Conserved charges and positivity of energy for Asymptotically locally AdS spacetimes

Kostas Skenderis
University of Amsterdam

based on
I. Papadimitriou, K.S., hep-th/0404176
hep-th/0505190
M. Cheng, K.S., hep-th/0506123

1. Introduction

In recent years there has been lots cross fertilization between theoretical physics and mathematics. One such arena is in the so-called AdS/CFT correspondence.

The AdS/CFT correspondence relates string theory on asymptotically locally AdS (AlAdS) spacetimes (bulk theory) to a quantum field theory (boundary theory) on the conformal boundary of AlAdS.

The existence of dualities often offers different perspectives on known problems leading to progress that otherwise would be difficult to come about.

In this talk we will discuss three (related) such instances:

1. Asymptotic solutions of Einstein equations with cosmological constant

2. Conserved charges in General Relativity

3. Positive energy conjectures

The AdS/CFT correspondence relates the on-shell gravitational action to the generating functional of QFT correlations functions. Making precise this map and showing that it is well-defined was the primary goal of our work. In this talk however we will focus on the spin-offs for GR and mathematical issues.

For the purpose of this talk we restrict our attention to pure gravity. Discussion of the modifications due to matter fields can be found in the papers.

All local results are valid for any sign of the cosmological constant and any signature. In our discussion we focus on negative cosmological constant and use Lorentzian signature.

2. Asymptotic Solutions

Gravitational action:

$$S = \int_M d^{d+1}x\sqrt{-G}(R - 2\Lambda)$$

Field equation:

$$R_{\mu\nu} = -dG_{\mu\nu}$$

Asymptotic solutions of this equation were obtained 20 years ago by [Fefferman,Graham (1985)]:

$$ds^2 = \frac{dz^2}{z^2} + \frac{1}{z^2} g_{ij}(x,z)dx^i dx^j$$

where

$$\begin{aligned} g_{ij}(x,z) &= g_{(0)ij} + z^2 g_{(2)ij} + \ldots + \\ &\quad + z^d \left(\log z^2 h_{(d)ij} + g_{(d)ij}\right) + \ldots \end{aligned}$$

near $z = 0$.

We call spacetimes with such asymptotics "*Asymptotically locally AdS*" (AlAdS) spacetimes.

Variational problem

Is the variational problem well-posed with a conformal structure kept fixed at infinity?

I.e. do the boundary terms in the variation of the action cancel when the following boundary condition is used?

$$\delta g_{(0)ij}(x) = 2\delta\sigma(x) g_{(0)ij}(x) \qquad \text{on } \partial M$$

The answer depends on $[g_{(0)}]$. More precisely, the answer depends on the holographic conformal anomaly $\mathcal{A}[g_{(0)}]$, a conformal density of weight d determined from $g_{(0)}$.

• $\mathcal{A} \equiv 0$ in odd dimensions

• In $d = 4$:

$$\mathcal{A} = E_4 + Weyl^2$$

where E_4 is the Euler density.

• There are similar formulae in higher even dimensions.

- $\mathcal{A} \equiv 0$

The variational problem is well posed, provided one adds a certain number of boundary terms in the action.

- $\mathcal{A} \neq 0$

The variational problem is NOT well-posed. One has to choose a representative $g_{(0)}$ of the conformal structure, but one can add boundary terms such that the dependence of the representative is governed by the anomaly $\mathcal{A}$:

$$\delta S = \int_{\partial M} \mathcal{A} \delta \sigma$$

In both cases the boundary terms needed for the variational problem are *exactly* equal to the counterterms determined in earlier work by requiring finiteness of the on-shell action.

Asymptotic solutions

A very efficient way to obtain asymptotic solutions and the renormalized on-shell action is by using a "Hamiltonian formalism" where the role of time is played by the radial coordinate `[Papadimitriou, Skenderis (2004)]`.

We write the metric as ($z = e^{-r}$)

$$
\begin{aligned}
ds^2 &= dr^2 + \gamma_{ij}(x,r)dx^i dx^j \\
&= dr^2 + e^{2r}(g_{(0)ij} + e^{-2r} g_{(2)ij} + \cdots)dx^i dx^j
\end{aligned}
$$

The boundary conditions at ∂M imply that the radial derivative is to leading order equal to the dilatation operator,

$$\partial_r = \delta_D(1 + \mathcal{O}(e^{-r}))$$

where δ_D generates infinitesimal constant Weyl transformations, e.g.

$$\delta_D \gamma_{ij} = 2\gamma_{ij}.$$

The (radial) canonical momentum is given by

$$\pi^{ij} = K^{ij} - \gamma^{ij} K$$

where

$$K_{ij} = \frac{1}{2} \partial_r \gamma_{ij}.$$

is the extrinsic curvature of the $r = const$ surfaces.

The idea is now to express all quantities in terms of eigenfunctions of the dilatation operator,

$$K^i_j = \delta^i_j + K^i_{(2)j} \cdots + K^i_{(d)j} + (-2r)\tilde{K}^i_{(d)j}$$

where

$$\delta_D K^i_{(n)j} = -n K^i_{(n)j}, \quad \delta_D \tilde{K}^i_{(d)j} = -d \tilde{K}^i_{(d)j},$$

$$\delta_D K^i_{(d)j} = -d K^i_{(d)j} - 2 \tilde{K}^i_{(d)j}$$

• $K_{(n)ij}$ are equivalent to $g_{(n)ij}$:
$K_{(n)ij} = -\frac{n}{2} g_{(n)ij} +$ lower

• All $K_{(n)ij}, n < d$ and the trace and divergence of $K_{(d)ij}$ are locally determined in terms of $g_{(0)}$. Remaining components of $K_{(d)ij}$ are NOT determined by asymptotics.

• Conformal anomaly: $\mathcal{A} = K_{(d)} = \gamma^{ij} K_{(d)ij}$.

• $\tilde{K}_{(d)ij}$ is equal to metric variation of the integrated conformal anomaly:

$$\tilde{K}_{(d)ij} = \frac{1}{\sqrt{-\gamma}} \frac{\delta}{\delta \gamma^{ij}} \int d^d x \sqrt{-\gamma} \mathcal{A}$$

[de Haro, Solodukhin, Skenderis (2000)]

• The regulated on-shell action is given by

$$S[g_{(0)}; \epsilon] = \int_{r>\epsilon} d^{d+1} x \sqrt{-G} (R - 2\Lambda)$$
$$= \int_{r=\epsilon} d^d x \sqrt{-\gamma} \left(\sum_{2n<d} \frac{1}{(2n-d)} K_{(2n)} - \epsilon K_{(d)} \right) + S_{\text{ren}}$$

[Henningson, Skenderis (1998)]

• Rernormalized 1-point function:

$$T_{ij} \equiv K_{(d)ij} - K_{(d)}\gamma_{ij} = \frac{1}{\sqrt{-g_{(0)}}}\frac{\delta S_{\text{ren}}}{\delta g_{(0)}^{ij}}$$

It satisfies

$$\nabla^i T_{ij} = 0, \quad T_i^i = \mathcal{A}$$

$\Rightarrow$ $g_{(0)ij}$ and T_{ij} are conjugate variables.

There is a natural symplectic form in the space of solutions.

• n-point functions:

Derivatives of the "Dirichlet-to-Neunman" map:

$$\frac{\delta^{n-1} K_{(n)ij}}{\delta g_{(0)i_1 j_1} \cdots \delta g_{(0)i_n j_n}}$$

encode n-point functons of the dual QFT. If one would know the non-linear map from to $g_{(0)}$ to $K_{(d)ij}$ one would solve the dual QFT!!!

2-point and 3-point functions around certain backgrounds where computed in the physics literature

[Bianchi, Freedman, Skenderis (2001)]
[Skenderis (2002)]
[Bianci, Prisco, Mück (2004)]
[Papadimitriou, Skenderis (2004)]

One can also express T_{ij} in terms of the coefficients appearing in the FG expansion:

$$T_{ij} = -\frac{d}{2}(g_{(d)ij} + X^{(d)}_{ij}(g_{(0)})).$$

where

$$\begin{aligned} X^{(2k+1)}_{ij} &= 0 \\ X^{(2)}_{ij} &= -g_{(0)ij}\,\mathrm{Tr}\, g_{(2)} \\ X^{(4)}_{ij} &= -\frac{1}{8}g_{(0)ij}[(\mathrm{Tr}\, g_{(2)})^2 - \mathrm{Tr}\, g^2_{(2)})^2] \\ &\quad -\frac{1}{2}(g^2_{(2)})_{ij} + \frac{1}{4}g_{(2)ij}\,\mathrm{Tr}\, g_{(2)} \\ X^{(6)}_{ij} &= ... \end{aligned}$$

[de Haro, Solodukhin, Skenderis (2000)]

2. Conserved Charges in AlAdS

The issue of conserved charges in GR, especially of mass, has been a controversial issue.

Most definitions lead to infinite answers and some form of "background subtraction" is often used.

Previous definitions of conserved charges for Asymptotically AdS spacetimes (AAdS) (i.e. conformal infinity is represented by the Einstein universe) such as the ones by [Teitelboim-Henneaux], [Ashtekar-Magnon] are essentially of this form.

We present here a definition a definition of conserved charges for AlAdS spacetimes with the following properties. The conserved charges

1. are derived from first principles

2. are finite by construction

3. are intrinsic to the spacetime (no background subtraction)

4. when specialized to AAdS agree with previous definitions

5. satisfy the first law and the quantum statistical relation

6. New positive energy conjectures can be formulated

Construction

1. Associated to every AlAdS spacetime there is a conserved energy momentum tensor

$$T_{ij} = K_{(d)ij} - K_{(d)}\gamma_{ij}.$$

It satisfies

$$\nabla^i T_{ij} = 0, \quad T^i_i = \mathcal{A}$$

due to the bulk field equations.

Definition An *asymptotic conformal Killing vector* is a bulk vector ζ^μ with the following asymptotics

$$\zeta^r = \mathcal{O}(e^{-dr}), \qquad \zeta^i = \xi^i(x)(1 + \mathcal{O}(e^{-(d+2)r})$$

where $\xi^i(x)$ is a conformal Killing vector of $g_{(0)}$.

Asymptotic conformal Killing vectors are in 1-1 correspondence with asymptotic Killing vectors of the bulk metric.

2. If the spacetime possesses an asymptotic CKV ζ^μ then there exist a corresponding conserved charge given by

$$Q = \int_{C \cap \partial M} dS_i T^i_j \xi^j$$

where C is a Cauchy surface.

[The charges associated with conformal isometries are conserved only if the conformal anomaly vanishes.]

Derivation

1. Using Noether theorem.

Associated to every asymptotic CKV there is a global symmetry that leaves invariant the renormalized action. Using standard methods one can extract from this the corresponding conserved charges.

2. Covariant phase space methods (Wald)

The “Hamiltonians” H_ξ that generate global symmetries must satisfy Hamilton’s equations

$$\delta H_\xi = \omega$$

where ω is the symplectic form in covariant phase space. Integrating this equation leads to conserved charges.

3. T_{ij} is the Brown-York stress energy tensor associated with the renormalized action.

Thermodynamics

We now specialize to stationary black hole space-times possibly possesing a number of additional commuting isometries.

One can establish in complete generality:

1. The quantum statistical relation
$$\beta(M - TS - \Omega_i J_i) = S^E_{ren}$$
where $\Omega_i = \Omega_i^H - \Omega_i^\infty$, the relative angular velocity. M is defined wrt a non-rotating frame. S^E_{ren} is the Euclidean on-shell renormalized action.

2. The first law of black hole mechanics
$$\delta M = T\delta S + \Omega_i \delta J_i$$
The variation δ:
(i) need not respect the symmetries of the solution,
(ii) must preserve the boundary conditions.

3. Positive Energy Conjectures

It is well known that there is a positive energy theorem associated with AAdS spacetimes [Gibbons, Hull, Warner (1983)].

Under which condition the mass of AlAdS spacetime is bounded from below?

The existence of new stable AlAdS spacetimes would have profound implications via the AdS-CFT correspondence. For example, it would imply the existence of stable strongly coupled QFTs in curved backgrounds.

To investigate this question we computed a regulated version of the spinorial Witten-Nester energy for AlAdS spacetimes.

The Witten-Nester spinorial energy

[Witten (1981), Nester (1981)]

Consider the antisymmetric tensor field

$$E^{\mu\nu} = \bar{\epsilon}\Gamma^{\mu\nu\rho}\nabla_\rho\epsilon + c.c.$$

where

$$\nabla_\mu\epsilon = (D_\mu + \frac{1}{2}\Gamma_\mu)\epsilon$$

and D_μ is the standard covariant derivative.

Using $E^{\mu\nu}$ we can construct a *conserved* current

$$j^\mu = \nabla_\nu E^{\nu\mu}$$

and an associate conserved charge

$$E_{WN} = \int dS_\mu j^\mu = \int d^dx\sqrt{g}\eta_\mu\nabla_\nu E^{\mu\nu}$$

where we integrate over a Cauchy surface and η^μ is its unit normal.

Positivity

Using Einstein's equations one finds by direct computation

$$\eta_\mu \nabla_\nu E^{\nu\mu} = (\nabla^\alpha \epsilon)^\dagger (\nabla_\alpha \epsilon) - (\Gamma^\alpha \nabla_\alpha \epsilon)^\dagger (\Gamma^\beta \nabla_\beta \epsilon)$$

where α run over all values except t.
It follows that

$$E_{WN} \geq 0$$

if the Witten equation

$$\Gamma^\alpha \nabla_\alpha \epsilon = 0, \qquad \alpha \neq t$$

is satisfied.

Furthermore,

$$E_{WN} = 0 \quad \Leftrightarrow \quad \nabla_\alpha \epsilon = 0$$

i.e. the geometry admits a Killing spinor.

Using Stokes' theorem we can express E_{WN} in terms of a boundary integral

$$E_{WN} = \int dS_{\mu\nu} E^{\mu\nu}$$

Our task is now to evaluate this expression for general asymptotically locally (spin) AdS spacetimes.

Asymptotic Witten spinors

To evaluate E_{WN} we need to know the asymptotics of ϵ. This is obtained by solving asymptotically the Witten equation. To do so we express all quantities in terms of eigenfunctions of the dilatation operator:

$$\nabla = \nabla_{(0)} + \nabla_{(1)} + ... + \nabla_{(d)} + (-2r)\tilde{\nabla}_{(d)} + ..$$

$$\epsilon = \epsilon_{(m)} + \epsilon_{(m+1)} + ... + \epsilon_{(m+d)} + (-2r)\tilde{\epsilon}_{(m+d)} + ..$$

where indices in parenthesis indicate dilatation weight.

Inserting in the Witten equation one obtains:

- $m = -\frac{1}{2}$
- $\epsilon_{(n-\frac{1}{2})}$ and $\tilde{\epsilon}_{(d-\frac{1}{2})}$ are uniquely determined in terms of $\epsilon_{(-1/2)}$
- $\epsilon_{(d-\frac{1}{2})}$ is undetermined.

Witten-Nester energy

We regulate the Witten-Nester energy by considering the integration surface at $r = \epsilon$, as in our earlier discussion. Inserting the asymptotic solution for the Witten spinors one obtains

$$E_{WN} = \int_{r=\epsilon} dS_{\mu\nu} E^{\mu\nu}$$

$$= \int_{r=\epsilon} d^{d-1}x\sqrt{\gamma}[E_{(1)}+E_{(3)}+...+E_{(d-1)}-\epsilon\tilde{E}_{(d-1)}]$$

- All $E_{(k)}$ and $\tilde{E}_{(d-1)}$ are local functions of $\epsilon_{(-\frac{1}{2})}$.

- $\tilde{E}_{(d-1)}$ is present only when d is even and it is proportional to $\tilde{K}_{(d)ij}$.

- All terms but the one involving $E_{(d-1)}$ diverge as $\epsilon \to \infty$.

To obtain a well-defined Witten-Nester energy we require that all divergent terms vanish. This imposes restrictions on the asymptotic geometry, i.e. only AlAdS with certain conformal structures admit positive Witten-Nester energy.

• d even
$\tilde{E}_{(d-1)} = 0 \quad \Leftarrow$ boundary manifold is conformally Einstein

• $E_{(1)} = 0$ is satisfied if the boundary manifold admits a spinor $\epsilon_{(-\frac{1}{2})}$ satisfying

$$\left((\Gamma^a D_a)^2 + \frac{d-1}{2} K^j_{(2)a} \Gamma_j \Gamma^a\right) \epsilon_{(-\frac{1}{2})} = 0,$$

where

$$K_{(2)ij} = \frac{1}{d-2}(R_{ij} - \frac{1}{2(d-1)} R \gamma_{ij})$$

Up to $d = 4$ (i.e. $AlAdS_5$) these are all asymptotic conditions that need to be analyzed.

• $E_{(k)} = 0$ for all AAdS spacetimes, in agreement with earlier work.

Witten-Nester vs holographic energy

Suppose now that the asymptotic geometry is such that all divergent terms vanish, and $\epsilon_{(-\frac{1}{2})}$ can be chosen such that

$$\xi^i = \bar{\epsilon}_{(-\frac{1}{2})}\Gamma^i\epsilon_{(-\frac{1}{2})}$$

is a timelike Killing vector ξ^i.
Then the following hold:

• d odd

$$M = E_{WN} \quad \Rightarrow \quad M \geq 0$$

• d even

$$E_{WN} = M - E_0 \quad \Rightarrow \quad M \geq E_0$$

where E_0 is a bounded quantity.

This would provide a positive energy theorem if Witten spinors with these asymptotics exist globally.

- $d=2$

$$E_0 = \int dx \sqrt{g} |\partial_x \epsilon_{(-\frac{1}{2})}|^2$$

- $d=4$

$$E_0 = -\frac{1}{6} \int d^3x \sqrt{g} |K^j_{a(2)} \Gamma^a \Gamma^j \epsilon_{(-\frac{1}{2})}|^2$$

Recall that $E_{WN} = 0$ when the background admits a Killing spinor. The E_{WN} measures the energy of the system relative to this background. The holographic energy, however, is not necessarily zero when the background admits Killing spinors.

Indeed, even AdS_3 and AdS_5 have non-zero energy. In both cases, the energy is due to the conformal anomaly and is equal to the Casimir energy of the dual CFT.

The term E_0 off-sets the energy to take into account the conformal anomaly.

Open questions

There lots of open problems and progress could be achieved, especially if there is additional input from mathematics. In particular, one would like to:

1. Classify conformally Einstein 4-manifolds that admit spinors satisfying

$$\left((\Gamma^a D_a)^2 + \frac{d-1}{2} K^j_{(2)a} \Gamma_j \Gamma^a\right) \epsilon_{(-\frac{1}{2})} = 0,$$

2. Understand the dependence of the Witten-Nester energy of spin structures.

3. Study the global existence of Witten spinors with such asymptotics.

[An example where the asymptotics conditions can be satisfied but global Witten spinors do not exist is the case of the AdS soliton.]

4. There are positive energy theorems for certain AAdS spacetimes (domain-wall spacetimes) with specific matter fields present.
[Townsend (1985), Skenderis,Townsend (1999), Freedman, Nunez, Schnabl, Skenderis (2003)].

It would be interesting to extend our analysis to include matter fields. Such extension would lead to a systematic search for stable backgrounds supported by matter fields. The existence of such stable background is motivated by the AdS-CFT correspondence.