Gradient RG Flow for Non Linear Sigma Model (NLSM)

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The Physics of Perelman's "Entropy"

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Some Important References:

- G Perelman math.DG/0211159
- AB Zamolodchikov JETP Lett 43 (1986) 730.
- AA Tseytlin Phys Lett B194 (1987) 63
- H-D Cao, R Hamilton, T Ilmanen math.DG/0404165.
- N Sesum math.DG/0410062.

Classical Physics

Study Critical Points of the Action S.

e.g. Point Particle : $S = \int_{t_1}^{t_2} dt \left(\frac{1}{2} m v^2 - V(x) \right)$

$$\delta S = 0 \implies m \frac{d\vec{v}}{dt} = -\vec{\nabla} V \quad 2^{\text{nd}} \text{ Law}$$

Scalar Field in M^4 with Quartic Interaction

$$S = \int_M \sqrt{-\eta}\, d^4x \left[\eta^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{m^2}{2} \phi^2 - \frac{1}{4!} \lambda \phi^4 \right]$$

Coupling Constants

$$\implies (\nabla^\mu \nabla_\mu - m^2) \phi = \frac{1}{12} \lambda \phi^3$$

Quantum Physics

Study the Wiener - Feynman Measure

$$Dx \; e^{-\frac{1}{i\hbar} S[x]} \quad \text{--- Quantum Mechanics}$$

$$D\phi \; e^{-\frac{1}{i\hbar} S[\phi]} \quad \text{--- Quantum Field Theory}$$

Interested in <u>integrating polynomials</u> of ϕ, $\partial\phi$, ... against this measure.

These are the <u>Correlation Functions</u>.

Generating Functional Trick:

Define: $Z[J] = \int D\phi \, \exp\left(-S[\phi] + \int J\phi\right)$

Take: $\dfrac{\delta^m}{\delta J^m} Z[0]$ to get m-point correlation function

$\therefore$ Study $Z[J]$.

(Partially) Defining the Measure

* Parametrize fields ϕ in terms of Fourier Modes, $\vec{k}$ = "momentum"

* $D\phi := \prod' d\phi(k)$

  But then $Z[J]$, correlation functions diverge.

* Define cut-off Λ and cut-off measure

$$[D\phi]_\Lambda = \prod_{|k|<\Lambda} d\phi(k)$$

$$Z_\Lambda[J] = \int [D\phi]_\Lambda \, e^{-S[\phi] + \langle J\phi \rangle}$$

But are low-energy predictions of theory now sensitive to arbitrary cut-off Λ ?

Renormalization Group Flow

* Split the Fields: $\phi(k) = \hat{\phi}(k) + \tilde{\phi}(k)$

$$\hat{\phi} = \begin{cases} \phi(k) & , \quad b\Lambda \le |k| \le \Lambda \\ 0 & , \quad 0 \le |k| < b\Lambda \end{cases}$$

$$\tilde{\phi} = \begin{cases} 0 & , \quad b\Lambda \le |k| \le \Lambda \\ \phi(k) & , \quad 0 \le |k| < b\Lambda \end{cases}$$

for some $0 < b < 1$

* Split the Generating Function:

$$Z_\Lambda[J] = \int \underbrace{D\tilde{\phi}}_{= [D\tilde{\phi}]_{b\Lambda}} \, e^{-S[\tilde{\phi}] + \langle J\tilde{\phi}\rangle} \underbrace{\int D\hat{\phi} \, e^{-\left(S[\tilde{\phi}+\hat{\phi}] - S[\tilde{\phi}] + \langle J\hat{\phi}\rangle\right)}}_{\text{Do this integral.}}$$

$$= \int [D\phi]_{b\Lambda} \, e^{-S_{eff}[\phi] + \langle J_{eff}\, \phi\rangle}$$

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e.g. Scalar Field:

$$S_{eff} = \int_M \Big[\tfrac{1}{2}(1+\delta Z) D\phi\cdot D\phi + \tfrac{1}{2}(m^2 + \delta m^2)\phi^2 + \tfrac{1}{4!}(\lambda + \delta\lambda)\phi^4 + \delta C (D\phi\cdot D\phi)^2 + \delta D \phi^6 + \dots \Big] d^n x$$

β - Functions

$$\frac{d}{db} Z_\Lambda[J] = 0$$ since b did not enter definition of $Z_\Lambda[J]$.

But: $$Z_\Lambda[J] \equiv \underbrace{\tilde{Z}_\Lambda[J']}_{\substack{\text{depends on rescaled}\\ \text{fields and coupling constants,}\\ \text{which depend on b.}}}$$ by definition of $\tilde{Z}_\Lambda[J']$

$$\Rightarrow \quad 0 = \frac{\partial \tilde{Z}_\Lambda}{\partial \phi'_i} \frac{\partial \phi'^i}{\partial b} + \frac{\partial \tilde{Z}_\Lambda}{\partial \lambda'_k} \frac{\partial \lambda'_k}{\partial b}$$

λ'_k = rescaled coupling constants

$\frac{\partial \lambda'_k}{\partial b} = \beta_k$ = beta functions

Beta functions $= 0$

$\Downarrow$

$\tilde{Z}_\Lambda$ invariant under field rescalings

$\Downarrow$

scale invariant theory

$\Downarrow$ (sometimes)

Conformal Field Theory.

Rescaling

Define: $k' := \frac{k}{b}$, $x' = bx$, so $k' \cdot x' = k \cdot x$.

and $\phi(k;x) = \phi(k';x')$

Z can now be written in terms of k' with original cut-off Λ:

$$\tilde{Z}_\Lambda[J'] = \int [D\phi']_\Lambda \, e^{-\tilde{S}[\phi'] + \langle J'\phi' \rangle}$$

$$\equiv Z_\Lambda[J]$$

Tildes / primes indicate fields and coupling constants now depend on scale b

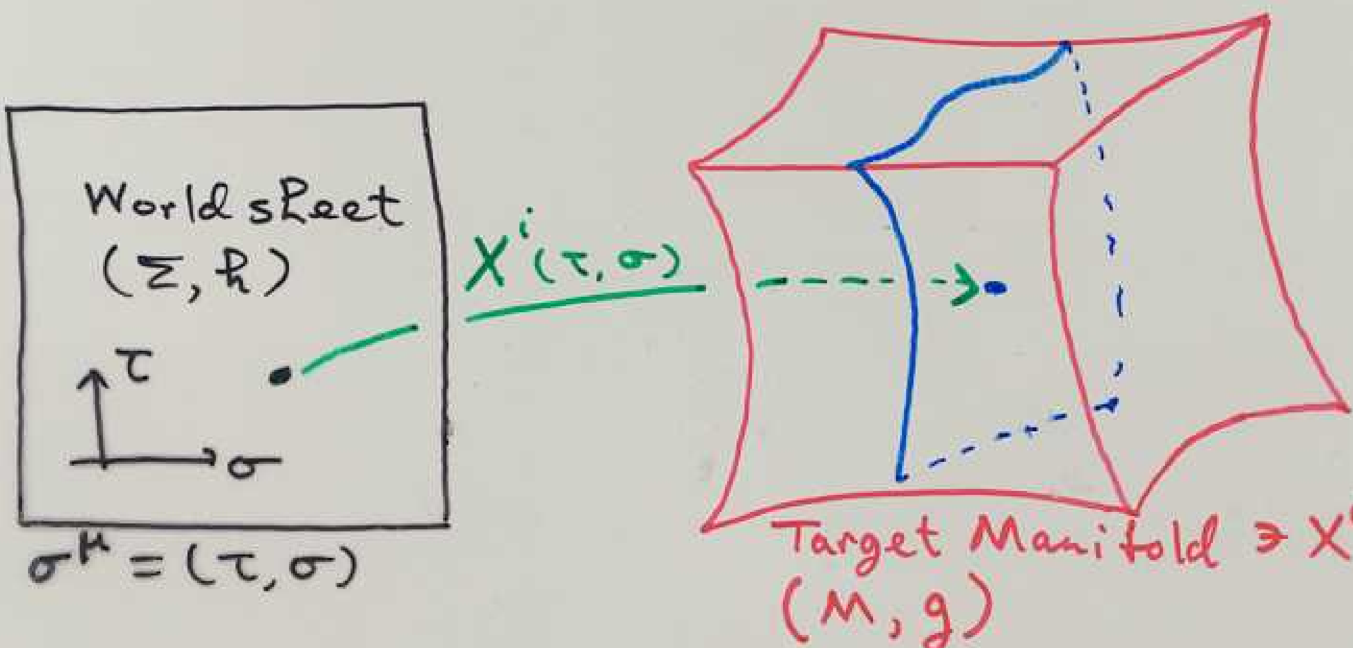
e.g. $$\phi' = \left[b^{2-\eta}(1+\delta J) \right]^{1/2} \phi$$

$$m'^2 = \frac{m^2 + \delta m^2}{(1+J)\, b^2}, \quad \text{etc.}$$

This dependence is called:

Renormalization Group Flow

Bosonic NonLinear Sigma Model



NonLinear

N scalar fields on Σ.

$$S[x] = \frac{1}{\alpha'} \int_\Sigma \sqrt{h}\, d^2\sigma \; h^{\mu\nu} \underbrace{g_{ij}}_{} (X) \frac{\partial X^i}{\partial \sigma^\mu} \frac{\partial X^j}{\partial \sigma^\nu}$$

"String scale (squared)"

coupling constants
= target manifold metric!

$$+ \frac{1}{\alpha'} \int_\Sigma \sqrt{h}\, d^2\sigma \; \epsilon^{\mu\nu} B_{ij}(X) \frac{\partial X^i}{\partial \sigma^\mu} \frac{\partial X^j}{\partial \sigma^\nu}$$

Anti-symmetric B-field on M
(Coupling Constants on Σ)

$$+ \frac{1}{2} \int_\Sigma \Phi(X)\, K_G(h) \sqrt{h}\, d^2\sigma$$

dilaton

Gauss Curvature of h.

RG Flow of the Coupling Constants

$t = \log b$

$$\frac{\partial g_{ij}}{\partial t} = -\alpha' \left[R_{ij} + 2 \nabla_i \nabla_j \Phi - \frac{1}{4} H_{ik\ell} H_j{}^{k\ell} \right] + \mathcal{O}(\alpha'^2)$$

$$=: -\beta^g_{ij}$$

$$\frac{\partial B_{ij}}{\partial t} = \alpha' \left[\frac{1}{2} \nabla^k H_{kij} - H_{kij} \nabla^k \Phi \right] + \mathcal{O}(\alpha'^2)$$

$$=: -\beta^B_{ij} \quad , \quad H = dB$$

$$\frac{\partial \Phi}{\partial t} = \frac{26-n}{6} + \alpha' \left[\frac{1}{2} \Delta \Phi - |\nabla \Phi|^2 + \frac{1}{24} |H|^2 \right] + \mathcal{O}(\alpha'^2)$$

$$=: -\beta^\Phi$$

Set $n = 26$.
Drop $\mathcal{O}(\alpha'^2)$ terms.

} 1-loop approx. or order α' flow

Set $\Phi = 0$, $B = 0$ $\forall t$ to get

$$\frac{\partial g_{ij}}{\partial t} = -\alpha' R_{ij}$$

Ecker and Huisken? Herencamp 1974

DeCoupling the Dilaton

Transform g_{ij} by t-dependent diffeo.

Transform $B \mapsto B + d\omega$, $\omega = t$-dependent gauge f^{n}

Get:

$$\boxed{\begin{aligned} \frac{\partial g_{ij}}{\partial t} &= -\alpha'\left(R_{ij} + \nabla_i \nabla_j \psi - \tfrac{1}{4} H_{ik\ell} H_j{}^{k\ell}\right) \quad \text{---} (1) \\ \frac{\partial B_{ij}}{\partial t} &= \tfrac{1}{2}\alpha'\left(\nabla^k H_{kij} - H_{kij} \nabla^k \psi\right) \quad \text{---} (2) \end{aligned}}$$

ψ generates an arbitrary diffeomorphism.

Dilaton Φ decouples from RHS above.

$$\frac{\partial \Phi}{\partial t} = \frac{1}{2}\alpha'\left(\Delta \Phi - \nabla \Phi \cdot \nabla \psi + \frac{1}{12}|H|^2\right)$$

and, by the way,

$$\frac{\partial H}{\partial t} = \frac{1}{2}\alpha'\left(\Delta_{LB} \Phi - d\langle H, \nabla \psi\rangle\right)$$

1-Loop RG Flow as Gradient Flow

$$F[g,B,\psi] := \alpha' \int_M dV \left(R + |\nabla\psi|^2 - \frac{1}{12}|H|^2 \right) e^{-\psi}$$

Dilaton Action — if ψ were dilaton.

= Functional on "Space of Coupling Constants (g,B)" $\times$ $C^\infty(M)$

$$= \alpha' \int_M e^{-\psi/2} \left(R - \frac{1}{12}|H|^2 - 4\Delta \right) e^{-\psi/2}\, dV$$

Minimize F over ψ such that $\int_M e^{-\psi} dV = 1$.

$$\lambda(t) := \lambda[g(t), B(t)] = \inf_{\psi \,|\, \int_M e^{-\psi} dV = 1} F[g,B,\psi]$$

Equivalent to solving Schrödinger Problem

$$\left(-4\Delta_t + R_t - \frac{1}{12}|H_t|^2 \right) u_t = \lambda(t)\, u_t$$

$u_t = e^{-\psi/2}$ has no nodes

$\Rightarrow$ ground state

$$\int_M u_t^2\, dV \equiv \int_M e^{-\psi} dV = 1$$

Variational Derivative of F:

$$\frac{dF}{ds} =$$

$$\int_M \left(-R^{ij} - \nabla^i \psi \nabla^j \psi + \frac{1}{4} H^i{}_{kl} H^{jkl}\right) \frac{\partial g_{ij}}{\partial s} e^{-\psi} dV \checkmark$$

$$+ \int_M \frac{1}{2}\left(\nabla_k H^{kij} - H^{kij} \nabla_k \psi\right) \frac{\partial B_{ij}}{\partial s} e^{-\psi} dV \checkmark$$

$$+ \int_M \underbrace{\left(R - \frac{1}{12}|H|^2 + 2\Delta\psi - |\nabla\psi|^2\right) e^{-\psi/2}}_{\substack{\shortparallel \\ \lambda_s e^{-\psi/2}}} \left(\frac{1}{2} g^{ij} \frac{\partial g_{ij}}{\partial s} - \frac{\partial \psi}{\partial s}\right) e^{-\psi/2} dV \, ?$$

$\lambda_s e^{-\psi/2}$ <u>if</u> $e^{-\psi/2} =: u$ is the ground state eigenfunction of $-4\Delta_s + R_s - \frac{1}{12}|H_s|_s^2$ all along the variation.

$$\Downarrow$$

$$F_s[g, B, \psi] = \lambda_s[g, B].$$

In that case, last integral is

$$\int_M \lambda \left(\frac{1}{2} g^{ij} \frac{\partial g_{ij}}{\partial s} - \frac{\partial \psi}{\partial s}\right) e^{-\psi} dV$$

$$= \lambda \frac{d}{ds} \int_M e^{-\psi} dV = \lambda \frac{d}{ds}(1)$$

$$= 0.$$

Variational Derivative of λ

$$\frac{d\lambda}{ds} = \left\langle \text{Grad}\,\lambda, \begin{pmatrix} \partial g_{ij}/\partial s \\ \partial B_{ij}/\partial s \end{pmatrix} \right\rangle$$

$$= \int_M e^{-\psi} dV\, g^{ik} g^{jl} \left[\left(\text{RHS of (1)}\right)_{ij} \frac{\partial g_{kl}}{\partial s} + \left(\text{RHS of (2)}\right)_{ij} \frac{\partial B_{kl}}{\partial s} \right]$$

from last slide

$$\Rightarrow \quad \text{Grad}\,\lambda = \begin{pmatrix} \text{RHS of (1)} \\ \text{RHS of (2)} \end{pmatrix}$$

$$= \begin{pmatrix} -R_{ij} - \nabla_i \nabla_j \psi + \frac{1}{4} H_{ikl} H_j{}^{kl} \\ \frac{1}{2} \nabla^k H_{kij} - \frac{1}{2} H_{kij} \nabla^k \psi \end{pmatrix}$$

so the order α' RG flow, with the diffeomorphism potential ψ fixed as above, is the gradient flow of λ.

Results

1. λ is monotonic along flow.

2. Fixed point of flow $\equiv$ stationary point of λ

$$R_{ij} + \nabla_i \nabla_j \psi = \frac{1}{4} H_{ikl} H_j{}^{kl}$$

$$\nabla^k \left(e^{-\psi} H_{kij} \right) = 0 \quad , \quad H = dB$$

3. If $|H|^2$ is constant over M, then $\psi = 0$ (generalizes $H = 0$ case due to Bourguignon).

4. At a fixed point, $\lambda = \frac{1}{6} \int_M |H|^2 e^{-P} dV$

5. There are no periodic or homoclinic flows other than the fixed points.

<u>Proof:</u> Periodic $\Rightarrow$ $\lambda(t_1) = \lambda(t_2)$ for some $t_1 < t_2$.

Monotonicity $\Rightarrow$ $\overset{\|}{\lambda}(t)$ for all $t \in [t_1, t_2]$

$\Rightarrow$ $\lambda'(t) = 0$ along flow

$\Rightarrow$ $\mathrm{RHS}(1) = \mathrm{RHS}(2) = 0$ — " —

$\Rightarrow$ Fixed Point.

Similar for Homoclinic Case.

6. We can compute Hessian of λ, study stability.

Def$^{\underline{ns}}$: $h_{ij} := \dfrac{\partial g_{ij}}{\partial s}$, $\beta_{ij} := \dfrac{\partial B_{ij}}{\partial s}$

$$v_h := \mathrm{tr}_g h - 2 \frac{\partial \varphi}{\partial s}$$

Special Case: $\underbrace{\text{Second Variation about a Ricci-flat fixed point.}}$

$$\Downarrow$$

$$Ric = 0 \implies H = 0, \ \varphi = 0$$

We get

$$\left. \frac{d^2 \lambda_s}{ds^2} \right|_{s=0} = \frac{1}{2} \int_M \Big[h^{ij} \Delta_L h_{ij} + 2 |\mathrm{div}(h)|^2 - |\nabla v_h|^2 - \frac{1}{3} |d\beta|^2 \Big] dV$$

where $\Delta v_h = \nabla^i \nabla^j h_{ij}$

cf. Cao, Hamilton, and Ilmanen
Math. DG/0404165
for pure Ricci flow case.

$$\lambda''_{s=0} = \int_M \left[h^{ij} (\mathcal{L} h)_{ij} - \frac{1}{3} |d\beta|^2 \right] dV$$

Study: $\mathcal{L} h := \frac{1}{2} \Delta_L h + \mathrm{div}^* \mathrm{div}\, h + DD\sigma_h$

Know: $\mathcal{L} h \leq 0$ unless h is transverse traceless

$\mathcal{L} h < 0$ unless —"— or due to a diffeo or homothety

$\mathcal{L} h = \frac{1}{2} \Delta_L h$ if h is transverse traceless.

Flat Tori:

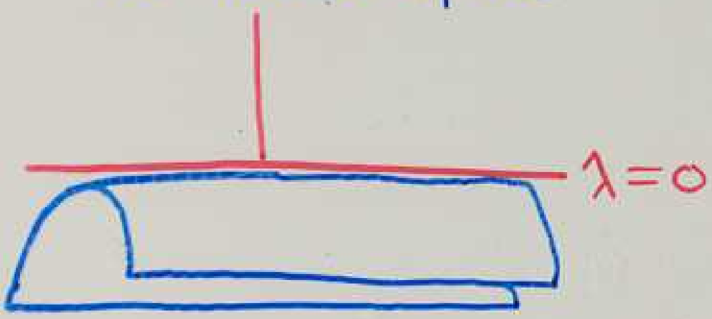
For h_{ij} transverse traceless then

$$\lambda''_{s=0} = \frac{-1}{2}\int_M \left[3\,|D_{(i}h_{jk)}|^2 + \frac{1}{3}|d\beta|^2 \right] dV$$

$\Rightarrow$ $\lambda''_{s=0} = 0$ only if h_{ij} is a Killing tensor and β is closed.

* Otherwise, the eigenvalues do not accumulate at zero.

* The Killing tensor directions integrate to give the moduli space



Linearly Stable and Isolated

since $\lambda \geq 0$ at FP and $\lambda > 0$ at any non-Ricci-flat FP.

K3 Surfaces

* Closed, $\dim_{\mathbb{C}}(K3) = 2$.

* $c_1(K3) = 0$

* No global holomorphic 1-form

* Siu: Every K3 admits a Kähler metric

* Yau: Each Kähler Class of a K3 contains
  a (unique) Ricci-flat Kähler metric

* Todorov: 57-dim moduli space of
  Ricci-flat metrics at each Kähler Ricci-flat
  metric; $\Delta_L < 0$ for all <u>other</u>
  transverse-traceless deformations h_{ij}.

$\Rightarrow$ Linearly Stable.

Zamolodchikov's C-Theorem

Complex Coordinates: $z = x + iy$, $\bar{z} = x - iy$

$$z\bar{z} = x^2 + y^2 = r^2$$

Stress-Energy: $T := T_{zz} := T_{\mu\nu}\, dz^\mu dz^\nu$

$$\bar{T} := T_{\bar{z}\bar{z}} := T_{\mu\nu}\, d\bar{z}^\mu d\bar{z}^\nu$$

$$\Theta := 4\,T_{z\bar{z}} = 4\,T_{\bar{z}z} = T^\mu_{\ \mu}$$

Conservation $T^{\mu\nu}_{\ \ ;\nu} = 0$: $\partial_{\bar{z}} T + \frac{1}{4}\partial_z \Theta = 0$

$$\partial_z \bar{T} + \frac{1}{4}\partial_{\bar{z}}\Theta = 0$$

On general grounds, assuming rotational symmetry:

$$\langle T(z,\bar{z})\,T(0,0)\rangle = F(z\bar{z})/z^4$$

$$\langle \Theta(z,\bar{z})\,T(0,0)\rangle = \langle T(z,\bar{z})\,\Theta(0,0)\rangle = G(z\bar{z})/z^3\bar{z}$$

$$\langle \Theta(z,\bar{z})\,\Theta(0,0)\rangle = H(z\bar{z})/z^2\bar{z}^2$$

Apply the first conservation equation to these to get:

$$z\bar{z}\,\Big(\underbrace{2F(z\bar{z}) - G(z\bar{z}) - \tfrac{3}{8}H(z\bar{z})}_{=:\,C(z\bar{z})}\Big)' = -\frac{3}{4}H(z\bar{z})$$

Reflection Positivity $\Rightarrow$ $H(z\bar{z}) \geq 0 \Rightarrow C'(z\bar{z}) \geq 0$,

and $= 0$ iff $\Theta = 0$.