Cycling Cycles: Dynamics near a Heteroclinic Network

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Introduction

A dynamical system

$$\dot{x} = f(x)$$

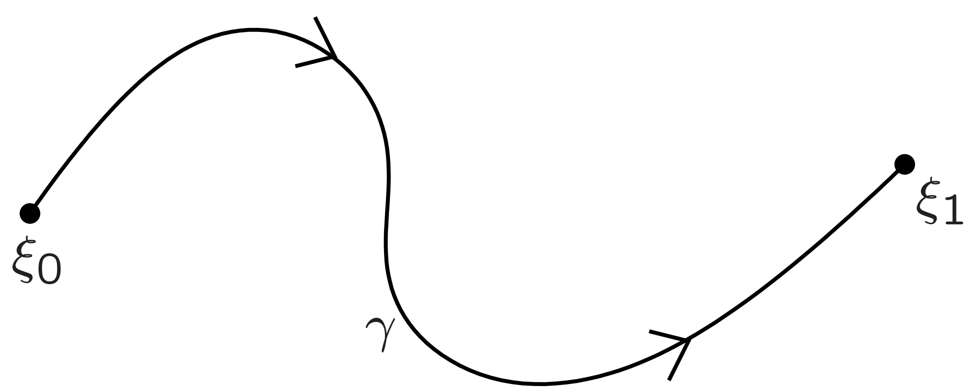
defines a flow $\phi_t(x)$ for $t \in \mathbb{R}$ and $x \in X$, we take $X = \mathbb{R}^n$.

ξ_0 an equilibrium $\Rightarrow f(\xi_0) = 0$.

A heteroclinic connection, $\gamma : \xi_0 \to \xi_1$ has

$$\phi_t \to \xi_1 \quad \text{as} \quad t \to \infty$$

$$\phi_t \to \xi_0 \quad \text{as} \quad t \to -\infty$$

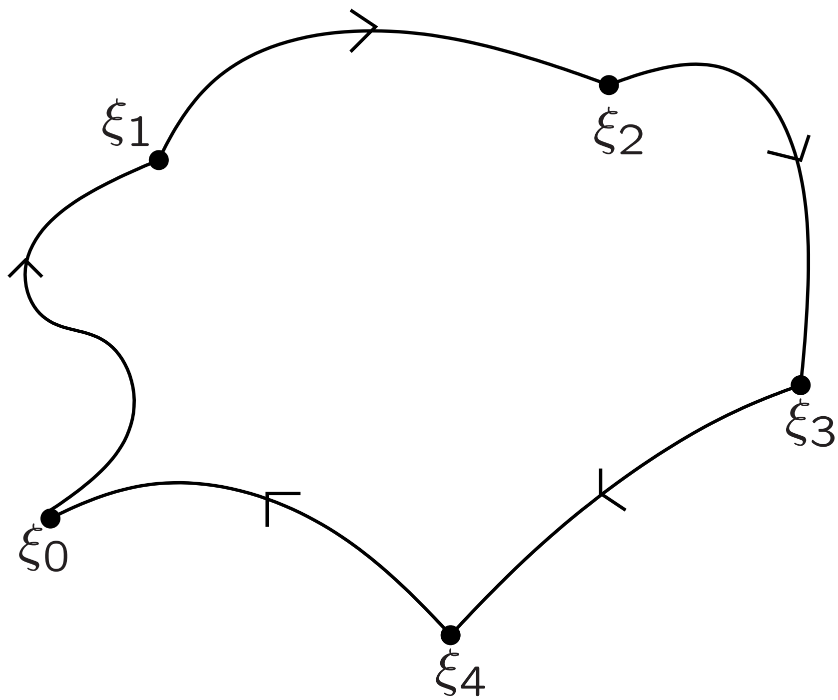


A heteroclinic cycle is a collection of equilibria

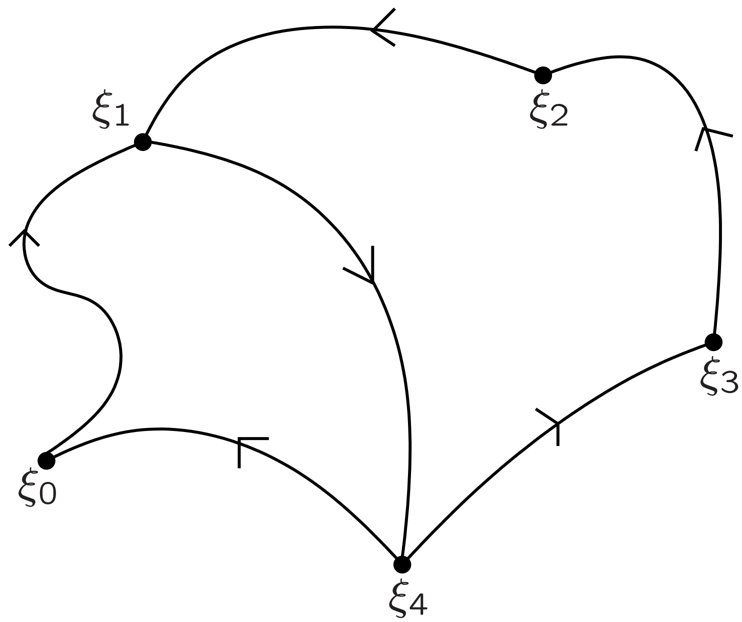
$$\{\xi_0, \xi_1, \ldots, \xi_n \equiv \xi_0\}$$

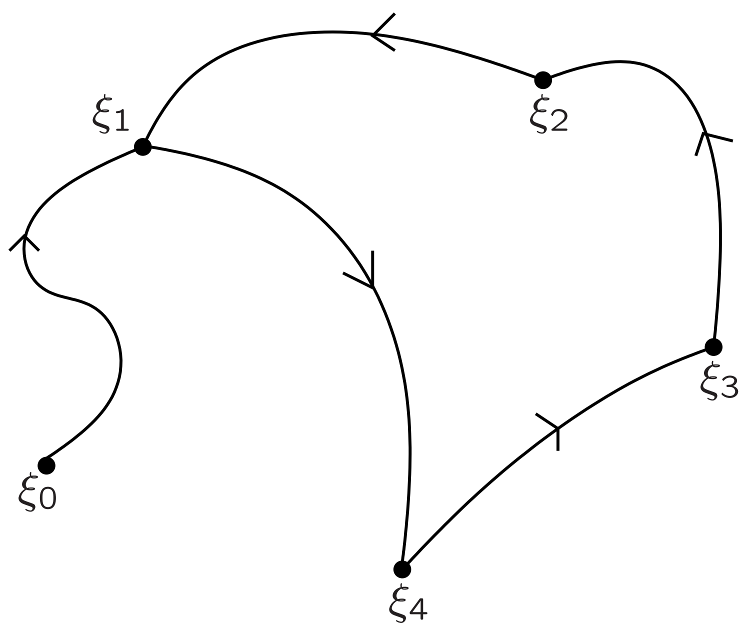
and heteroclinic connections

$$\{\gamma_i : \xi_i \to \xi_{i+1}\}$$



A heteroclinic network is a collection of equilibria $\{x_i\}$ and heteroclinic connections $\{\gamma_j\}$ which is chain recurrent.





The Guckenheimer-Holmes cycle in $\mathbb{R}^3$ has symmetries

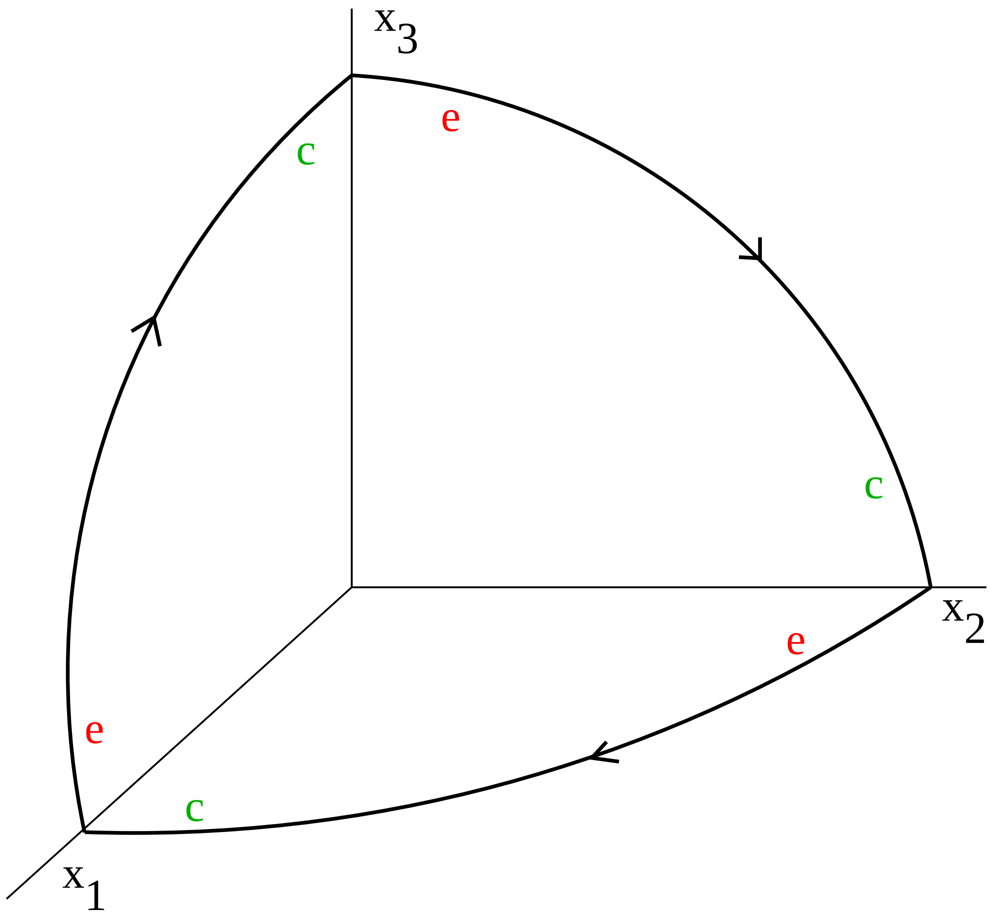
$$\mathbb{Z}_3 \ltimes (\mathbb{Z}_2)^3$$

$$\begin{aligned}
\kappa : (x_1, x_2, x_3) &\to (-x_1, x_2, x_3) \\
\rho : (x_1, x_2, x_3) &\to (x_2, x_3, x_1)
\end{aligned}$$

Most general equivariant equations, truncated at third order:

$$\begin{aligned}
\dot{x}_1 &= x_1(1 - \mathbf{X}^2 + e x_2^2 - c x_3^2) \\
\dot{x}_2 &= x_2(1 - \mathbf{X}^2 + e x_3^2 - c x_1^2) \\
\dot{x}_3 &= x_3(1 - \mathbf{X}^2 + e x_1^2 - c x_2^2)
\end{aligned}$$

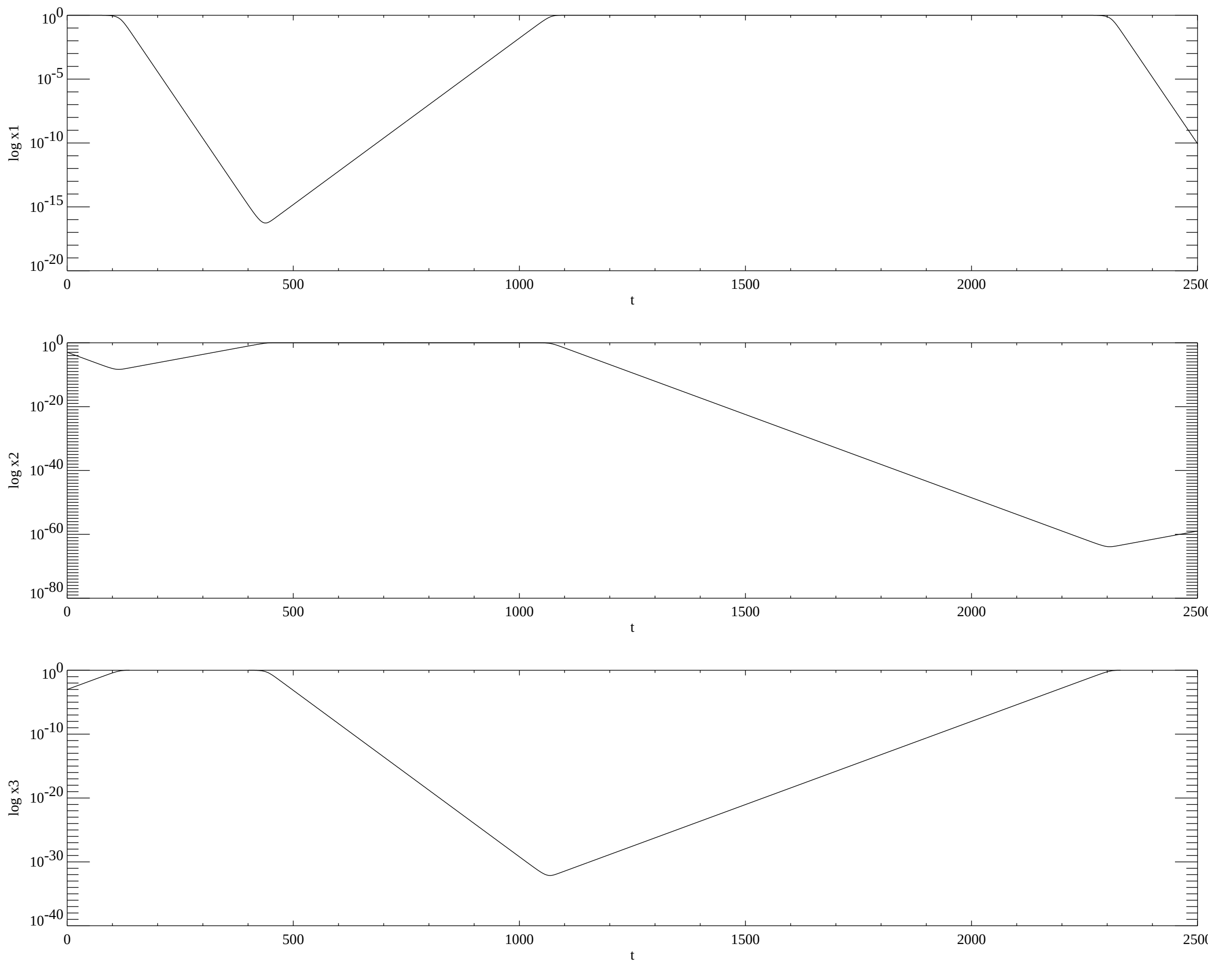
$$\mathbf{X}^2 = \sum_{i=1}^{3} x_i^2$$



Cycle is asymptotically stable if

$$c > e > 0$$

J. Guckenheimer and P. Holmes, Structurally stable heteroclinic cycles. *Math. Proc. Camb. Phil. Soc.* **103** (1988), 189-192.



$e = 0.5, \ c = 1.$

A heteroclinic network

We form a heteroclinic network by coupling together two stable G-H cycles in $\mathbb{R}^6$, with the symmetry

$$\mathbb{Z}_3 \ltimes (\mathbb{Z}_2)^6$$

acting reducibly on $\mathbb{R}^6$, with the action:

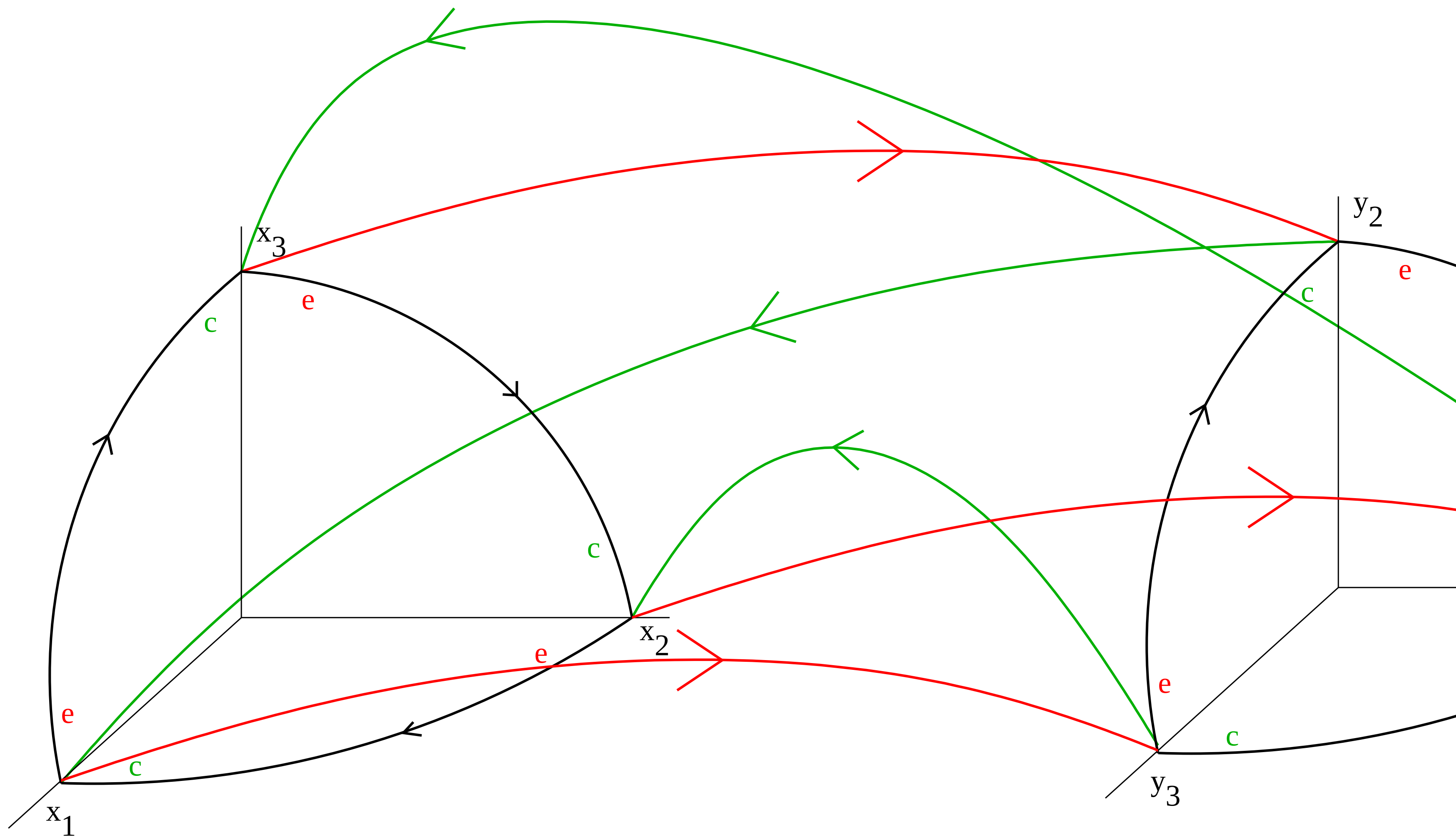
$$\begin{aligned}
\kappa_x : (x_1, x_2, x_3, y_1, y_2, y_3) &\to (-x_1, x_2, x_3, y_1, y_2, y_3) \\
\kappa_y : (x_1, x_2, x_3, y_1, y_2, y_3) &\to (x_1, x_2, x_3, -y_1, y_2, y_3) \\
\rho : (x_1, x_2, x_3, y_1, y_2, y_3) &\to (x_2, x_3, x_1, y_2, y_3, y_1)
\end{aligned}$$

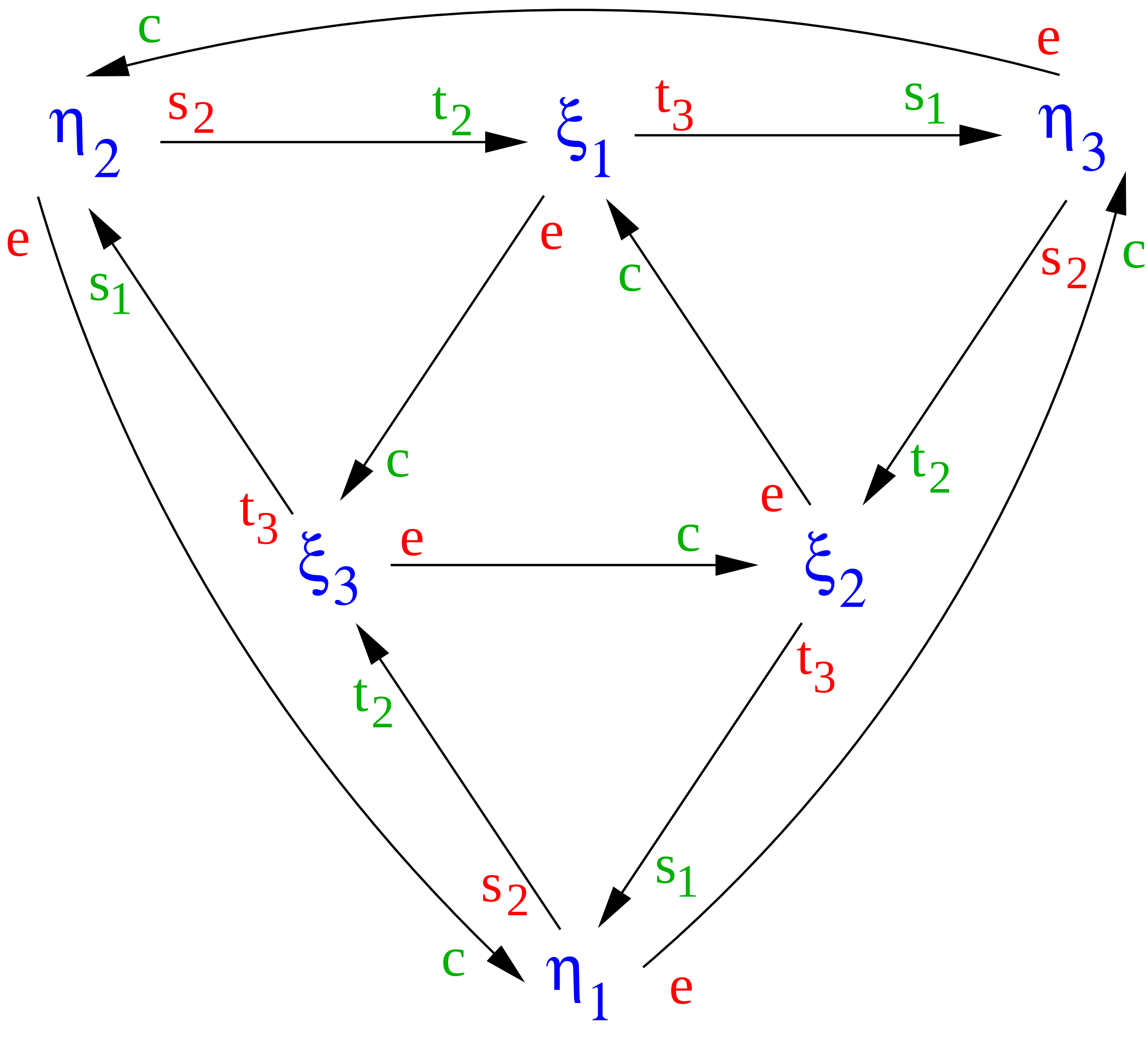
Most general equivariant equations, again truncated at third order:

$$
\begin{aligned}
\dot{x}_1 &= x_1(1 - \mathbf{X}^2 + ex_2^2 - cx_3^2 - s_3y_1^2 + s_2y_2^2 - s_1y_3^2) \\
\dot{x}_2 &= x_2(1 - \mathbf{X}^2 + ex_3^2 - cx_1^2 - s_3y_2^2 + s_2y_3^2 - s_1y_1^2) \\
\dot{x}_3 &= x_3(1 - \mathbf{X}^2 + ex_1^2 - cx_2^2 - s_3y_3^2 + s_2y_1^2 - s_1y_2^2) \\
\dot{y}_1 &= y_1(1 - \mathbf{X}^2 + ey_2^2 - cy_3^2 - t_1x_1^2 + t_3x_2^2 - t_2x_3^2) \\
\dot{y}_2 &= y_2(1 - \mathbf{X}^2 + ey_3^2 - cy_1^2 - t_1x_2^2 + t_3x_3^2 - t_2x_1^2) \\
\dot{y}_3 &= y_3(1 - \mathbf{X}^2 + ey_1^2 - cy_2^2 - t_1x_3^2 + t_3x_1^2 - t_2x_2^2)
\end{aligned}
$$

Pick $c > e > 0$ and then the G-H cycles in the $\mathbf{x}$ and $\mathbf{y}$ subspaces are stable.

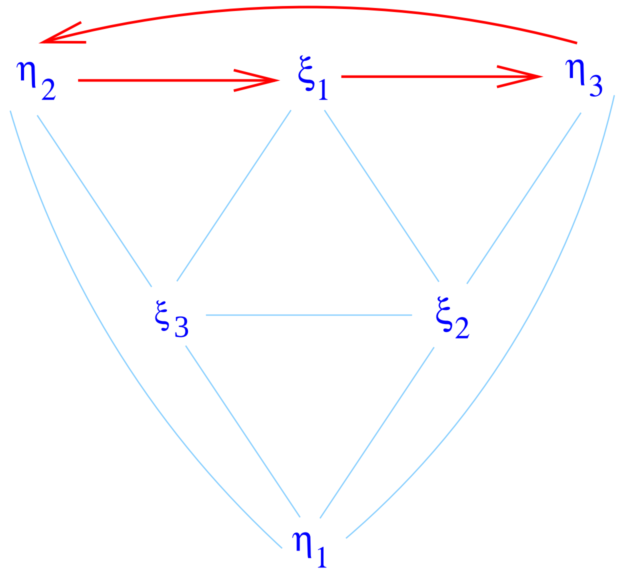
$t_i, s_i > 0 \Rightarrow$ a heteroclinic network.


x1
x2
x3
y2
y3
e
e
e
e
e
e
c
c
c
c
c
c

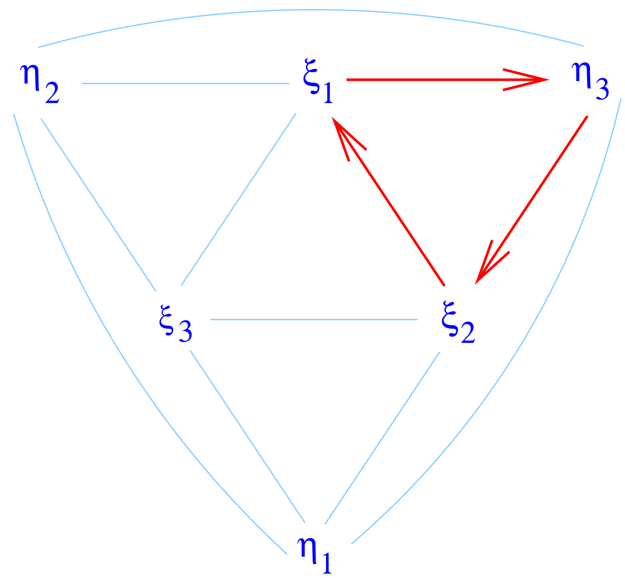


c
e
η2
s2
t2
ξ1
t3
s1
η3
e
s1
e
c
s2
c
c
t2
t3
ξ3
e
c
e
ξ2
t2
t3
s2
s1
c
η1
e

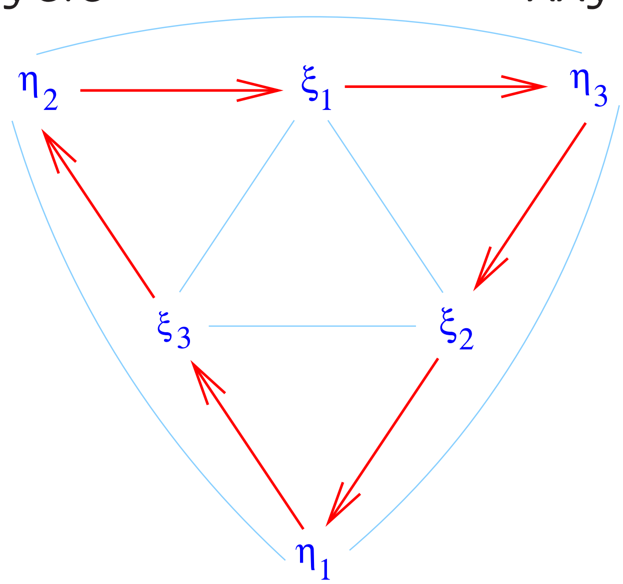
The heteroclinic network has many new types of cycle within it:



yyx-cycle



xxy-cycle



6-cycle

An invariant set Σ, is *essentially asymptotically stable (e.a.s.)* if there exists a set $\mathcal{A}$ such that given any number $a \in (0, 1)$, and any neighbourhood $\mathcal{U}$ of Σ, there is an open neighbourhood $\mathcal{V} \subset \mathcal{U}$ of Σ such that:

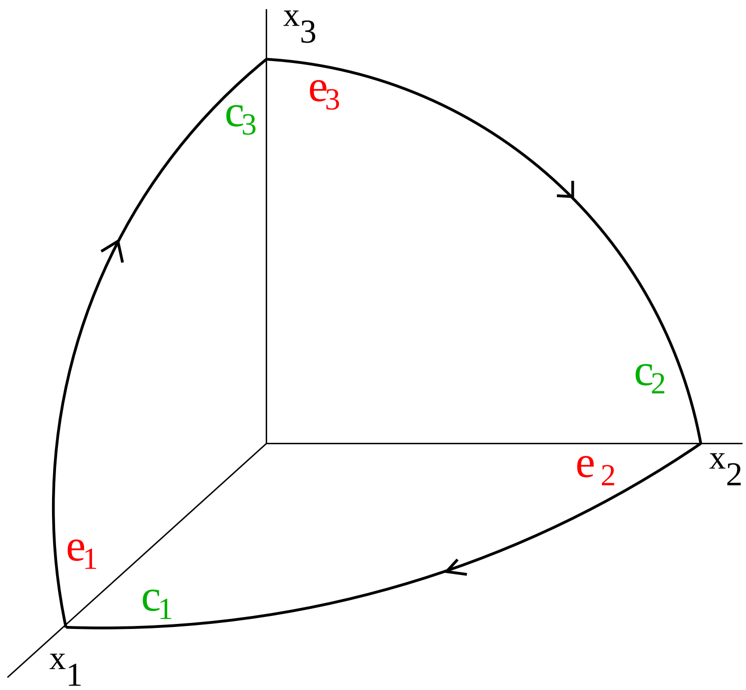
1. All trajectories starting in $\mathcal{V}/\mathcal{A}$ remain in $\mathcal{U}$,

2. All trajectories starting in $\mathcal{V}/\mathcal{A}$ are asymptotic to Σ,

3. $\mu(\mathcal{V}/\mathcal{A})/\mu(\mathcal{V}) > a$, where μ is Lebesgue measure.

If only 2 and 3 are satisfied, we say that Σ is *essentially quasi-asymptotically stable*.

I. Melbourne, An example of a non-asymptotically stable attractor. *Nonlinearity* **4** (1991), 835-844.



T_i is time spent near equilibrium on axis x_i.

Considering x_2 tells us that

$$c_1 T_1 = e_3 T_3$$

and

$$T_3 = \tfrac{c_1}{e_3} T_1 \quad T_2 = \tfrac{c_3}{e_2} T_3 \quad T_1' = \tfrac{c_2}{e_1} T_2$$

so

$$T_1' = \frac{c_1 c_2 c_3}{e_1 e_2 e_3} T_1$$

Consider a direction transverse to this cycle, x_4.

For a trajectory starting near x_1, what happens to x_4?

$$x_4 \to x_4 \mathrm{e}^{\nu_1 T_1}$$

$$t_1 T_1 + t_3 T_3 + t_2 T_2 = \left(t_1 + t_3 \frac{c_1}{e_3} + t_2 \frac{c_1 c_3}{e_2 e_3} \right) T_1 = \nu_1 T_1$$

Stability $\Rightarrow \nu_1 < 0$.

But if we start near x_3, then we get

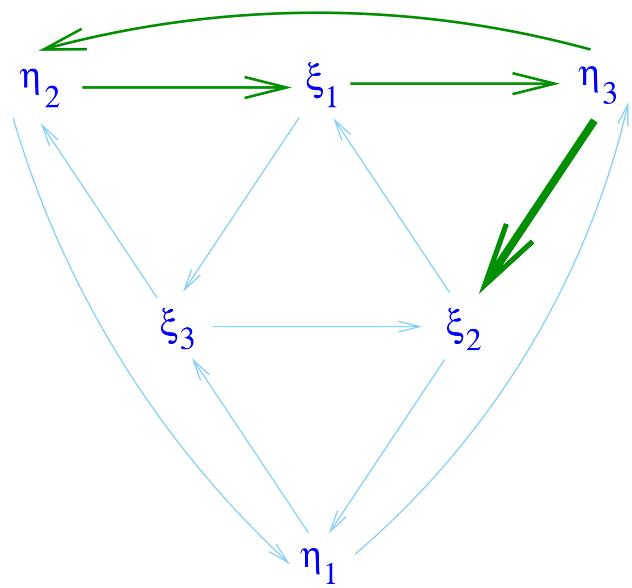
$$t_3 T_3 + t_2 T_2 + t_1 T_1 = \left(t_3 + t_2 \frac{c_3}{e_2} + t_1 \frac{c_2 c_3}{e_1 e_2} \right) T_3$$

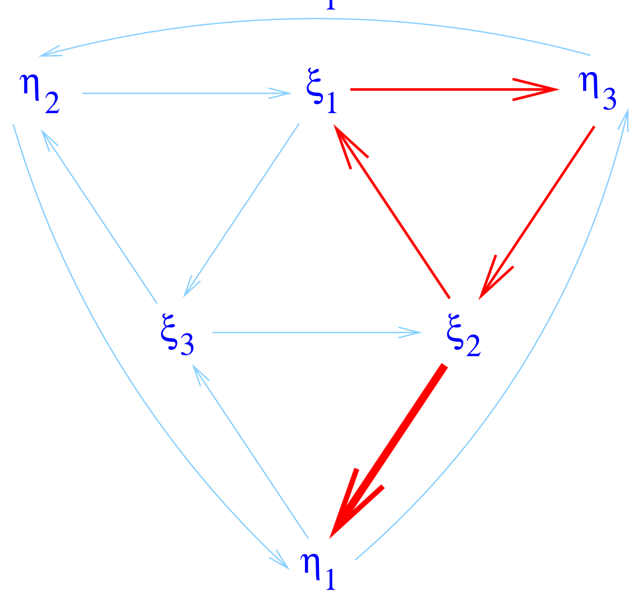
also different if start near x_2.

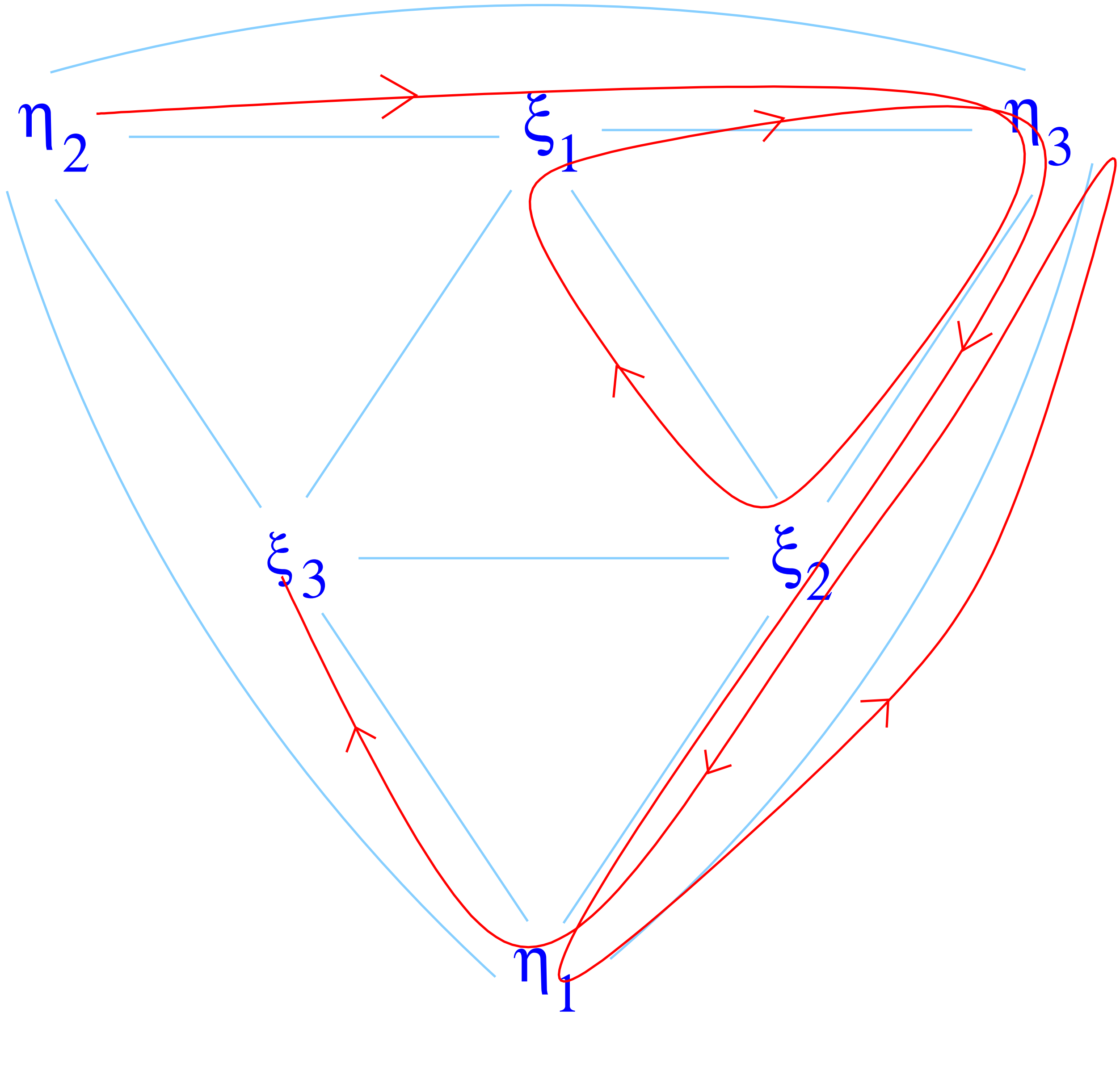
$\Rightarrow$ three conditions for each transverse direction.

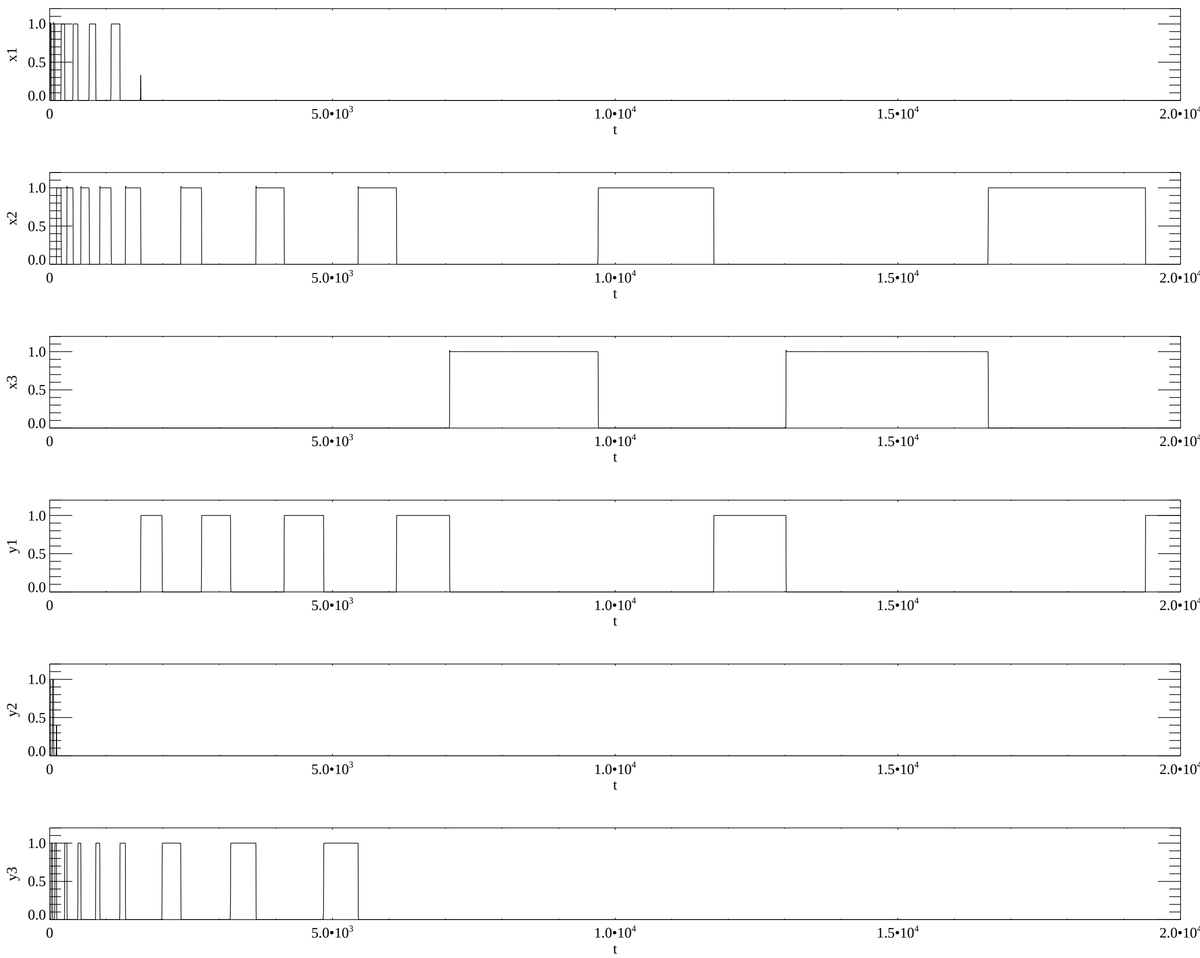
Cycling Cycles

If both types of cycle are unstable to a transverse direction we can see *cycling cycles*.






η_2
ξ_1
η_3
ξ_3
ξ_2
η_1



Numerical data from an integration of the equations.

	eq.		time		growth/decay factor of x_1
	η_3		T_0		
	ξ_2		$\frac{s_1}{e}T_0$		
0	ξ_1	n_0	$\frac{s_1^2}{es_2}T_0$	$\Delta_{n_0}\nu T_0$	
	η_3		$\delta^\star T_0$		$-s_1\delta^{\star n_0}T_0$
	ξ_2		T_1		eT_1
	η_1		$\frac{s_1}{e}T_1$		$-s_3\frac{s_1T_1}{e}$
1	η_3	n_1	$\frac{s_1^2}{es_2}T_1$	$\Delta_{n_1}\nu T_1$	$-s_1\frac{s_1^2T_1}{es_2}$
	ξ_2		$\delta^\star T_1$		$e\delta^\star T_1$
	η_1		T_2		$-s_3T_2$
	ξ_3		$\frac{s_1}{e}T_2$		$-c\frac{s_1T_2}{e}$
2	ξ_2	n_2	$\frac{s_1^2}{es_2}T_2$	$\Delta_{n_2}\nu T_2$	$s_2\frac{s_1^2T_2}{es_2}$
	η_1		$\delta^\star T_2$		$-s_3\delta^\star T_2$
	ξ_3		T_3		$-cT_3$
	η_2		$\frac{s_1}{e}T_3$		$s_2\frac{s_1T_3}{e}$
3	η_1	n_3	$\frac{s_1^2}{es_2}T_3$	$\Delta_{n_3}\nu T_3$	$-s_3\frac{s_1^2T_3}{es_2}$
	ξ_3		$\delta^\star T_3$		$-c\delta^\star T_3$
4	η_2		T_4		s_2T_4

The length of time spent on each cycle,

$$\mathcal{T}_i = T_i + \Delta_{n_i} \nu T_i$$

satisfies

$$\begin{aligned} \mathcal{T}_i = A_1(n_{i-1})\mathcal{T}_{i-1} + A_2(n_{i-2})\mathcal{T}_{i-2} \\ + A_3(n_{i-3})\mathcal{T}_{i-3} + A_4(n_{i-4})\mathcal{T}_{i-4} \end{aligned}$$

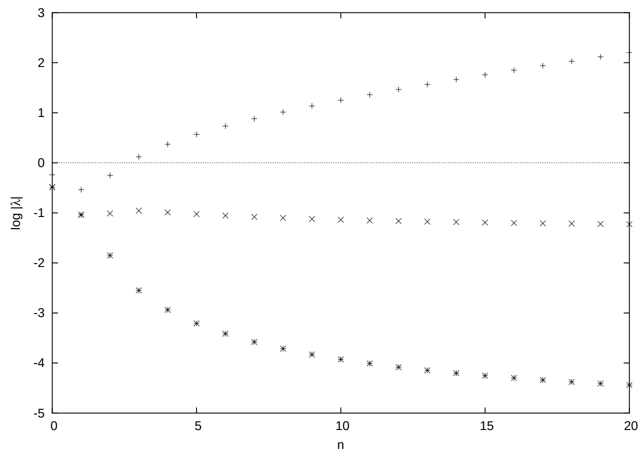
if $n_i = n$ it has solution

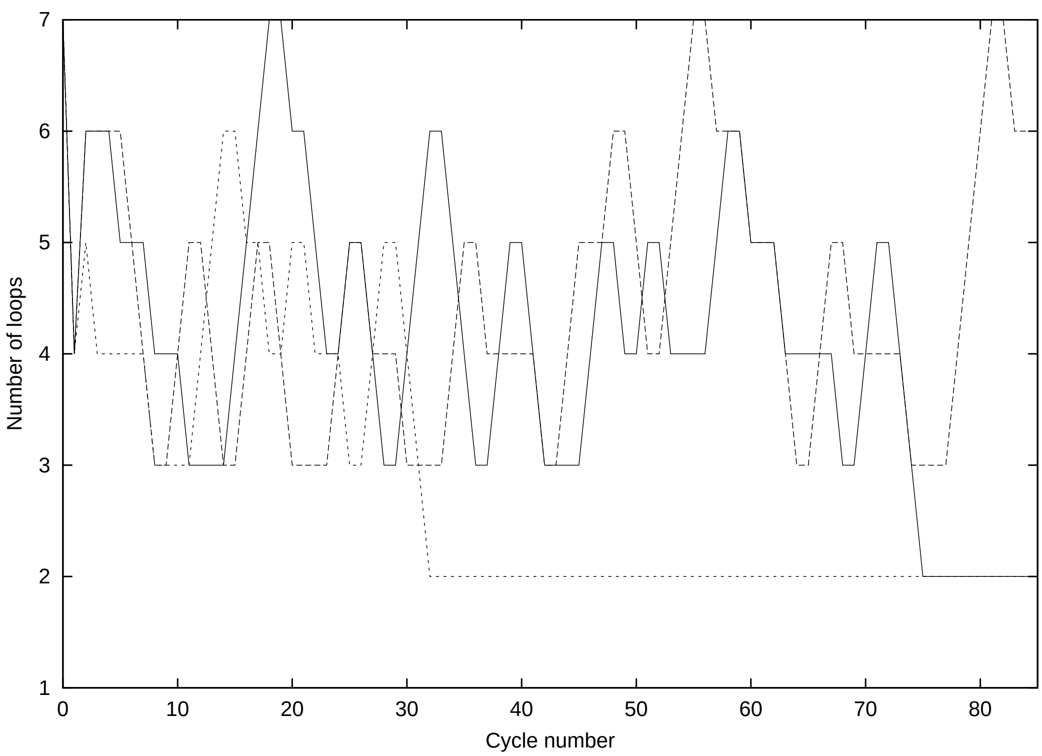
$$\frac{\mathcal{T}_i}{\mathcal{T}_{i-1}} \to \rho_1$$

where ρ_1 is the (unique) positive solution of

$$\rho^4 - A_1\rho^3 - A_2\rho^2 - A_3\rho - A_4 = 0$$

We can calculate the stability of solutions





Different initial conditions can give different n.

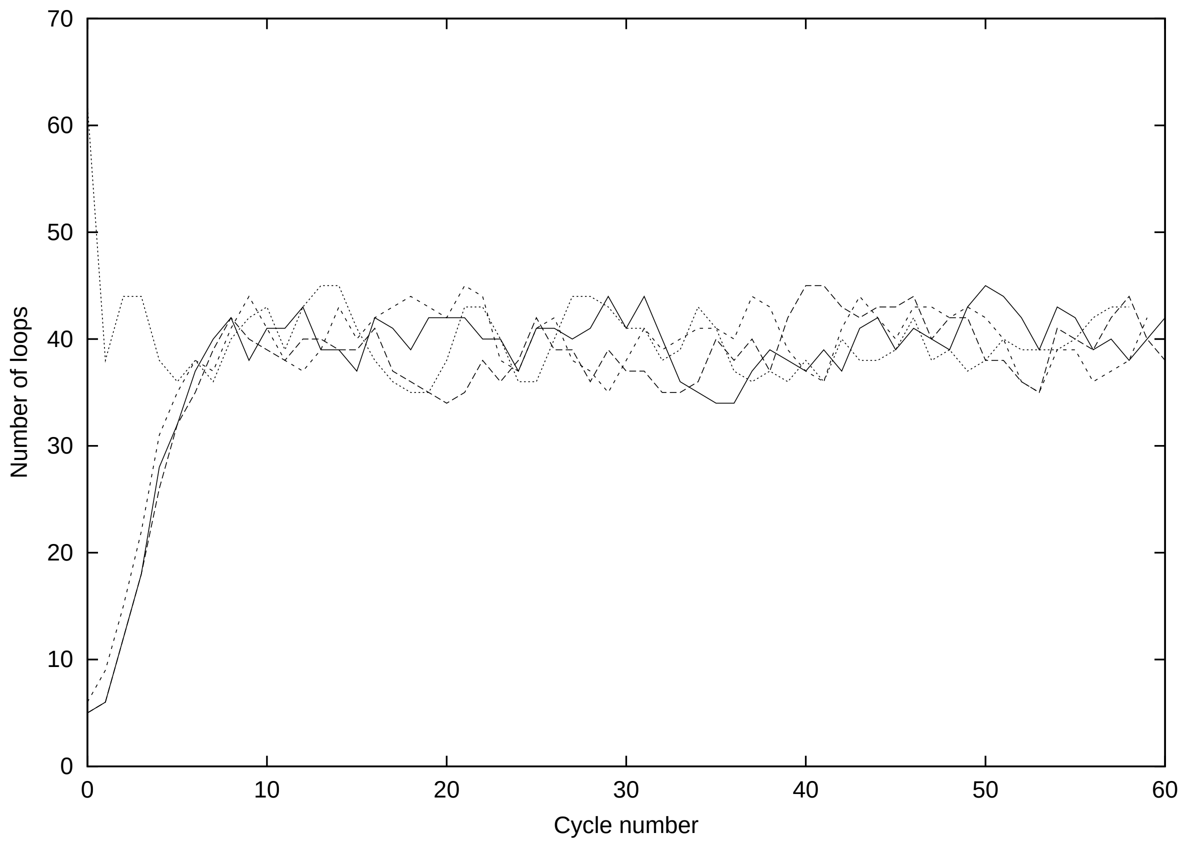
The ratio $r_i = \frac{T_i}{T_{i-1}}$ can be calculated analytically, or from the numerical integrations.

n	2	5
Analytic	2.66521546892	3.910207341
Num int	2.66521546933	3.910207

$$c = 1, \quad e = 0.5, \quad t_1 = s_3 = 0.8$$

$$t_2 = s_1 = 1.0, \quad t_3 = s_2 = 1.3$$

We can also have irregular cycling in regions where there are no stable solutions; trajectories with very close initial conditions diverge rapidly.



When n is not constant, but follows some sequence $\{n_i\}$, the recurrence relation is much more complicated:

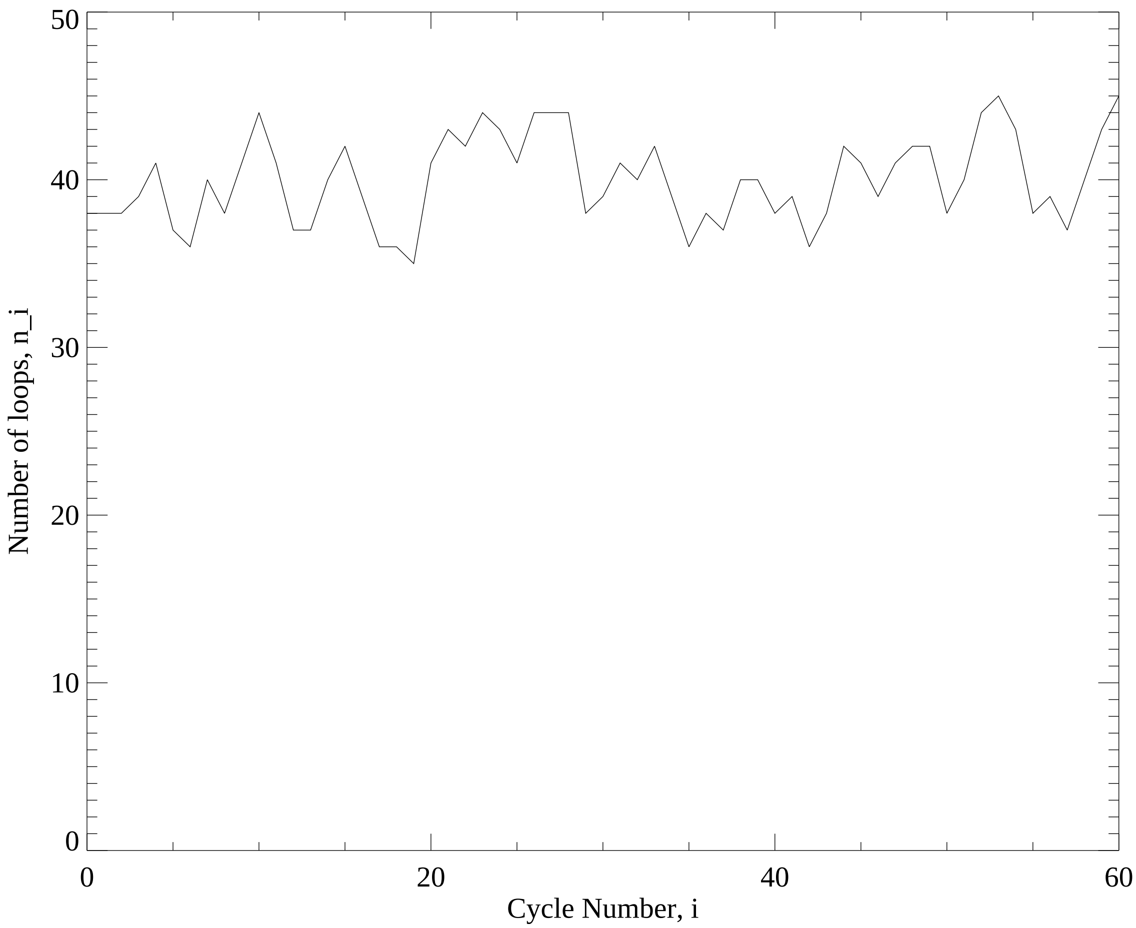
$$\begin{aligned} r_{i+1} &= A_1(n_i) + A_2(n_{i-1})\frac{1}{r_i} + A_3(n_{i-2})\frac{1}{r_i r_{i-1}} \\ &\quad + A_4(n_{i-3})\frac{1}{r_i r_{i-1} r_{i-2}} \end{aligned}$$

However, we can find an expression which determines the n_i:

$$\begin{aligned} G(n_{i+1}-1) &< A_2(n_i)\frac{1}{r_{i+1}} + A_3(n_{i-1})\frac{1}{r_{i+1} r_i} \\ &+ A_4(n_{i-2})\frac{1}{r_{i+1} r_i r_{i-1}} < G(n_{i+1}) \end{aligned}$$

where $G(n)$ is a strictly increasing function.

Iterating these two expressions together gives results which are qualitatively similar to the continuous integrations:



We can also combine the expressions to show that if all previous n, r are trapped within a band, then all future n, r will be also, if

$$
\begin{aligned}
\frac{A_2(n_{\mathrm{min}}) + \frac{A_3(n_{\mathrm{min}})}{r_{\mathrm{max}}} + \frac{A_4(n_{\mathrm{min}})}{r_{\mathrm{max}}^2}}{\frac{s_1}{e}\delta^{\star n_{\mathrm{min}}}} &> G(n_{\mathrm{min}}) \\
\frac{A_2(n_{\mathrm{max}}) + \frac{A_3(n_{\mathrm{max}})}{r_{\mathrm{min}}} + \frac{A_4(n_{\mathrm{max}})}{r_{\mathrm{min}}^2}}{\left(\frac{s_3}{c} + \frac{c}{s_2}\right)\delta^{\star n_{\mathrm{max}}}} &< G(n_{\mathrm{max}} - 1) \\
\frac{s_1}{e}\delta^{\star n_{\mathrm{max}}} &< r_{\mathrm{max}} \\
\left(\frac{s_3}{c} + \frac{c}{s_2}\right)\delta^{\star n_{\mathrm{min}}} &> r_{\mathrm{min}}
\end{aligned}
$$

Conclusions

- We find a robust heteroclinic network contained in a system of equations with some symmetries.

- Trajectories are found which are asymptotic to the entire network and may approach the network in a regular or an irregular way.

Further Work

- Adding in appropriate second order terms will break some of the heteroclinic connections, creating periodic orbits in their place. Some connections will remain however, and there may be more interesting behaviour.