).

Important to understand how network topology contrains the associated dynamics.

The combinatorial complexity is huge:
Aldosray and Stewart (2004) enumerate identical-edge homogeneous networks;
Leite (2005) studies three identical-edge homogeneous networks.

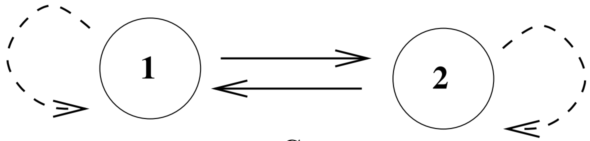
Topologically distinct networks have related dynamics.

Coupled cell networks

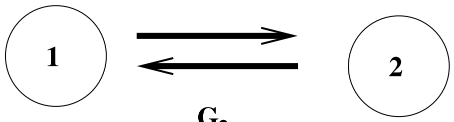
- Associated with each network is a class of differential equations on the total phase space P of the coupled cell system, which correspond to *admissible vector fields* on P.

- Non-isomorphic networks can correspond to the same space of admissible vector fields (for a suitable choice of cell phase spaces).

Example

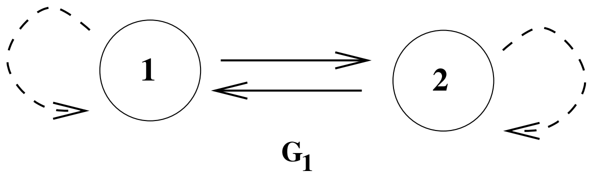


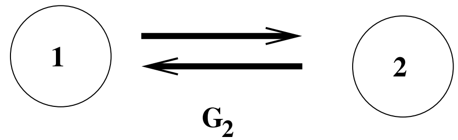
G_1



G_2

Example





Admissible vector fields for G_1:

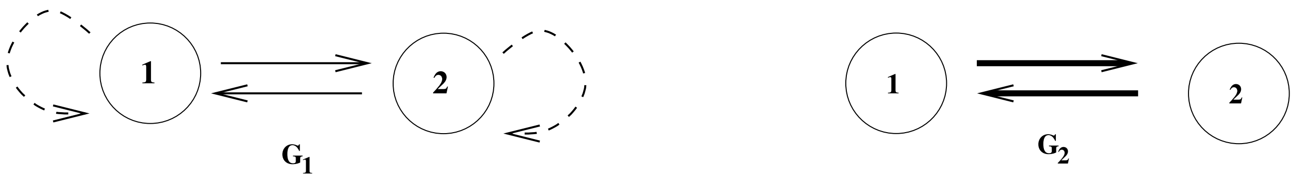
$H(x_1, x_2) = (h(x_1, x_1, x_2), h(x_2, x_2, x_1))$

Admissible vector fields for G_2:

$F(x_1, x_2) = (f(x_1, x_2), f(x_2, x_1))$

$h: \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k, \quad f: \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ are any smooth functions.

Example



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The set $\{H\}$ of all H is the same as the set $\{F\}$ of all F:

Given f we can set $h(x, y, z) = f(x, z)$

Given h we can set $f(a, b) = h(a, a, b)$

Groupoid equivariance

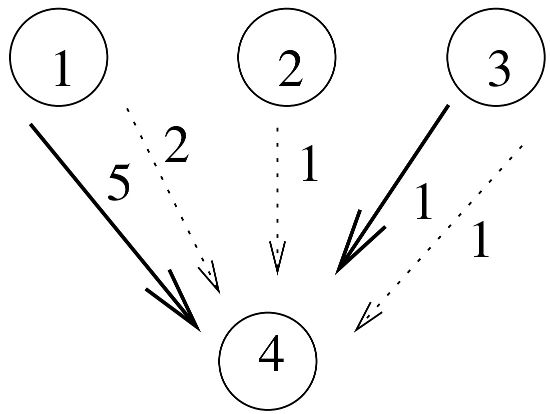
The admissible vector fields can be characterised in terms of an algebraic structure known as the *symmetry groupoid* of the network:

- The *input set* $I(c)$ of a cell c consists of all cells coupled to c.
- Two input sets are *isomorphic* if there is a bijection between the input sets that preserves coupling types.
- *The symmetry groupoid* consists of all such isomorphisms between pairs of cells.

Groupoid equivariance

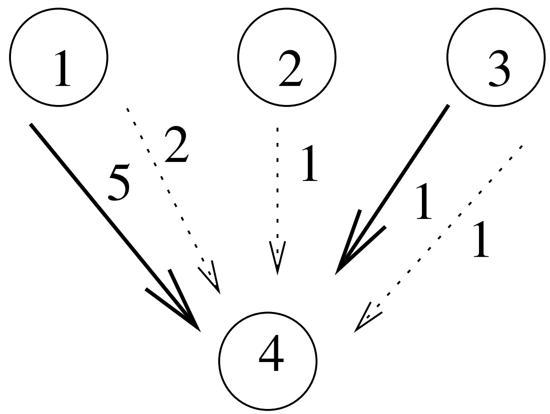
The admissible vector fields are those that are equivariant under a natural action of the groupoid on P, in a sense that generalizes the usual notion of equivariance under the action of a group.

Example



The number k attached to the right of each edge symbolizes k edges of that type.

Example



Admissible vector fields:

$$F(x_1, x_2, x_3, x_4) = \left(f(x_1), f(x_2), f(x_3), g\left(x_4, \overline{x_1^{(5)}, x_3}, \overline{x_1^{(2)}, x_2, x_3} \right) \right)$$

$f : \mathbb{R}^k \to \mathbb{R}^k, \quad g : (\mathbb{R}^k)^{11} \to \mathbb{R}^k$ are any smooth functions.

ODE-equivalence

Non-isomorphic networks giving rise to the same space of admissible vector fields (for a suitable choice of cell phase spaces) are said to be ODE-equivalent.

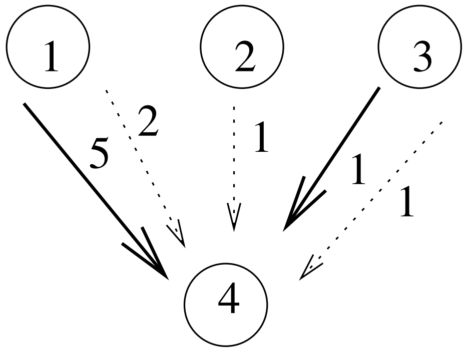
This phenomenon raises the prospect of simplifying the network structure while preserving some (in this case, all) of its dynamical features.

Linear equivalence

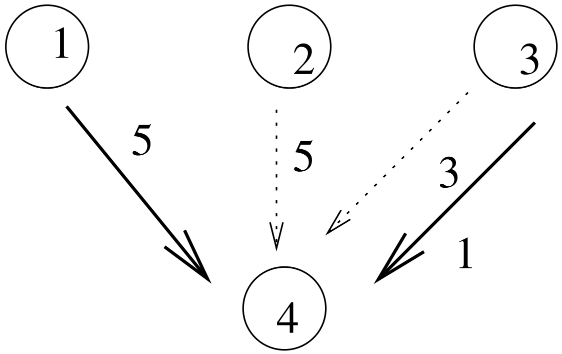
We make a start on this question: we determine necessary and sufficient conditions for networks to be ODE-equivalent. We prove:

- ODE-equivalence reduces to *linear equivalence*, where two networks (with suitably identified phase spaces) are linearly equivalent if they determine the same space of *linear* admissible vector fields.
- When deciding linear equivalence, it can without loss of generality be assumed that each cell phase space is 1-dimensional.

Example

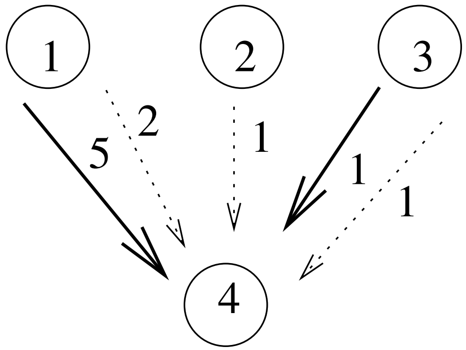


$\mathbf{G_1}$



$\mathbf{G_2}$

Example



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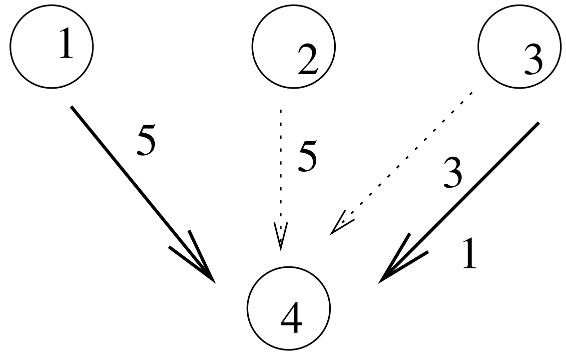

$\mathbf{G_2}$

Choose all cell phase spaces to be $P_c = \mathbb{R}$.

Linear G_1-admissible vector fields from $\mathbb{R}^4$ to $\mathbb{R}^4$ have the form:

$$(aY_1, aY_2, aY_3, bY_4 + c(5Y_1 + Y_3) + d(2Y_1 + Y_2 + Y_3))$$

where a, b, c, d are real constants.

Example

Linear G_2-admissible vector fields from $\mathbb{R}^4$ to $\mathbb{R}^4$ have the form:

$$(eY_1, eY_2, eY_3, hY_4 + j(5Y_1 + Y_3) + l(5Y_2 + 3Y_3))$$

where e, h, j, l are real constants.

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$$\mathbb{R}\{Y_4,\ 5Y_1 + Y_3,\ 2Y_1 + Y_2 + Y_3\} =$$
$$\mathbb{R}\{Y_4,\ 5Y_1 + Y_3,\ 5Y_2 + 3Y_3\}$$

The coupled cell networks G_1 and G_2 are ODE-equivalent.

Linear equivalence and minimality

Our results classify ODE-equivalence in terms of a linear algebra invariant; the characterization of linearly equivalent networks reduces to a combinatorial condition in linear algebra.

Aguiar and D. (2005) characterize the minimal networks for ODE-classes.

Formal definitions

Definition (Golubitsky *et al.*, 2005) A *coupled cell network* G consists of:

(a) A finite set $\mathcal{C} = \{1, \ldots, n\}$ of *nodes* (or *cells*).
An equivalence relation $\sim_C$ on the nodes in $\mathcal{C}$.
The *type* or *cell label* of cell c is the $\sim_C$-equivalence class $[c]_C$ of c.

(b) A finite set $\mathcal{E}$ of *edges* or *arrows*.
An equivalence relation $\sim_E$ on the edges in $\mathcal{E}$.
The *type* or *coupling label* of edge e is the $\sim_E$-equivalence class $[e]_E$ of e.

(c) Two maps $\mathcal{H} : \mathcal{E} \to \mathcal{C}$ and $\mathcal{T} : \mathcal{E} \to \mathcal{C}$.
For $e \in \mathcal{E}$ we call $\mathcal{H}(e)$ the *head* of e and $\mathcal{T}(e)$ the *tail* of e.
Equivalent edges have equivalent tails and heads.

$$\boxed{G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)}$$

Formal definitions

Definition (Golubitsky *et al.*, 2005)

(a) The *input set* of $c \in \mathcal{C}$ is

$$I(c) = \{e \in \mathcal{E} : \mathcal{H}(e) = c\}$$

(b) The relation $\sim_I$ of *input-equivalence* on $\mathcal{C}$ is defined by $c \sim_I d$ if and only if there exists a bijection $\beta : I(c) \to I(d)$ that preserves edge type: for every input edge $i \in I(c)$

$$i \sim_E \beta(i)$$

The set of all such bijections from cell c to cell d is $B(c, d)$.

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When $c = d$ we have

$$B(c, c) = \mathbf{S}_{K_1} \times \cdots \times \mathbf{S}_{K_{r(c)}}$$

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$\mathcal{B}_G = \dot{\bigcup}_{c,d \in \mathcal{C}} B(c, d)$ *symmetry groupoid of the network* G

Formal definitions

To each cell $c \in \mathcal{C}$ we associate a *cell phase space* P_c (a nonzero finite-dimensional real vector space). Assume

$$[c]_C = [d]_C \Longrightarrow P_c = P_d$$

The *total phase space* is

$$P = \prod_{c \in \mathcal{C}} P_c$$

with coordinate system

$$x = (x_c)_{c \in \mathcal{C}}$$

on P.

Formal definitions

Definition (Golubitsky *et al.*, 2005) Let G and $\mathcal{B}_G$ the symmetry groupoid. For a given choice of the P_c, a (smooth) vector field $f: P \to P$ is *$\mathcal{B}_G$-equivariant* if:

(a) *Domain Condition*: For any $c \in \mathcal{C}$ the component $f_c(x)$ depends only on the internal phase space variable x_c and the coupling phase space variables $x_{\mathcal{T}(I(c))}$:

$$\boxed{f_c(x) = \hat{f}_c\left(x_c, x_{\mathcal{T}(I(c))}\right)}$$

for some smooth function $\hat{f}_c: P_c \times P_{\mathcal{T}(I(c))} \to P_c$.

(b) *Pullback Condition*: For all $c, d \in \mathcal{C}$ and $\beta \in B(c, d)$

$$\boxed{\hat{f}_d\left(x_d, x_{\mathcal{T}(I(d))}\right) = \hat{f}_c\left(x_d, \beta^* \left(x_{\mathcal{T}(I(d))}\right)\right)}$$

for all $x \in P$.

Notation

Let G be a coupled cell network. For a given choice of the phase space P, define:

$\mathcal{F}_G^P$ the real vector space of all smooth G-admissible vector fields on P.

$\mathcal{P}_G^P$ the real vector space of the G-admissible polynomial vector fields on P.

$\mathcal{L}_G^P$ the real vector space of the G-admissible linear vector fields on P.

ODE-equivalence

Given a coupled cell network G_i and a choice of total phase space P_i for G_i, denote by $P_{i,c}$ the cell phase space corresponding to cell c of $\mathcal{C}_i$.

ODE-equivalence

Definition Two coupled cell networks G_1 and G_2 are *ODE-equivalent* if there is some bijection $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$ such that:

1. γ preserves cell-equivalence and input-equivalence.
2. If we choose cell phase spaces $P_c \neq 0$ for G_1, and define the corresponding choice of cell phase spaces for G_2 by $\boxed{P_{2,\gamma(c)} = P_{1,c}}$ so that the corresponding total phase spaces are

$$P_1 = \prod_{c \in \mathcal{C}_1} P_{1,c} \qquad P_2 = \prod_{c \in \mathcal{C}_1} P_{2,\gamma(c)}$$

then:

3. The following condition is satisfied

$$\boxed{\mathcal{F}_{G_1}^{P_1} = \mathcal{F}_{G_2}^{P_2}}$$

Linear equivalence

Definition Two coupled cell networks G_1 and G_2 are *linearly equivalent* if there is some bijection $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$ such that:

1. γ preserves cell-equivalence and input-equivalence.
2. If we choose cell phase spaces $P_c \neq 0$ for G_1, and define the corresponding choice of cell phase spaces for G_2 by $\boxed{P_{2,\gamma(c)} = P_{1,c}}$ so that the corresponding total phase spaces are

$$P_1 = \prod_{c \in \mathcal{C}_1} P_{1,c} \qquad P_2 = \prod_{c \in \mathcal{C}_1} P_{2,\gamma(c)}$$

then:

3. The following condition is satisfied

$$\boxed{\mathcal{L}_{G_1}^{P_1} = \mathcal{L}_{G_2}^{P_2}}$$

Results

Theorem Two coupled cell networks G_1 and G_2 are ODE-equivalent if and only if they are linearly equivalent.

Corollary The following conditions on two networks G_1, G_2 are equivalent:

(a) G_1 and G_2 are γ-linearly equivalent.

(b) With the identification $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$, the spaces $\mathcal{L}^P_{G_1}$ and $\mathcal{L}^P_{G_2}$ are equal for all P.

(c) With the identification $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$, the spaces $\mathcal{L}^P_{G_1}$ and $\mathcal{L}^P_{G_2}$ are equal when all cell phase spaces are taken to be .

Proof

Given two coupled cell networks G_1 and G_2 and a bijection $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$ preserving cell-equivalence and input-equivalence, together with a choice of total phase space P_1 for G_1 and P_2 for G_2 according to the above definitions, prove that:

- $\mathcal{F}_{G_1}^{P_1} = \mathcal{F}_{G_2}^{P_2} \iff \mathcal{P}_{G_1}^{P_1} = \mathcal{P}_{G_2}^{P_2}$
- $\mathcal{P}_{G_1}^{P_1} = \mathcal{P}_{G_2}^{P_2} \iff \mathcal{L}_{G_1}^{P_1} = \mathcal{L}_{G_2}^{P_2}$

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