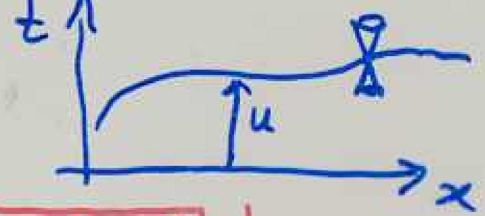
Area & mean curvature

Spacelike hypersurface

$M = \text{graph}\, u = \{(x, u(x)),\ x \in \Omega \subset \mathbb{R}^n\}$



$\text{Area}(M) = \int_\Omega \sqrt{1-|Du|^2}\, dx$

<u>Variational Problem</u>:

maximise $A_h(u) = \int_\Omega \left(\sqrt{1-|Du|^2} - hu\right) dx$

over all $u \in \text{Lip}(\Omega)$ with

boundary condition $u|_{\partial\Omega} = \varphi$

where $h(x,t)$, $h \in C^\infty(\Omega \times \mathbb{R})$

<u>Euler-Lagrange equation</u>

$$(*) \quad D_i\left(D_i u / \sqrt{1-|Du|^2}\right) = h(x, u(x))$$

<u>Dirichlet Problem</u>: $(*)$ with boundary condition $u(x) = \varphi(x)$, $x \in \partial\Omega$

Hypersurface geometry

Hypersurface $M = \text{graph}\, u \subset \mathbb{R}^{n,1}$

Tangent vectors $X_i = \partial_i + D_i u\, \partial_t$

Induced metric $g_{ij} = \delta_{ij} - D_i u\, D_j u$

Volume form $\sqrt{\det g_{ij}} = \sqrt{1 - |Du|^2}$

Inverse metric $g^{ij} = \delta_{ij} + \dfrac{D_i u\, D_j u}{1 - |Du|^2}$

Normal vector $N = \nu(\partial_t + D_i u\, \partial_i)$

Gradient factor $\nu = 1/\sqrt{1 - |Du|^2}$

Extrinsic curvature

$$K_{ij} = \langle X_i, \nabla_{X_j} N \rangle = -\langle \nabla_{X_i} X_j, N \rangle$$

$$= \nu D^2_{ij} u$$

Mean curvature

$$\begin{aligned} \operatorname{tr}_M K &= \nu\, g^{ij} D^2_{ij} u \\ &= D_i \left(D_i u / \sqrt{1 - |Du|^2} \right) \\ &= \frac{1}{\sqrt{1 - |Du|^2}} \left(\Delta_x u + \frac{D_i u\, D_j u\, D^2_{ij} u}{1 - |Du|^2} \right) \\ &= \sqrt{1 - |Du|^2}\; \Delta_M u \end{aligned}$$

Hypersurfaces in general spacetime

Time function $t \in C^\infty(V^{n+1})$

∇t is past-timelike

$T = -\alpha \nabla t$ reference unit future timelike vector

$\alpha^{-2} = -g^V(\nabla t, \nabla t)$

Mean curvature

$$\Delta_M u = \alpha^{-1} v\, H_M + \mathrm{div}_M \nabla t$$

height $u = t|_M$, gradient factor $v = -g^V(T,N)$

Second Variation of Area:

variation $F : M \times (-\varepsilon, \varepsilon) \to V$
$X = F_*(\partial_s)$

$$\Rightarrow \frac{d}{ds} H(s) = -\Delta_M\, g^V(X,N) + g(X^T, \nabla^M H_M) + g^V(X,N)\left(|K|^2 + \mathrm{Ric}^V(N,N)\right)$$

$\Rightarrow$ (choose $X = T$, $v = -g^V(T,N)$)

$$\Delta_M v = v\left(|K|^2 + \mathrm{Ric}^V(N,N)\right) - g(T, \nabla^M H_M) + \frac{d}{ds} H(s)$$

Alternate formula for $\frac{d}{ds} H(s)$

using Killing tensor

$$\mathcal{L}_X g^{v} = L(X) = 2X_{(\mu|\nu)}$$

If X is a Killing vector $L(X)=0 \Rightarrow \frac{d}{ds} H(s)=0$

$$\Rightarrow \frac{d}{ds} H(s) = \frac{1}{2} \nabla_N L(X)(e_i, e_i) - \nabla_{e_i} L(X)(N, e_i) - L(X)(e_i, e_j)\, K(e_i, e_j) - \frac{1}{2} H\, L(X)(N,N)$$

where H = mean curvature of M,

$H(s)$ = m.c. of $M_s = F(M, s)$

e_i = orthonormal frame on M

N = future unit normal to M

∇ = spacetime covariant derivative

Gradient estimates

Suppose $F = F(x,t,T) \in C^1(T^*M)$

$g^V \in C^2(V)$

If M has prescribed mean curvature

$$H_M = F(x, u(x), N(x))$$

then we have apriori estimates for v

case(i) $\partial M = \emptyset$, M compact

$$v \leq 2\exp(K \operatorname{osc}_M u)$$

for some $K = K(\|g^V\|_{C^2}, \|F\|_{C^1})$

case(ii) $\partial M \neq \emptyset$, M compact.

$$v \leq 2\exp(K \operatorname{osc}_M u) \sup_{\partial M} v$$

case(iii) $\partial M \neq \emptyset$, M compact and $\|H_{\partial M}\|_{C^0} < \infty$

$$v \leq 2\exp(K \operatorname{osc}_M u)$$

provided $u|_{\partial M} = 0$, $K = K(\|g^V\|_{C^2}, \|F|_M\|_{C^1}, \|H_{\partial M}\|_{C^0})$

<u>Method</u>: apply maximum principle to test function $f = Ku + \log v$

(where K is a constant to be determined)

At a max point of f:

(i) $\nabla^M f = 0$, $\Delta_M f \le 0$ interior max.

or (ii) $D_n f \ge 0$, $D_A f = 0$ boundary max

(n = outer normal, e_A tang ∂M)

(i) $\Rightarrow$ $0 = K v \nabla u + \nabla v$

$$0 \ge K v \Delta u - K^2 v |\nabla u|^2 + \Delta v$$

Now use $\Delta u, \Delta v$ formulae $\Rightarrow$

$$\Delta u \ge -Cv^2, \quad \Delta v \ge \boxed{v|A|^2} - C(v^3 + v^2|A|)$$

and $|\nabla v|^2 \le (1+\varepsilon) v^2 \lambda_1^2 + C v^4$

$$\Rightarrow v^2|A|^2 \ge (1+\tfrac{1}{2n})|\nabla v|^2 - Cv^4$$

$$\ge (1+\tfrac{1}{2n}) K^2 v^2 |\nabla u|^2 - Cv^4$$

$$\Rightarrow 0 \ge \tfrac{1}{2n} K^2 v |\nabla u|^2 - CKv^3$$

Thus if we choose $K > 2nC$ then $v \le 2$ ...

(ii) $q \in \partial M \Rightarrow u(q) = 0 \Rightarrow n(q) \propto \pm \nabla^M u$ ...

Results

Dirichlet problem:

given boundary mfld Σ^{n-1} s.t. $\exists$ strictly spacelike $\tilde{M}$ s.t. $\partial\tilde{M} = \Sigma$, and m.c. function $F \in C^1(TV)$, then $\exists$ hypersurface M

$$H_M = F(\cdot, N(\cdot)) = F|_M$$

(assuming $\tilde{M}$ compact, $\|F\|_{C^1} < \infty$ and spacetime metric $\|g\|_{C^2} < \infty$.)

Variational problem:

maximise $\text{area}(M) - \int_{\text{Vol}(M_0, M)} F\, dv$

over all achronal hypersurfaces M with $\partial M = \partial M_0$, where $F \in C^1(V)$

Then, $\exists$ M with $H_M = F|_M$, strictly spacelike except on singular null set $\Xi \subset M$,

$\Xi = \bigcup_{\gamma}$ null geodesics γ with endpts $\in \partial M$

(If M_0 is acausal then $\Xi = \emptyset$.)

Uniqueness

[Brill. Flaherty 76]

Suppose M_0, M_1

compact hypersurfaces,

either $\partial M_0 = \emptyset = \partial M_1$ or $\partial M_0 = \partial M_1$,

and $H_{M_0} \leq H_{M_1}$ and Timelike Convergence

Condition $Ric(T,T) \geq 0$ $\forall$ timelike T.

Then $M_1 \subset I^-(M_0)$ and $M_1 = M_0$, $H_1 = H_0$

if $M_1 \cap M_0 \neq \emptyset$.

Follows from variation of arclength,

applied to maximal length timelike curve

from M_0 to M_1

$$
\begin{aligned}
0 &\geq \frac{d^2}{ds^2} L(s) \\
&\geq \int_0^1 Ric(\gamma', \gamma')\, dt + H_1(\gamma(1)) - H_0(\gamma(0))
\end{aligned}
$$

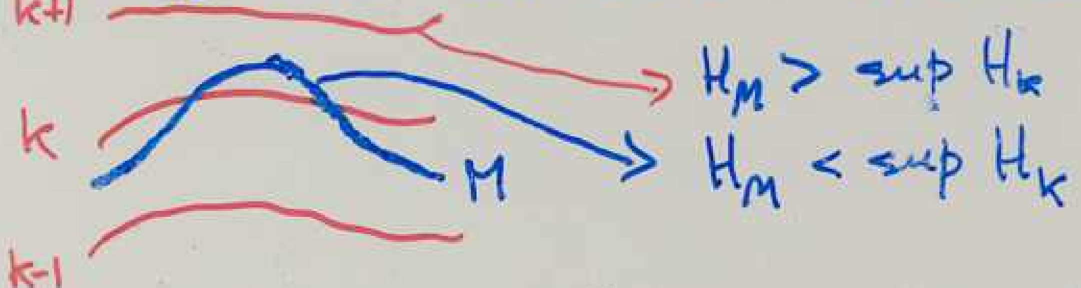
A <u>future</u> <u>crushing</u> <u>singularity</u> is a seq. of (compact) Cauchy surfaces S_k, $k=1,2,\dots$, such that

$$S_{k+1} \subset I^+ S_k \qquad k = 1,2,\dots$$

with mean curvatures H_k (not necessarily constant) s.t.

$$\lim_{k\to\infty} \sup_{S_k} H_k = -\infty$$

The S_k form barriers to CMC surfaces



<u>NB</u> past crushing singularities are defined similarly

<u>NB</u> compactness can be weakened: cf crushing singularities in Schwarzschild and Kerr.

Asymptotically flat maximal

Spacetime $V \approx \mathbb{R}^n \times \mathbb{R}$, smooth,
AF in region $r = |x| \geq 10$, with
"Bounded geometry" in $\{(x,t) : r \leq 10\}$
Then $\exists$ M, maximal ($H_M = 0$), AF,
Cauchy surface.

Sketch

(1) solve DP for M_R, $\partial M_R = \{t=0, |x|=R\}$
and $\{M_R\}_{R\to\infty}$ converges if $\exists$ uniform
height bound.

(2) radial barriers w_R with $H(w_R) \leq -c/r^3$
$\Rightarrow$ either $u = u_{max}$ inside $r < 100$, or u
controlled by barriers & ok.

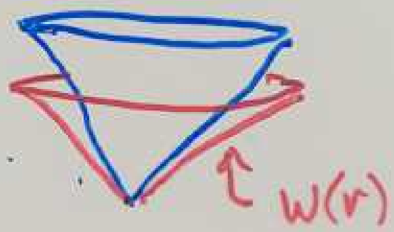
(3) Use t-coord s.t. $H(t=0) \leq -c/r^3$, test f^n
$$\int_{r < \frac{1}{2}\max(u)} \psi(u) \cdot (H - H(t=0)) \leq \cdots \leq C \quad \& \quad \psi \geq 1$$
$\Rightarrow$ $\log \max(u) \leq C$.

Removable Singular Points.

The radial mean curvature eq^{n}

$$\frac{1}{r^2}\left(\frac{r^2 w'}{\sqrt{1-w'^2}}\right)' = H(w)$$

has solutions with cone-like singular pts



$$w(r) = \int_0^r \frac{ds}{\sqrt{1+Ks^4}}$$

(these are Born-Infeld particles)

Theorem If $\exists\, \delta > 0$ st.

$$|u(x)| \le (1-\delta)|x|, \quad \forall x \in \dot{B}_1(0)$$

and $H(u) = h$ constant. Then $u \in C^\infty(B_1)$ (ie the singular pt $x=0$ is removable.)

Application: If $\exists\, p \in V$ st $V - I(p)$ is compact, then $\exists$ a C^∞ Cauchy surface with constant mean curvature.

Corollary If $V - I(p)$ is compact, then either V splits, or V is geodesically incomplete.

Problems.

① Clarify the coordinate conditions for AF maximal [B84] and CMC in approx Schw. [AI 92]

② Schwarzschild does not satisfy the bounded geometry conditions for maximal. Find conditions which ensure a suitable height bound.

③ Find conditions which guarantee a height bound for boosted spacetimes (or boosted surfaces) – or with interior containing multiple BHs & boosted matter.

④ Show ∃ maximal AF Cauchy in "approx" Reisner-Nordstrom extremal spacetimes.