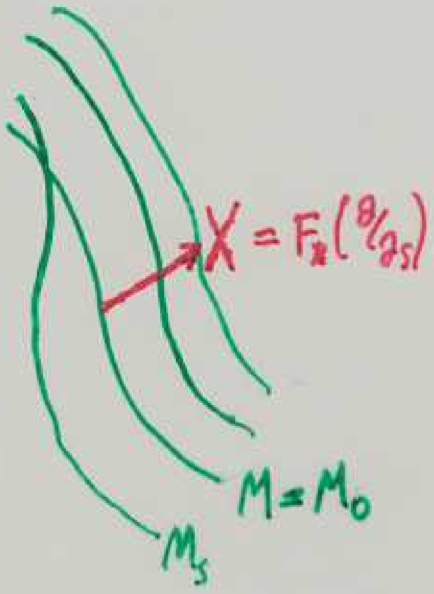
1^{st} & 2^{nd} Variation of Area



Immersion $M^n \to V^{n+1}$

Variation $F : M \times I \to V$

Variation vector field

$$X = F_*(\partial_s)$$

Compute derivatives of area of $M_s = F(M, s)$

e_i, $i = 1, .., n$ orthonormal frame on M_0

$E_i(s)$ Lie transport of e_i by vector field X:

$E_i(0) = e_i$, $L_X E_i = [X, E_i] = 0$

Volume n-vector

$$E_1(s) \wedge \ldots \wedge E_n(s) = \sqrt{\det g_{ij}}\; e_1 \wedge \ldots \wedge e_n$$

$$g_{ij}(s) = g^V(E_i, E_j)$$

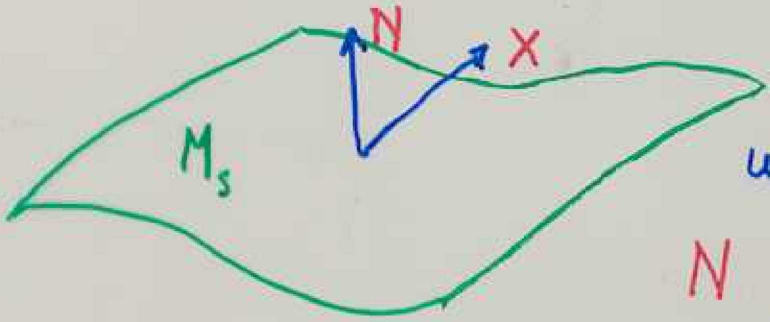
Area of M_s:

$$\text{area}(s) = \int_M \sqrt{\det g_{ij}(s)}\; dv_M$$

$$\frac{d}{ds}\,\text{area}(s) = \int_M g^{ij} L_X g_{ij} \cdot \tfrac{1}{2}\sqrt{\det g}$$

using the Lie derivative

$$L_X g_{ij} = g(e_i, \nabla_{e_j} X) + g(e_j, \nabla_{e_i} X)$$



with unit normal N satisfying

$$g(N,N) = \varepsilon = \begin{cases} -1 & \text{Lorentzian} \\ +1 & \text{Riemannian} \end{cases}$$

Decompose X into tangential & normal parts.

$$X = X^T + X^\perp$$

$$X^\perp = \varepsilon\, g(X,N)\, N$$

$\Rightarrow$

$$\tfrac{1}{2} g^{ij} L_X g_{ij} = \operatorname{div}_M X^T + \varepsilon H\, g(X,N)$$

where the mean curvature is

$$H = g^{ij}\, g(e_i, \nabla_{e_j} N)$$

$$\Rightarrow \frac{d}{ds} \text{area}(M_s) = \int_{M_s} \text{div}_{M_s} X \sqrt{\det g(s)}$$

$$= \int_{M_s} \left(\text{div}_{M_s} X^T + \phi H \right) \sqrt{g(s)}$$

Apply Stokes' thm with $X^T = 0$ on ∂M shows critical pts of area (under all variations fixing ∂M) are "minimal"

$$H = 0.$$

Define the second fundamental form

$$A_{ij} = g(e_i, \nabla_{e_j} N)$$

Differentiate again:

$$\frac{d^2}{ds^2} \text{area}(s) \Big|_{s=0} = \int_M (\text{div}_M X)^2 + D_X \left(g^{ij} g(E_i, \nabla_{E_j} X) \right)$$

$$= \int_M (\text{div}_M X)^2 - L_X g^{ij} g(e_i, \nabla_{e_j} X)$$

$$+ \text{div}_M \nabla_X X + g(\nabla_{e_i} X, \nabla_{e_i} X) + (R(X, e_i) X, e_i)$$

$$= \int_M (\text{div}_M X)^2 + |\nabla_i^\perp X|^2 + \text{div}_M \nabla_X X$$

$$- g(e_i, \nabla_{e_j} X) g(e_j, \nabla_{e_i} X) - g(R(e_i, X) X, e_i)$$

If $H = 0$ on M & $X = 0$ on ∂M, then

$$\frac{d^2}{ds^2} \operatorname{area}(M_s)\Big|_{s=0}$$

$$= \int_M |\nabla_i^\perp X^\perp|^2 - g(X,N)^2 |A|^2 - g(R(e_i, X^\perp)X^\perp, e_i)$$

Recalling $X^\perp = \phi N$ gives the Stability form:

$$\frac{d^2}{ds^2} \operatorname{area} = \int_M \varepsilon |\nabla\phi|^2 - \phi^2 (|A|^2 + \operatorname{Ric}(N,N))$$

Corollary: Any $H=0$ spacelike hypersurface in a spacetime satisfying TCC ($\operatorname{Ric}(N,N) \geq 0 \ \forall \ \underset{T}{\text{timelike}}$) is stable maximal ie $\forall \phi \in C_0^1(M)$, $\frac{d^2}{ds^2}\operatorname{area} \leq 0$.

The Riemannian case is more interesting: a stable minimal surface satisfies the stability inequality

$$\int_M |\nabla\phi|^2 - \phi^2 (|A|^2 + \operatorname{Ric}(N,N)) \geq 0$$

for all $\varphi \in C^1(M)$, $\phi = 0$ on ∂M.

2^{nd} Variation formula.

Instead, by direct calculation of $D_X H_X = \frac{\partial}{\partial s} H(s)$,

$$\varepsilon \Delta_M \phi = -\phi \left(|A|^2 + Ric(N,N) \right) - D_X H_X + D_{X^T} H$$

where $Ric = Ric^V$, $V^{n+1} \supset M^n$, and the final 2 terms give $-\phi D_N H_X$.

Gauss equation:

$$R^V - R^M = \varepsilon \left(|A|^2 - H^2 + 2 Ric(N,N) \right)$$

$$\Rightarrow \boxed{2 \Delta_M \phi = \left[R^M - R^V - \varepsilon \left(2 D_N H_X + |A|^2 + H^2 \right) \right] \phi}$$

where H_X is variation wrt $X^\perp = \phi N$.

Application: 2^{nd} VF gives linearised mean curvature: If M is CMC, $H_M =$ constant, then

$$\frac{\partial}{\partial s} H(s) \Big|_{s=0} = -\left(\varepsilon \Delta + |A|^2 + Ric(N,N) \right) \phi$$

is the linearised M.C. for variation vector $X = \phi N + X^T$ and $L = \varepsilon \Delta + |A|^2 + Ric(N,N) \geq 0$ (if TCC satisfied). IFT $\Rightarrow \exists$ CMC foliation.

Application:

Gradient bounds for prescribed mean curvature.

Dirichlet problem:

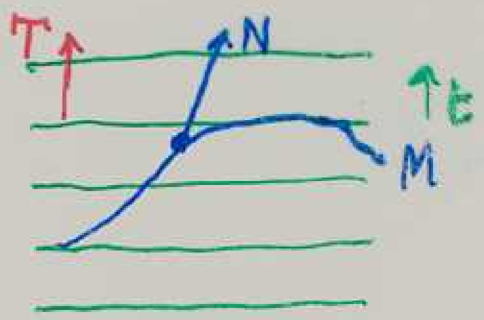
Find M spacelike with

$H_M = F$ (prescribed)

∂M prescribed.

Geometry:

time function t



height function $u = t|_M$

reference future (unit timelike) $T = \dfrac{-\nabla t}{|\nabla t|}$

tilt function $\nu = -g(T, N) \sim \dfrac{1}{\sqrt{1 - |Du|^2}}$

Mean curvature relation

$$\Delta_M u = \alpha^{-1} \nu H_M + \mathrm{div}_M(\nabla t)$$

Quasi-linear elliptic PDE $\to$ require uniform ellipticity

$\to$ require apriori bound on ν.

Gradient bounds for PMC spacelike.

reference future $T = -\alpha \nabla t$ (unit timelike)

for time function t

Height function $u = t|_M$

$$\Delta_M u = \frac{1}{\alpha} \nu H_M + \mathrm{div}_M \nabla t$$

where $\nu = -\langle T, N \rangle \sim \frac{1}{\sqrt{1-|Du|^2}}$

2VF $\Rightarrow$ $\Delta_M \nu = \nu(|A|^2 + Ric(N,N))$

$$- \langle T, \nabla^M H \rangle + D_T H_T$$

Control $D_T H_T$ via the Killing form

$$\mathcal{L}_X g(u,v) = \langle \nabla_u X, v \rangle + \langle \nabla_v X, u \rangle$$

$$\Rightarrow \boxed{\begin{aligned} D_X H_X &= \tfrac{1}{2} \nabla_N \mathcal{L}_X g(e_i, e_i) - \nabla_{e_i} \mathcal{L}_X g(N, e_i) \\ &\quad - \mathcal{L}_X g_{ij} A^{ij} - \tfrac{1}{2} H \mathcal{L}_X g(N,N) \end{aligned}}$$

Now apply maximum principle to test function $e^{Ku}\nu$, K large

$\Rightarrow$ apriori bounds for ν

Thm: If $H_M = F$,

$u|_{\partial M} = 0$

then $v \leq 2\exp(K \sup |u|)$

Proof sketch

let q be max pt for $e^{Ku} v$

case 1: $q \in \operatorname{Int} M \Rightarrow 0 = K v \nabla u + \nabla v$

and $0 \geq K v \Delta u - \boxed{K^2 v |\nabla u|^2} + \Delta v$

Now, H, Δu, Δv eq^{ns} $\Rightarrow$

$\Delta u \geq -C v^2$

$\Delta v \geq v|A|^2 - C v^3 - C v^2 |A|$

and $|\nabla v|^2 \leq (1+\varepsilon) v^2 \lambda_1^2 + C v^4$

$\Rightarrow v^2 |A|^2 \geq \left(1 + \frac{1}{2n}\right) |\nabla v|^2 - C v^4$

$\geq \left(1 + \frac{1}{2n}\right) K^2 v^2 |\nabla u|^2 - C v^4$

$\Rightarrow 0 \geq \frac{1}{2n} K^2 v |\nabla u|^2 - C K v^3$

so $K > 2nC \Rightarrow v(q) \leq 2$

case 2: $q \in \partial M \Rightarrow u(q) = 0$

use bnd on $H_{\partial M}$ ...

Einstein-Hilbert Lagrangian

E-H action

$$\delta \int_{V^4} R(g^V)\, dv_V = \int G_{ab}\, \delta g^{ab}\, dv_V$$

variation gives Einstein eq^{ns} $G_{ab} = 0$.

Derive 2^{nd} order pde $G_{ab} = 0$ because

$$R^V dv_V = dA + Q$$

where $Q = Q(\partial g, \partial g)$ is quadratic in ∂g.

this identity is 2^{nd} VF in disguise.

Suppose M_0, M_1 are spacelike, with

$$\partial M_0 = \partial M_1$$

$$\int_{V(0,1)} R^V dv_V = \int_0^1 \int_{M_s} \Big\{ \big(R^M + |A|^2 + H^2\big)\alpha - 2\Delta_M \alpha \Big\}\, dv_{M_s}\, ds$$

where $\alpha = |\nabla t|^{-1}$ is the lapse, and

$X = \partial_t = \alpha N + \beta^i \partial_i$ is the variation vector.

ADM Hamiltonian evolution

EL eq^{ns} from reduced EH Lagrangian

$$L = \alpha\left(R^M + |K|^2 + (\mathrm{tr} K)^2\right) dv_g$$

conjugate momentum

$$\pi^{ij} = \frac{\delta L}{\delta \dot{g}_{ij}} = \sqrt{g}\left(K^{ij} - \mathrm{tr} K\, g^{ij}\right)$$

$\Rightarrow$ Hamiltonian $\mathcal{H} = \int_M \left(\pi \cdot \dot{g} - L\right)$

$$H_{ADM} = -\int_M \Phi_\alpha(g, \pi)\, \xi^\alpha$$

where
$\xi = (\alpha, \beta^i)$ is the lapse-shift

Constraint
$$\begin{cases} \Phi_0 = \left(R^M - |K|^2 + (\mathrm{tr} K)^2\right)\sqrt{g} \\ \Phi_i = -2\left(\nabla^j K_{ij} - \nabla_i \mathrm{tr} K\right)\sqrt{g} \end{cases}$$

these still contain $\partial^2 g^r$ terms; exactly these terms are unbounded (non-L^1 as $r \to \infty$)

$$\Rightarrow \mathcal{H}(g, \pi; \xi) = \mathbb{P}_{ADM\,\alpha}\, \xi^\alpha_\infty - \int_M \Phi_\alpha \xi^\alpha$$

generates the evolution on the phase space

$$\mathcal{F} = \left(\mathring{g} + W^{2,2}_{-1/2}\right) \times W^{1,2}_{-3/2}$$

Raychaudhuri eq$^{\underline{n}}$ & singularities

Given: null hypersurface $\mathcal{N}$

null generator $L \subset \mathcal{N}$

null tangent ℓ (eg geodesic param$_s$).

orthogonal, spacelike, surface-forming

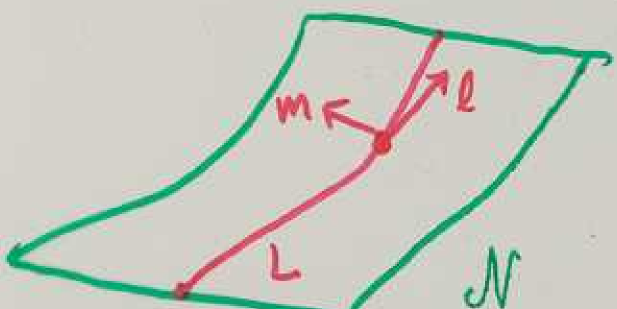
$$e_1, e_2 \perp \ell, \quad m = \frac{1}{\sqrt{2}}(e_1 - i e_2)$$

Geometry of $\mathcal{N}$:

null 2FF $\quad B_{ij} = g(e_i, \nabla_{e_j} \ell)$

expansion $\quad \theta = tr_2 B = 2B(m, \bar{m})$ = null m.c. $= 2\rho_{NP}$

shear $\quad \sigma = B - \frac{1}{2}\theta g = 2B(m, m)$



Raychaudhuri eq^n

$$\frac{d\theta}{ds} = -\frac{1}{2}\theta^2 - |\sigma|^2 - \mathrm{Ric}^V(\ell, \ell)$$

Consequence: If Null Converg. Cond$^{\underline{n}}$, and

if $\theta(0) < 0$, then $\theta(s) \to -\infty$ as $s \nearrow -2/\theta(0)$

$\Longrightarrow$ L is future null inextendible.

Quasi-spherical metrics

Given: Riemannian (M, g) with a mean-convex foliation Σ_r, $r \in (r_0, \infty)$

so $H_r = H(\Sigma_r) > 0$, and metric form

$$g = u^2 dr^2 + r^2 \mathring{g}_{ab} \, d\theta^a d\theta^b$$

$$H_r = \frac{1}{ru}\left(2 + r\partial_r(\log \det \mathring{g})\right) > 0$$

Gives 2^{nd} VF with $N = u^{-1}\partial_r$

$$R(g) = -2 D_N H + 2K_\Sigma - |A|^2 - H^2 - 2u^{-1}\Delta_r u$$

Idea: write $H_r = \frac{1}{ru} h$, $h = 2 + r\partial_r \log\det \mathring{g}$

so the terms $D_N H = u^{-1}\partial_r(u^{-1} \cdot h/r) \sim -u^{-3}\partial_r u \ldots$

$\Rightarrow$ prescribed $R(g)$ is $\boxed{\text{parabolic for } u}$:

$$h r \partial_r u - u^2 \Delta_r u = -r^2 u \partial_r(h/r) - u^3 K(g)$$
$$+ \frac{r^2 u}{2}\left(H_0^2 + |A_0|^2\right) + \frac{1}{2} u^3 r^2 R(g)$$

and the parabolic initial condition $u(r_0)$

determines the boundary mean curvature $H(r_0)$

Applications

① Construct $R(g) = 0$ (or prescribed $R(g)$) on $\mathbb{R}^3$ which is AF, and isometric to $(B(0,1), \delta)$ for $r < 1$,

or on $\mathbb{R}^3 \setminus B(0,1)$ with totally geodesic boundary at $r=1$ (but not Schwarzschild)

method

Choose inner geometry $(B(0,2), \delta)$

or Schw, $m = 1/2$, $r \leq 2m+1$, choose any extension g_r, $r \geq 2$ which C^∞ extends g_r, $r \leq 2$

bndy condition $u(r=2) = {}^{h}/_{r} H_r \big|_{r=2}$,

solve the parabolic eq^n with $R(g) = 0$.

② Construct IMCF examples:

choose g_r s.t. $\partial_r \det g_r = 0$

then variation vector $\partial_r = uN \Rightarrow u = -\frac{2}{r} \cdot \frac{1}{H}$

is IMCF ($-2/r$ is rescaling only)

Shih-Tam positivity

Given (Ω^3, g_Ω), $R(g_\Omega) \geq 0$, bndy Σ and bounday geometry (H_Σ, g_Σ)

assume convexity $K(g_\Sigma) > 0$, $H_\Sigma > 0$

Thm: the BY mass

$$m_{BY} = \oint_\Sigma (H_0 - H_\Sigma)\, dv_\Sigma \geq 0$$

where H_0 is m.c of $\Phi(\Sigma)$, where

$$\Phi : (\Sigma, g_\Sigma) \to \mathbb{R}^3 \quad \text{is isometry.}$$

and $m_{BY} = 0$ iff (Ω, g_Ω) is flat.

Sketch $\Sigma_0 = \Phi(\Sigma)$, gaussian coords on $\mathbb{R}^3$

$$|dx|^2 = dr^2 + g_r \Rightarrow H_0(r)$$

$$ds^2 = u^2 dr^2 + g_r \Rightarrow H(r)$$

choose warping function u to solve the $R(ds^2) = 0$ parabolic eq$^{\underline{n}}$, with initial condition $u(0) = H_0 / H_\Sigma$. Then $g_\Omega \cup ds^2$ AF, has $R \geq 0$, and $\boxed{\frac{d}{dr} \oint_{\Sigma_r} (H_0(r) - H(r))\, dv_r \leq 0.}$

Now apply PMT.

???

Schrödinger - Lichnerowicz

(M, g) asymptotically flat

$$\int_M -|D\psi|^2 + |\nabla\psi|^2 + \frac{1}{4} R |\psi|^2$$

$$= \oint_{\partial M} \langle \psi, (D_{\partial M} + \frac{1}{2} H_{\partial M}) \psi \rangle$$

Note: 1) Similar structure to 2VF

2) Boundary term $= \oint \mu(\psi)$

$$\mu(\psi) = \langle \psi, \gamma^{[ij]} \nabla_j \psi \rangle * (e_i)$$