1

Sectional Curvature and General Relativity.

1 2-Spaces.

Let (M,g) be a space-time and let $p \in M$. Let G_p be the 4-dimensional Grassmann manifold of all 2-dimensional subspaces (2-spaces) of the tangent space T_pM to M at p. Also let S_p, T_p and N_p denote the subsets of G_p consisting, respectively, of its spacelike, timelike and null members. Then S_p and T_p are diffeomorphic 4-dimensional (open) connected submanifolds of G_p and N_p is a closed compact connected 3-dimensional submanifold of G_p and is the boundary of S_p and of T_p. Finally, let $\overline{G_p} = S_p \cup T_p$ denote the submanifold of non-null members of G_p.

Define the Grassmann bundle G on M by $G = \bigcup_{p\in M} G_p$ and the non-null Grassmann bundle $\overline{G}$ by $\overline{G} = \bigcup_{p\in M} \overline{G}_p$ [1, 2]

Now let B_p denote the 6-dimensional vector space of all bivectors at p, regarded as the manifold $\mathbb{R}^6$. For $a, b \in B_p$, $a \neq 0 \neq b$, write $a \sim b \Leftrightarrow b = \lambda a$ $(0 \neq \lambda \in \mathbb{R})$. Then $\sim$ is an equivalence relation on B_p which gives rise to the quotient manifold PB_p (projective bivectors at $p \approx P\mathbb{R}^5$). Let SB_p denote the collection of simple bivectors at p. Then SB_p is a 5-dimensional submanifold of G_p and $\sim$ extends naturally to SB_p to give PSB_p (projective simple bivectors at p) the structure of a 4-dimensional manifold.

The manifold structures PSB_p and G_p on the collection of 2-spaces at p are <u>diffeomorphic</u>. Thus one can choose to view the collection of 2-spaces at p either geometrically (via G_p) or algebraically (via PSB_p) [2, 3]

2. The Sectional Curvature Function

Let $F \in \overline{G_p}$ be a non-null 2-space at p with corresponding (projective) bivector F^{ab}.

Define the <u>sectional curvature</u> of F (at p) by

$$\sigma_p(F) = \frac{R_{abcd} F^{ab} F^{cd}}{2 G_{abcd} F^{ab} F^{cd}} = \frac{R_{abcd} F^{ab} F^{cd}}{2 F_{ab} F^{cd}} \qquad (1)$$

$$\left(2G_{abcd} = g_{ac} g_{bd} - g_{ad} g_{bc}\right)$$

<u>Notes</u> (i) The denominator in (1) vanishes $\Leftrightarrow$ F null and hence the restriction of σ_p to $\overline{G_p}$. Then one can clearly extend σ_p to a map $\sigma : \overline{G} \to \mathbb{R}$.

(ii) Consider the collection of geodesics starting from p and with initial direction in F. The collection of points on these geodesics is (locally and rather roughly speaking) a 2-dimensional submanifold of M with an induced metric whose Gauss curvature at p is $\sigma_p(F)$.

(iii) In the positive definite case, σ_p is defined on the whole of the (compact) manifold G_p and is less interesting (?) than the Lorentz case (where $\overline{G_p}$ is not compact).

(iv) If $\sigma_p : \overline{G_p} \to \mathbb{R}$ is a <u>constant function</u> then, at p,

$$R_{abcd} \propto G_{abcd}$$

One now asks if the map $\sigma_p : \overline{G}_p \to \mathbb{R}$ can be extended continuously to any member of N_p. The answer is, that, if it can then σ_p is a constant map defined on G_p. So if σ_p is <u>not</u> a constant map it has a well defined maximum domain $\overline{G}_p$ and thus, in this way, knowledge of σ_p determines N_p. From knowledge of N_p, it can be shown that the null cone of g at p is determined [4].

Now suppose g and g' are Lorentz metrics on M which give rise to the same sectional curvature function σ. The knowledge of σ (and hence of its domain at each $p \in M$) and the remark in the above paragraph show that g and g' have the same null cone at each $p \in M$. Hence g and g' are conformally related on M; $g' = \phi g$, $\phi : M \to \mathbb{R}$. The equality of the sectional curvatures of g and g' then show that $R'_{abcd} = \phi^2 R_{abcd}$ (and $R'_{ab} = \phi R_{ab}$) and finally that $C'^{a}{}_{bcd} = \phi C^{a}{}_{bcd}$.

So $g'_{ab} = \phi g_{ab}$ and $C'^{a}{}_{bcd} = \phi C^{a}{}_{bcd}$ and it follows that either $C' = C = 0$ or $\phi = 1$ at each $p \in M$. More precisely [4, 5]

Theorem.

If σ_p is, for no $p \in M$, a constant function, then one may decompose M as

$$M = U \cup V \cup A \tag{2}$$

where U, V are open in M, A is closed and has empty interior in M and where $g' = g$ on U and where, on V, g and g' are, locally, conformally related, conformally flat plane waves.

Corollary.

If g and g' have the same sectional curvature function and g is non-flat and vacuum, then $g' = g$ on M.

[Here, non-flat means that the curvature tensor does not vanish over any non-empty open subset of M]

These results show that, at least in the vacuum case, the sectional curvature function may serve as an alternative variable to the metric for the gravitational field in general relativity.

3 Critical Point Structure.

For a space-time (M, g) we have the curvature decomposition

$$R_{abcd} = C_{abcd} + E_{abcd} + \frac{R}{6} G_{abcd} \qquad (3)$$

where R is the Ricci scalar and

$$E_{abcd} = \tilde{R}_{a[c}\, g_{d]b} + \tilde{R}_{b[d}\, g_{c]a} \quad , \quad \tilde{R}_{ab} = R_{ab} - \frac{R}{4} g_{ab} \qquad (4)$$

$$(\Rightarrow E^{c}{}_{acb} = \tilde{R}_{ab})$$

Thus we have the duality relations

$$^{*}C = C^{*}, \quad ^{*}G = G^{*}, \quad ^{*}E = -E^{*}$$

$$\Rightarrow \; ^{*}R^{*} + R = 2E \quad \text{and} \quad R - {}^{*}R^{*} = 2\left(C + \frac{R}{6} G\right) \qquad (5)$$

Now define two functions f_p and $h_p : S_p \to \mathbb{R}$ by

$$f_p(F) = \sigma_p(F) - \sigma_p(\overset{*}{F}), \qquad h_p(F) = \sigma_p(F) + \sigma_p(\overset{*}{F}) \qquad (6)$$

where $\overset{*}{F}$ is the dual of the (projective) bivector F (or the orthogonal complement of the 2-space F). Then

$$f_p(F) = \frac{E_{abcd} F^{ab} F^{cd}}{G_{abcd} F^{ab} F^{cd}} \, , \qquad h_p(F) = \frac{\left(C_{abcd} + \frac{R}{6} G_{abcd}\right) F^{ab} F^{cd}}{G_{abcd} F^{ab} F^{cd}} \qquad (7)$$

<u>Note</u> The function f_p uniquely determines E and hence $\tilde{R}$ ~~at~~ p and h_p uniquely determines R and C at p.

One can now examine the critical points of f_P and h_P which give information on the algebraic type of the Weyl tensor C and the energy-momentum tensor T. [6,7]

Function f_P		Function h_P	
No of crit pts	Segre type of T	No of crit pts	Petrov type of C
0	$\{31\}$, $\{(31)\}$	0	III
1	$\{211\}$, $\{2(11)\}$ $\{z\bar{z}11\}$, $\{z\bar{z}(11)\}$	1	II
3	$\{1,111\}$	3	I
∞	$\{(21)1\}$, $\{(211)\}$ All degeneracies of $\{1,111\}$	∞	N, D

<u>Note</u> M is an Einstein space $\Leftrightarrow \tilde{R} = 0 \Leftrightarrow E = 0$
$\Leftrightarrow \sigma_P(F) = \sigma_P(\overset{*}{F})$ for each $F \in \bar{G}_P$ [8]
$\Leftrightarrow f_P$ is the zero function
So for vacuum space-times, $f_P \equiv 0$ and $h_P \equiv 2\sigma_P$.

Remarks.

(i) Let $l \in T_pM$ be null and let $F \in S_p$ be such that $g(l, v) = 0$ for each $v \in S_p$. Then F is called a wave surface to l at p. [For a given l, the collection of wave surfaces $W_p(l)$ to l at p is a submanifold of G_p diffeomorphic to $\mathbb{R}^2$.]

Now suppose M is a non-flat vacuum space time and, for a null $l \in T_pM$, σ_p is a constant (in fact bounded is enough) function on $W_p(l)$. Then either $C(p) = 0$ or $C(p)$ is algebraically special in the Petrov classification with l a repeated principal null direction of $C(p)$. If this property of σ_p holds over an open subset U of M on which C is nowhere zero and l a null vector field on U then l defines a null geodesic shearfree congruence [9]

(ii) The result that $\sigma_p(F) = \sigma_p(\overset{*}{F})$ $\forall F \in S_p$ is equivalent to the Einstein space condition can be generalised [3]. Also, it can be shown that the condition $\sigma_p(F) + \sigma_p(\overset{*}{F})$ is a constant independent of F is equivalent to the Weyl tensor vanishing at p. [GH and L MacNay - unpublished].

4 Sectional Curvature Preserving Symmetries

Let $\psi : U \to V$ be a local diffeomorphism on M, for open subsets U, V of M. Let $p \in U$, let $q = \psi(p)$ and if $F \in G_p$ is spanned by $x, y \in T_pM$ let $F' \in G_q$ be spanned by $\psi_*(x)$ and $\psi_*(y)$. Call ψ sectional curvature preserving [4] if $\sigma_p(F) = \sigma_q(F')$ in the sense that either each is defined and they are equal, or neither is defined. It follows (in an obvious sense) that ψ_* maps N_p bijectively onto N_q and then that ψ_* maps T_p and S_p bijectively onto T_q and S_q, respectively. Thus ψ preserves the null cone of g and so $\psi^* g = \lambda g$ for some function $\psi : U \to \mathbb{R}$. It follows that ψ is conformal.

Now let X be a vector field on M. Call X sectional curvature preserving if each of its local flow diffeomorphisms is sectional curvature preserving. It follows that X is a conformal vector field.

Thus $\mathcal{L}_X g = 2\chi g$ for $\chi : M \to \mathbb{R}$ and it can also be shown that

$$\mathcal{L}_X R_{abcd} = 4\chi R_{abcd}, \quad \mathcal{L}_X R_{ab} = 2\chi R_{ab}, \quad \mathcal{L}_X C^a{}_{bcd} = 2\chi C^a{}_{bcd} \tag{8}$$

(These could also be deduced from the work in section 2).

From the previous work, any sectional curvature preserving map is an isometry (except, essentially, when (M,g) is a conformally flat plane wave) and then the above vector field X is Killing). Further, it can be shown that for conformally flat plane waves, a conformal vector field X $(\mathcal{L}_X g = 2\chi g)$ is sectional curvature preserving $\Leftrightarrow \mathcal{L}_X R_{ab} = 2\chi R_{ab}$. It also follows that, for such a plane wave [10].

$$ds^2 = a(u)(x^2+y^2)\,du^2 + 2\,du\,dv + dx^2 + dy^2 \tag{9}$$

the function $\chi = \chi(u)$. Also, for (9), one has $R_{ab} = B(u)\,l_a l_b$ $(l_a = u_{,a})$ and if B is assumed nowhere zero, a conformal vector field X with $\mathcal{L}_X g = 2\chi(u) g$ is sectional curvature preserving $\Leftrightarrow$ $\dot{\chi} = -\chi B$.

It is remarked that examples of <u>distinct</u> (conformally related) plane waves with the same sectional curvature functions can be constructed [2,5,10].

As for the collection of sectional curvature preserving vector fields, say $S(M)$ of a plane wave M, it follows that $S(M)$ is a Lie algebra (since $X \in S(M)$ $\Leftrightarrow$ $\mathcal{L}_X g = 2\psi g$ <u>and</u> $\mathcal{L}_X R_{ab} = 2\psi R_{ab}$) and that $\dim S(M) \geq 6$ (since the Killing algebra $K(M)$ satisfies $\dim K(M) \geq 6$ — and for <u>homogeneous</u> plane waves $S(M) = K(M)$ with $\dim S(M) = \dim K(M) = 7$). In fact, $\dim S(M) \leq 8$ (and examples with $\dim S(M) = 7, 8$ exist)

5 Quadric Surfaces.

For $p \in M$ consider first the complexified tangent space $\mathbb{C}^4$ at p and then the set $P\mathbb{C}^3$ of all complex directions at p. Then define the set (the fundamental quadric) $\mathcal{B}$ of all complex null directions at p by (c.f [11])

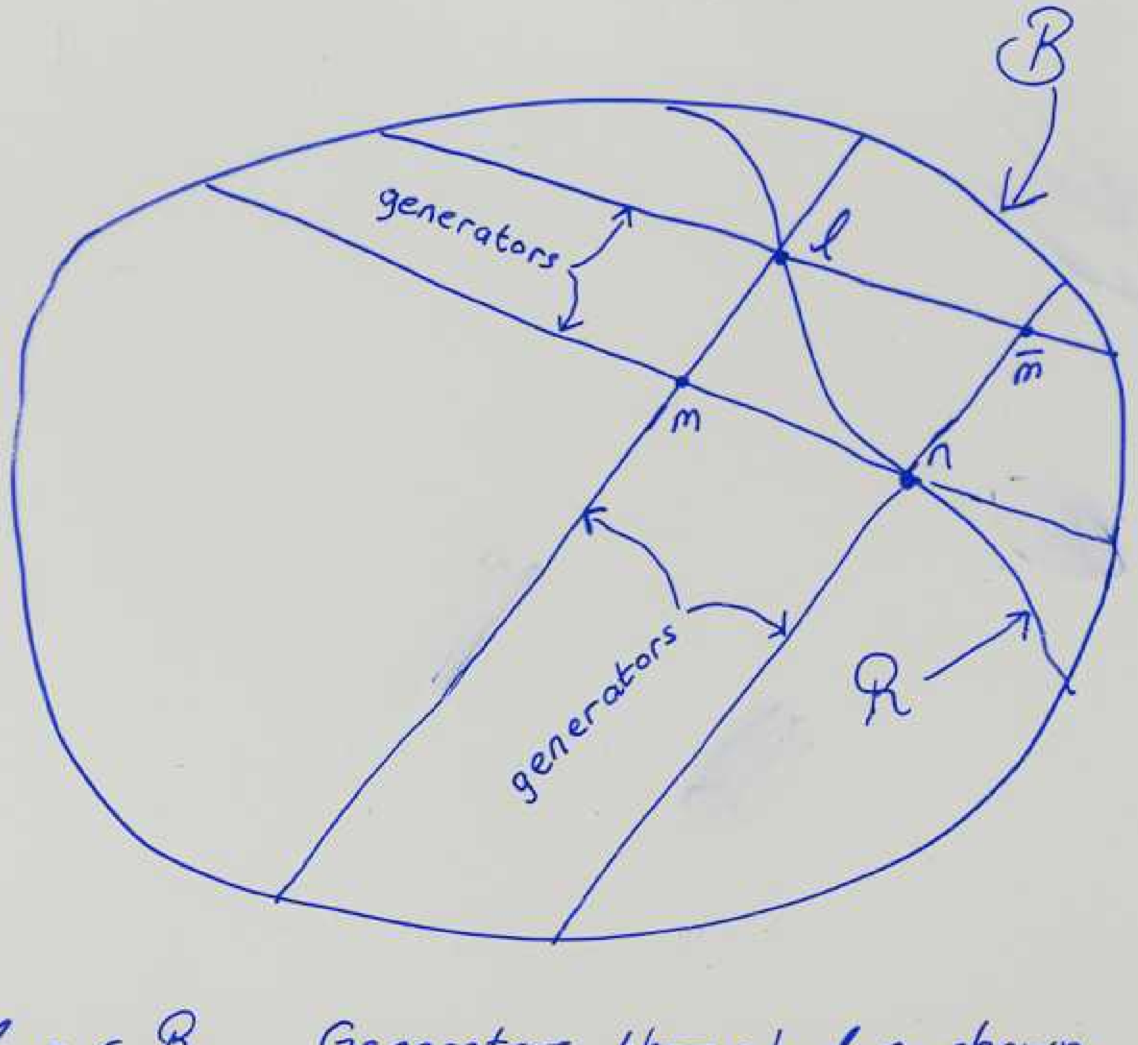
$$g_{ab} x^a x^b = 0 \tag{10}$$

where the x^a are the homogeneous coordinates in $P\mathbb{C}^3$. Let $\mathcal{R} \subseteq \mathcal{B}$ denote the reality section, the set of all complex multiples of all real null vectors at p.

Then $\mathcal{B}$ and $\mathcal{R}$ are compact submanifolds of $P\mathbb{C}^3$ and $\mathcal{B} \approx S^2 \times S^2$ and $\mathcal{R} \approx S^2$.

If $m \in \mathcal{B} \setminus \mathcal{R}$ say $m = x + iy$ $x, y \in T_p M$ then $m^a m_a = 0 \Leftrightarrow x^a x_a = y^a y_a$ and $x^a y_a = 0$.
So the real and imaginary parts of (any complex multiple of) m determine the same 2-space in S_p. (and similarly for $\bar{m} \in \mathcal{B} \setminus \mathcal{R}$). In fact, this gives rise to a 2:1 map $f : \mathcal{B} \setminus \mathcal{R} \to S_p$ which is a double cover of S_p. Then the discontinuous group action $m \to \bar{m}$ on $\mathcal{B} \setminus \mathcal{R}$ shows that $S_p \approx$ a quotient manifold $\tilde{\mathcal{B}}$ of $\mathcal{B} \setminus \mathcal{R}$.

Now σ_p is uniquely determined by its values on S_p. Thus σ_p may be regarded as a map $\sigma_p : \mathcal{B} \to \mathbb{R}$ or a map $\tilde{\mathcal{B}} \to \mathbb{R}$. The use of $\mathcal{B}$ and its associated geometry is often useful for visualising σ_p [12].



$l, n \in \mathcal{R}$ Generators through l, n shown.

$m = x + iy \quad \bar{m} = x - iy$

x, y span member of S_p.

$(l, n, m, \bar{m})$ is a "complex null tetrad" at p.

References

[1] GH and A.D. Rendall. Gen. Rel. Grav. 19, (1987), 771.

[2] GH. "Symmetries and Curvature Structure in General Relativity" World Sci 2004

[3] GH and L. MacNay. Class Q. Grav 22, (2005), 1493.

[4] GH. Gen. Rel. Grav. 16, (1984), 79.

[5] B. Ruh. Math Zts, 189, (1985), 371.

[6] J. A. Thorpe. J. Math. Phys. 10, (1969), 1.

[7] W. J. Cormack and GH. J. Phys A 12, (1979), 55.

[8] J Ehlers and W Kundt in "Gravitation; an introduction to current research" ed. L. Witten, Wiley, 1962.

[9] GH. Zeits Naturforsch. 33a, (1978), 659.

[10] GH, A. D. Hossack and J. R. Pulham J. Math. Phys 33, (1992), 1408.

[11] R. Penrose. "Gravitatsya" A. Z. Petrov Festschrift Naukdumka, Kiev, (1972) 203.

[12] G. H. Proceedings of the 1st Hungarian Relativity Workshop. ed Z. Perjés Balatonszeplak, (1985), 141.