Curvature Integrals and the Gauss-Bonnet Theorem on Poincaré-Einstein spaces

Pierre Albin

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Poincaré-Einstein metrics

(M, g) is Poincaré-Einstein if

- M is the interior of a compact manifold with bdy, $\overline{M}$
- g is Einstein
- $\bar{g} = x^2 g$ extends to a metric on $\overline{M}$

Poincaré-Einstein metrics

- Conformal Geometry

  g determines a conformal class of metrics on ∂M

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- Physics

  AdS/CFT correspondence
  Requires a *renormalized* volume

Renormalized Volume

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Renormalized Volume

If x is a *special* bdf, and ...

- if M is even dimensional

$$\bar{g} = dx^2 + \bar{g}^{(0)} + x^2 \bar{g}^{(2)} + ..(\text{even powers}).. + x^{m-1} \bar{g}^{(m-1)} + \dots$$

- if M is odd dimensional

$$\bar{g} = dx^2 + \bar{g}^{(0)} + x^2 \bar{g}^{(2)} + ..(\text{even powers})..$$
$$+ x^{m-1} \bar{g}^{(m-1)} + x^{m-1} (\log x)\, \bar{g}^{(m-1,1)} + \dots$$

Renormalized Volume

Thus the volume form

$$\mathsf{dvol}_g = \left(\frac{\det G_x}{\det \bar{g}^{(0)}} \right)^{1/2} \frac{\mathsf{dvol}_{\bar{g}(0)} dx}{x^m},$$

Renormalized Volume

Thus the volume form

$$\mathsf{dvol}_g = \left(\frac{\det G_x}{\det \bar{g}^{(0)}} \right)^{1/2} \frac{\mathsf{dvol}_{\bar{g}(0)} dx}{x^m},$$

has an expansion

$$\left(\frac{\det G_x}{\det \bar{g}^{(0)}} \right)^{1/2} = 1 + v^{(2)} x^2 + \ldots \text{(even powers)} \ldots$$

$$+ v^{(m-1)} x^{(m-1)} + O(x^m)$$

Renormalized Volume

$$ {}^{R}\mathrm{Vol} := \underset{\varepsilon=0}{\mathsf{FP}} \int_{x \geq \varepsilon} \mathsf{dvol}_{g} $$

Renormalized Volume

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$$\zeta_x(z) = \int_M x^z \, \mathrm{dvol}_g$$

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$$\zeta_{\hat{x}}(z) - \zeta_x(z) = \int_M \left(e^{z\omega} - 1\right) x^z \,\mathrm{dvol}_g$$

$$= z \int_M \left(\frac{e^{z\omega} - 1}{z}\right) x^z \,\mathrm{dvol}_g$$

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$$\xrightarrow[z=0]{FP} \int_{\partial M} \left(\omega \, \mathrm{dvol}_g\right)_{(-1)}$$

Curvature Integrals

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The proof consists in a careful analysis of the expansions of the curvature and its covariant derivatives.

Gauss-Bonnet Theorem

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Theorem 2. *On an even dimensional PE manifold,*

$$ {}^{R}\!\!\int_M \mathrm{Pff} = \chi(M). $$

Gauss-Bonnet Theorem

Proof. For any special bdf, x, consider the Chern-Gauss-Bonnet theorem on the manifold with boundary, $M_\varepsilon := \{x \geq \varepsilon\}$

$$\int_{M_\varepsilon} \mathrm{Pff} + \int_{x=\varepsilon} \Gemini = \chi(M_\varepsilon) = \chi(M)$$

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Previously known in four dimensions, [Anderson]

$$\frac{1}{8(2\pi)^2}\int |W|^2 + \frac{3}{(2\pi)^2}{}^R\mathrm{Vol} = \chi(M),$$

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$$\frac{1}{8(2\pi)^2}\int |W|^2 + \frac{3}{(2\pi)^2}{}^R\mathrm{Vol} = \chi(M),$$

and for hyperbolic manifolds, [Epstein]

$$\frac{(-1)^{m/2}}{2^{m/2}(2\pi)^{m/2}}\frac{m!}{(m/2)!}{}^R\mathrm{Vol} = \chi(M).$$

Gauss-Bonnet Theorem

There is also a recent result of Chang-Qing-Yang,

$$\int \widetilde{W} + (-1)^{\frac{m}{2}} \frac{\Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m+1}{2}}} {}^{R}\mathrm{Vol} = \chi\left(M\right).$$

Gauss-Bonnet Theorem

Lemma 3. *On an m-dimensional Einstein manifold, the Pfaffian is given by*

$$\frac{1}{(2\pi)^{m/2}} \sum_{k=0}^{m/2} \frac{(2k)!}{k!} \left(\frac{s_g}{2m(m-1)} \right)^k W_{m-2k}\left(\mathfrak{W}\right) \, \mathit{dvol}$$

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Here $W_\ell(\mathfrak{W})$ is the $\ell^{\rm th}$ Weyl volume of tube invariant (aka Lipschitz-Killing curvature) evaluated at the Weyl curvature.

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On a closed manifold,

$$\Omega^{even} \xrightarrow{d+\delta} \Omega^{odd}$$

is an elliptic operator whose index is the Euler characteristic.

Gauss-Bonnet as an Index Theorem

On a PE manifold, [Mazzeo]

$$\mathcal{H}^k_{L^2}(M) = \begin{cases} H^k(M, \partial M) & \text{if } k < \frac{m}{2} \\ \text{infinite dimensional} & \text{if } k = \frac{m}{2} \\ H^k(M) & \text{if } k > \frac{m}{2} \end{cases}$$

In every case, there is a spectral gap at zero.

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In every case, there is a spectral gap at zero. Furthermore, the structure of the projector onto harmonic $\frac{m}{2}$ forms allows its renormalized trace to be defined.

Gauss-Bonnet as an Index Theorem

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- As $t \longrightarrow \infty$, the heat kernel converges to the projection onto the kernel of the Laplacian.

- As $t \longrightarrow 0$, the local index theorem shows that the (local) supertrace of the heat kernel converges to the Pfaffian.

Gauss-Bonnet as an Index Theorem

Proposition 4. *On an even dimensional PE manifold, for any fixed* $t > 0$*, the super-trace of the heat kernel at time* t *has a renormalization independent of the choice of special bdf.*

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The proof consists in parlaying mapping properties of the resolvent and the expansion of its integral kernel into information about the expansion of the heat kernel through the Laplace transform.

Gauss-Bonnet as an Index Theorem

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$$ {}^{R}\!\!\int_M \mathrm{Pff} = {}^{R}\mathrm{Ind}(\eth_{GB}). $$

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We carry out the heat equation proof of the index theorem,

$$ {}^R\!\!\int \mathrm{Pff} + {}^R\eta = {}^R\mathrm{Ind}(\eth_{GB}), $$

and show directly that ${}^R\eta = 0$.

Gauss-Bonnet as an Index Theorem

Hence the Euler characteristic equals the renormalized index

$$\begin{aligned} {}^R\mathrm{Ind}\,(\eth_{GB}) = &\sum_{k<\frac{m}{2}} (-1)^k \dim H^k(M, \partial M) \\ &+ (-1)^{m/2}\ {}^R\dim \mathcal{H}^{m/2}_{L^2} \\ &+ \sum_{k>\frac{m}{2}} (-1)^k \dim H^k(M) \end{aligned}$$

Gauss-Bonnet as an Index Theorem

Hence the Euler characteristic equals the renormalized index

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{}^{R}\mathrm{Ind}\,(\eth_{GB}) = &\sum_{k<\frac{m}{2}} (-1)^k \dim H^k\,(M, \partial M) \\
&+ (-1)^{m/2}\ {}^{R}\dim \mathcal{H}^{m/2}_{L^2} \\
&+ \sum_{k>\frac{m}{2}} (-1)^k \dim H^k\,(M)
\end{aligned}
$$

Gauss-Bonnet as an Index Theorem

So $(-1)^{m/2}\ \ {}^{R}\dim \mathcal{H}^{m/2}_{L^2}$ is

$$\sum_{k\geq\frac{m}{2}}(-1)^k \dim H^k\left(M\right) - \sum_{k<\frac{m}{2}}(-1)^k \dim H^k\left(M,\partial M\right).$$

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For instance,

$${}^{R}\dim \mathcal{H}^{m/2}_{L^2}(\mathbb{H}^m) = 1.$$

Extensions and Generalizations

- Variation of PE structures
- Limiting index formulas for other asymptotically regular geometries
- Other operators and other manifolds

Variation of PE structures

Let g_s be a smooth family of PE metrics, and set $h = \partial_s \downharpoonleft_{s=0} g_s$.

- h_{ij} is even/odd mod x^m if the number of normal directions in $\{i, j\}$ is even/odd

$\Rightarrow$ Gradient of ${}^{R}Vol = -\frac{g^{(m-1)}}{4}$ [Anderson]

$\Rightarrow$ Gradient of $L = -\frac{g^{(m-1,1)}}{4}$ [Graham-Hirachi]

Variation of PE structures

The variation of the Weyl volume of tube invariants of Weyl curvature are given in terms of generalized Einstein tensors (Lovelock tensors).

We can verify directly that the variation of ${}^{R}\!\int \mathrm{Pff}$ vanishes.

Limiting Index Formulas

For asymptotically flat manifolds, it is not necessary to renormalize the integral of the Pfaffian.

$$\int \mathrm{Pff} + \sum_{q=0}^{m-1} C_q W_q(\partial M) = \chi(M)$$

Limiting Index Formulas

For asymptotically flat manifolds, it is not necessary to renormalize the integral of the Pfaffian.

$$\int \mathrm{Pff} + \sum_{q=0}^{m-1} C_q W_q(\partial M) = \chi(M)$$

$$\sum_{q=0}^{m-1} C_q W_q(\partial M) = \chi(M) - \chi_{L^2}(M)$$